

GEOMETRIC TRANSITIONS AND SYZ MIRROR SYMMETRY

ATSUSHI KANAZAWA SIU-CHEONG LAU

ABSTRACT. We prove that generalized conifolds and orbifolded conifolds are mirror symmetric under the SYZ program with quantum corrections. Our work mathematically confirms the gauge-theoretic assertion of Aganagic–Karch–Lüst–Miemiec, and also provides a supportive evidence to Morrison’s conjecture that geometric transitions are reversed under mirror symmetry.

1. INTRODUCTION

In [Mor], Morrison proposed that geometric transitions are reversed under mirror symmetry. A geometric transition is a birational contraction followed by a complex smoothing, or in the reverse way, applied to a Kähler manifold (see a nice review [Ros] by Rossi) . We will denote a geometric transition by $\widehat{X} \dashrightarrow X \rightsquigarrow \widetilde{X}$, where $\widehat{X} \dashrightarrow X$ is a birational contraction and $X \rightsquigarrow \widetilde{X}$ is a smoothing. The conjecture can be formulated as follows.

Conjecture 1.1 (Morrison [Mor]). *Let \widehat{X} and \widetilde{X} be Calabi–Yau manifolds, and suppose they are related by a geometric transition $\widehat{X} \dashrightarrow X \rightsquigarrow \widetilde{X}$. Suppose Y_1 and Y_2 are the mirrors of \widehat{X} and \widetilde{X} respectively. Then there exists a geometric transition $Y_2 \dashrightarrow Y \rightsquigarrow Y_1$ relating Y_1 and Y_2 .*

The present paper investigates mirror symmetry for geometric transitions of two specific types of local singularities, namely generalized conifolds and orbifolded conifolds. Let us first recall mirror symmetry for a conifold. A conifold is an isolated singularity defined by $\{xy - zw = 0\} \subset \mathbb{C}^4$. It is an important singularity appearing in algebraic geometry and also plays a special role in superstring theory. A folklore mirror symmetry for the conifold [Mor, Sze] that the deformed conifold is mirror symmetric to the resolved conifold can be refined in the framework of SYZ mirror symmetry as follows.

Theorem 1.2 (Conifold case of Theorem 3.1). *Let $\widehat{X} := \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \setminus D$ be the resolved conifold with a smooth anti-canonical divisor D removed, and*

$$\widetilde{X} := \{(x, y, z, w) \in \mathbb{C}^4 \mid xy - zw = 1\} \setminus (\{z = 1\} \cup \{w = 1\})$$

the deformed conifold with the anti-canonical divisor $\{z = 1\} \cup \{w = 1\}$ removed. Then \widehat{X} and \widetilde{X} are SYZ mirror to each other.

Although removing the divisors certainly does not affect the local geometry of the singularity, it is important when we discuss, for example, wrapped Fukaya categories and homological mirror symmetry¹.

¹We are grateful to Murad Alim for informing us about the importance of this issue.

We now focus on two natural generalizations of the conifold: generalized conifolds and orbifolded conifolds. For integers $k, l \geq 1$, a generalized conifold is given by

$$G_{k,l}^\sharp := \{(x, y, z, w) \in \mathbb{C}^4 \mid xy - (1+z)^k(1+w)^l = 0\}$$

and an orbifolded conifold is given by

$$O_{k,l}^\sharp := \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^5 \mid u_1v_1 - (1+z)^k = u_2v_2 - (1+z)^l = 0\}.$$

(We have made a change of coordinates, namely $z \mapsto 1+z$ and $w \mapsto 1+w$, for later convenience.) They reduce to the conifold when $k = l = 1$. The punctured generalized conifold is defined as $G_{k,l} := G_{k,l}^\sharp \setminus D_G$, where $D_G := \{z = 0\} \cup \{w = 0\}$ is a normal-crossing anti-canonical divisor of $G_{k,l}^\sharp$, and the punctured orbifolded conifold as $O_{k,l} := O_{k,l}^\sharp \setminus D_O$, where $D_O := \{z = 0\}$ is a smooth anti-canonical divisor of $O_{k,l}^\sharp$. As is the case of the conifold, their symplectic structures and complex structures are governed by the crepant resolutions and deformations respectively. The main theorem of the present paper is the following.

Theorem 1.3 (Theorem 3.1). *The punctured generalized conifold $G_{k,l}$ is mirror symmetric to the punctured orbifolded conifold $O_{k,l}$ in the sense that the deformed punctured generalized conifold $\widetilde{G}_{k,l}$ is SYZ mirror symmetric to the resolved punctured orbifolded conifold $\widetilde{O}_{k,l}$, and the resolved punctured generalized conifold $\widehat{G}_{k,l}$ is SYZ mirror symmetric to the deformed punctured orbifolded conifold $\widehat{O}_{k,l}$.*

$$\begin{array}{ccccc}
 \widetilde{G}_{k,l} & \leftarrow \text{wavy} & G_{k,l} & \leftarrow \text{dotted} & \widehat{G}_{k,l} \\
 \uparrow \text{SYZ} & & \uparrow \text{MS} & & \uparrow \text{SYZ} \\
 \widehat{O}_{k,l} & \text{dotted} \rightarrow & O_{k,l} & \text{wavy} \rightarrow & \widetilde{O}_{k,l}
 \end{array}$$

According to Theorem 1.3 the mirror duality of the conifold is purely caused by the fact that the conifold can be seen as either a generalized or an orbifolded conifold. The mirror duality of $G_{k,l}^\sharp$ and $O_{k,l}^\sharp$ has previously been studied by physicists Aganagic, Karch, Lust and Miemiec in [AKLM], where they use gauge theory and brane configurations. Our work mathematically confirms their gauge-theoretic assertion, and also provides a supportive evidence to Morrison's conjecture that geometric transitions are reversed under mirror symmetry.

In the present paper, we use the framework introduced by the second author with Chan and Leung [CLL] for defining SYZ mirror pairs. Namely, generating functions of open Gromov–Witten invariants of fibers of a Lagrangian fibration were used to construct the complex coordinates of the mirror. The essential ingredient is wall-crossing of the generating functions, which was first studied by Auroux [Aur]. We can also bypass symplectic geometry and employs the Gross–Siebert program [GS] which uses tropical geometry instead for defining mirror pairs. This tropical approach was taken by Castano-Bernard and Matessi [CbBM14] in the study of conifold transitions for compact Calabi–Yau varieties. Although our work has some overlap with theirs, the methods and interests are quite different. They deal with simultaneous multiple conifold transitions while we discuss orbifold/generalized conifolds transitions. In this paper tropical geometry is not directly needed since symplectic geometry can be handled directly.

One crucial feature of the present work is the involutive property of SYZ mirror symmetry. Namely, taking SYZ mirror twice gets back to itself, which we believe is an important point but often overlooked in literatures. We exhibit this feature by carrying out the SYZ construction for all the four directions in Theorem 1.3, namely from $\widehat{G}_{k,l}$ to $\widehat{O}_{k,l}$, and from $\widehat{O}_{k,l}$ back to $\widehat{G}_{k,l}$; from $\widetilde{G}_{k,l}$ to $\widetilde{O}_{k,l}$, and from $\widetilde{O}_{k,l}$ back to $\widetilde{G}_{k,l}$. The SYZ construction from $\widehat{G}_{k,l}$ to $\widehat{O}_{k,l}$ is a bit tricky and we will discuss it in details. We will employ the various techniques developed in [Aur, CLL, AAK, Lau1].

Another interesting feature is the dependence of the choice of a Lagrangian fibration, namely a Lagrangian fibration has to be ‘compatible’ with the choice of an anti-canonical divisor, in the sense that the anti-canonical divisor is the preimage of the boundary of the base of the Lagrangian fibration, in order to obtain the corresponding mirror. For example, $\widehat{G}_{k,l}$ admits two different Lagrangian fibrations: the Gross fibration and a ‘doubled’ Gross fibration. The former is not compatible with the anti-canonical divisor D_G , and does not produce the orbifold conifold as its SYZ mirror. Choosing appropriate Lagrangian fibrations is a key step in our work.

Lastly, Theorem 1.3 not only unveils a connection between geometric transitions and SYZ mirror symmetry, but also yields many interesting problems and conjectures that naturally extend what is known for the conifolds. In fact, based on the local models studied in this paper, the second author recently confirmed Morrison’s conjecture for a class of geometric transitions of the Schoen’s Calabi–Yau threefolds [Lau2].

Structure of Paper. Section 2 introduces generalized conifolds and orbifolded conifolds and basic properties thereof. Section 3 begins with a review on the Lagrangian torus fibrations and the SYZ program. Then we prove the main theorem (Theorem 1.3) by carrying out the SYZ constructions. Section 4 discusses global geometric transitions and provides a few examples.

Acknowledgement. The authors are grateful to Murad Alim, Kwokwai Chan, Yu-Wei Fan, Conan Leung, Shing-Tung Yau for useful discussions and encouragement. A.K. was supported by the Harvard CMSA. S.C. L. appreciates the excellent environment provided by Harvard University in writing this paper. The work of S.C. L. was supported by Simons Collaboration Grant for Mathematicians at the time of publication of this paper.

2. GENERALIZED AND ORBIFOLDED CONIFOLDS

In this section, we introduce two natural generalizations of the conifold, namely generalized conifolds and orbifolded conifolds. These two singularities possess interesting geometries and were studied by physicists in the context of gauge theory, for instance in [KKV, AKLM, Mie].

2.1. Generalized conifolds $G_{k,l}^\sharp$. A toric Calabi–Yau manifold is a semi-projective toric manifold with trivial canonical bundle. In dimension three, they can be described by a lattice polytope $\Delta \subset \mathbb{R}^2$ whose vertices lie in the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$. Its fan is produced by taking the cone over $\Delta \times \{1\} \subset \mathbb{R}^3$. A crepant resolution of a toric Calabi–Yau threefold corresponds to a subdivision of Δ into standard

triangles², which gives a refinement of the fan. For instance, the total space of the canonical bundle K_S of a smooth toric surface S is a toric Calabi–Yau threefold. In this situation, the surface S is the toric variety \mathbb{P}_Δ whose fan polytope is Δ .

The condition that a toric Calabi–Yau threefold contains no compact 4-cycles is equivalent to the condition that the polytope Δ contains no interior lattice points. The lattice polygons without interior lattice point are classified, up to the action of $\mathrm{GL}(2, \mathbb{Z})$, into two types:

- (1) triangle with vertices $(0, 0), (2, 0), (0, 2)$,
- (2) trapezoid $\Delta_{k,l}$ with vertices $(0, 0), (0, 1), (k, 0), (l, 1)$ for $k \geq l \geq 0$ with $(k, l) \neq (0, 0)$ (Figure 1(a)).

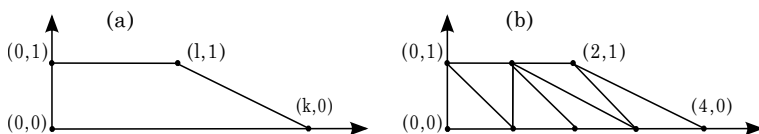


FIGURE 1. (a) Trapezoid $\Delta_{k,l}$, (b) Crepant resolution $\widehat{G_{4,2}^\#}$

The former is the quotient of \mathbb{C}^3 by the subgroup $(\mathbb{Z}_2)^2 \subset \mathrm{SL}(3, \mathbb{C})$ generated by the two elements $\mathrm{diag}(-1, -1, 1)$ and $\mathrm{diag}(1, -1, -1)$. In this paper, we are interested in the latter, which corresponds to the generalized conifold $G_{k,l}$ for $k \geq l \geq 1$. We do not consider the case $l = 0$, where the toric singularity essentially comes from the A_k -singularity in 2-dimensions³. The dual cone of the cone over the trapezoid $\Delta_{k,l}$ is spanned by the vectors

- (1) $\nu_1 := (1, 0, 0), \nu_2 := (0, -1, 1), \nu_3 := (-1, l - k, k), \nu_4 := (0, 1, 0)$

with relation $\nu_1 - k\nu_2 + \nu_3 - l\nu_4 = 0$. In equation the generalized conifold $G_{k,l}^\#$ is given by

$$G_{k,l}^\# := \{xy - z^k w^l = 0\} \subset \mathbb{C}^4.$$

The coordinates x, y, z, w correspond to the dual lattice points $\nu_1, \nu_3, \nu_2, \nu_4$ respectively. For $(k, l) \neq (1, 1)$, the generalized conifold $G_{k,l}^\#$ is a quotient of the conifold (which is given by $(k, l) = (1, 1)$) and has a 1-dimensional singular locus. A punctured generalized conifold is $G_{k,l} := G_{k,l}^\# \setminus D_G$, where $D_G = \{z = 1\} \cup \{w = 1\}$ is an anti-canonical divisor of $G_{k,l}^\#$. A crepant resolution $\widehat{G_{k,l}}$ of $G_{k,l}$ is called a resolved generalized conifold. We observe that $\widehat{G_{k,l}} = \widehat{G_{k,l}^\#} \setminus D_{\widehat{G}}$, where $D_{\widehat{G}}$ is an anti-canonical divisor of $\widehat{G_{k,l}^\#}$, and it uniquely corresponds to a maximal triangulation of the trapezoid $\Delta_{k,l}$ (Figure 1(b)). The resolved generalized conifold $\widehat{G_{k,l}}$ is endowed with a natural symplectic structure as an open subset of a smooth toric variety $\widehat{G_{k,l}^\#}$.

Proposition 2.1. *There are $\binom{k+l}{k}$ distinct crepant resolutions of $G_{k,l}$ (or equivalently $G_{k,l}^\#$).*

²A standard triangle in \mathbb{R}^2 is isomorphic to the convex hull of $(0, 0), (1, 0), (0, 1)$ under the $\mathbb{Z}^2 \times \mathrm{GL}(2, \mathbb{Z})$ -transformation.

³Mirror symmetry of this class of singularities is discussed in [Sze, Section 5].

Proof. There is a bijection between the crepant resolutions of $G_{k,l}$ and the maximal triangulations of $\Delta_{k,l}$. The assertion easily follows by induction with the relation $\binom{k+l+1}{k+1} = \binom{k+l}{k} + \binom{k+l}{k+1}$. \square

We may also smooth out the punctured generalized conifold $G_{k,l}$ by deforming the equation. The deformed generalized conifold $\widehat{G}_{k,l}$ is defined as

$$\widehat{G}_{k,l} := \left\{ (x, y, z, w) \in \mathbb{C}^2 \times (\mathbb{C} \setminus \{1\})^2 \mid xy - \sum_{i=0}^k \sum_{j=0}^l a_{i,j} z^i w^j = 0 \right\}$$

for generic $a_{i,j} \in \mathbb{C}$. The symplectic structure of $\widehat{G}_{k,l}$ is given by the restriction of the standard symplectic structure on $\mathbb{C}^2 \times (\mathbb{C} \setminus \{1\})^2$. We observe that the complex deformation space has dimension $(k+1)(l+1) - 3$ because three of the parameters can be eliminated by rescaling z, w and rescaling the whole equation. On the other hand, the Kähler deformation space has dimension $(k+1) + (l+1) - 3$, the number of linearly dependent lattice vectors in the polytope. It is the number of the exceptional \mathbb{P}^1 's and a Kähler form is parametrized by the area of these.

2.2. Orbifolded conifolds $O_{k,l}^\sharp$. Let X^\sharp be the conifold $\{xy - zw = 0\} \subset \mathbb{C}^4$. For $k \geq l \geq 1$, the orbifolded conifold $O_{k,l}^\sharp$ is the quotient of the conifold X^\sharp by the abelian group $\mathbb{Z}_k \times \mathbb{Z}_l$, where \mathbb{Z}_k and \mathbb{Z}_l respectively act by

$$(x, y, z, w) \mapsto (\zeta_k x, \zeta_k^{-1} y, z, w), \text{ and } (x, y, z, w) \mapsto (x, y, \zeta_l z, \zeta_l^{-1} w)$$

where ζ_k, ζ_l are primitive k -th and l -th roots of unity respectively (assume $\gcd(k, l) = 1$ for simplicity [AKLM]). Alternatively the orbifolded conifold $O_{k,l}^\sharp$ is realized as a hypersurface in \mathbb{C}^5 :

$$O_{k,l}^\sharp = \{u_1 v_1 = z^k, u_2 v_2 = z^l\} \subset \mathbb{C}^5.$$

The orbifolded conifold $O_{k,l}^\sharp$ is an example of a toric Calabi–Yau threefold and the corresponding polytope is given by the rectangle $\square_{k,l}$ with the vertices $(0, 0), (k, 0), (0, l), (k, l)$ (Figure 2(a)).

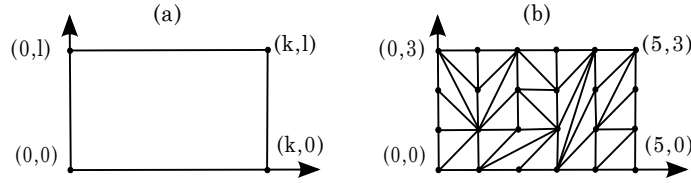


FIGURE 2. (a) Rectangle $\square_{k,l}$, (b) Crepant resolution $\widehat{O}_{5,3}^\sharp$

The dual cone of the cone over the rectangle $\square_{k,l}$ is spanned by the following vectors

$$v_1 := (1, 0, 0), v_2 := (0, -1, l), v_3 := (-1, 0, k), v_4 := (0, 1, 0)$$

with relation $lv_1 - kv_2 + lv_3 - kv_4 = 0$.

A punctured orbifolded conifold is $O_{k,l} := O_{k,l}^\sharp \setminus D_O$, where $D_O = \{z = 1\}$ is a smooth anti-canonical divisor of $O_{k,l}^\sharp$. Then a resolved orbifolded conifold $\widehat{O}_{k,l}$

is defined to be a crepant resolution of $O_{k,l}$. As before, $\widehat{O}_{k,l} = \widehat{O}_{k,l}^\sharp \setminus D_{\widehat{O}}$, where $D_{\widehat{O}}$ is a smooth anti-canonical divisor of the toric crepant resolution $\widehat{O}_{k,l}^\sharp$, and it corresponds to a maximal triangulation of the trapezoid $\square_{k,l}$ (Figure 2(b)). It has a canonical symplectic structure as an open subset of a smooth toric variety $\widehat{O}_{k,l}^\sharp$. In contrast to Proposition 2.1, it is a famous open problem to find the number of the crepant resolutions of the orbifolded conifold $O_{k,l}^\sharp$ [KZ]. The punctured orbifolded conifold $O_{k,l}$ can also be smoothed out by deforming the defining equations. Thus the deformed orbifolded conifold $\widetilde{O}_{k,l}$ is given by

$$\widetilde{O}_{k,l} := \left\{ (u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times (\mathbb{C} \setminus \{1\}) \mid u_1 v_1 = \sum_{i=0}^k a_i z^i, u_2 v_2 = \sum_{j=0}^l b_j z^j \right\}$$

for generic coefficients $a_i, b_j \in \mathbb{C}$. The symplectic structure of $\widetilde{O}_{k,l}$ is the restriction of the standard symplectic structure on $\mathbb{C}^4 \times (\mathbb{C} \setminus \{1\})$. The complex deformation space of $\widetilde{O}_{k,l}$ has dimension $(k+1) + (l+1) - 3$, while the Kähler deformation space has dimension $(k+1)(l+1) - 3$. Therefore the naive dimension counting is compatible with our claim that these two classes of singularities are mirror symmetric. We will formulate this mirror duality in a rigorous manner by using SYZ mirror symmetry in the next section.

3. SYZ MIRROR CONSTRUCTION

The Strominger–Yau–Zaslow (SYZ) conjecture [SYZ] provides a foundational geometric understanding of mirror symmetry. It asserts that, for a mirror pair of Calabi–Yau manifolds X and X^\vee , there exist Lagrangian torus fibrations $\pi : X \rightarrow B$ and $\pi^\vee : X^\vee \rightarrow B$ which are fiberwise-dual to each other. In particular, it suggests an intrinsic construction of the mirror X^\vee by fiberwise dualizing a Lagrangian torus fibration on X . This is motivated by *T-duality* studied by string theorists.

The SYZ program has been carried out successfully in the semi-flat case [Leu] in which the discriminant locus of the fibrations is empty. When singular fibers are present, quantum corrections by open Gromov–Witten invariants of the fibers are necessary, and they exhibit wall-crossing phenomenon. Wall-crossing of open Gromov–Witten invariants was first studied by Auroux [Aur]. Later on [CLL] gave an SYZ construction of mirrors with quantum corrections, which will be used in this paper. In algebro-geometric context, the Gross–Siebert program [GS] gives a reformulation of the SYZ program using tropical geometry, which provides powerful techniques to compute wall-crossing and scattering order-by-order. In this paper we will use the symplectic rather than the tropical approach.

We will first give a quick review of the setting of [CLL] for SYZ with quantum corrections in Section 3.1. We say that X is *SYZ mirror symmetric* to Y if Y is produced from X as a SYZ mirror manifold by this SYZ mirror construction. The later parts of this section prove the following main theorem.

Theorem 3.1. *The punctured generalized conifold $G_{k,l}$ is mirror symmetric to the punctured orbifolded conifold $O_{k,l}$ in the sense that the deformed generalized conifold $\widetilde{G}_{k,l}$ is SYZ mirror symmetric to the resolved orbifolded conifold $\widehat{O}_{k,l}$, and the resolved generalized conifold $\widehat{G}_{k,l}$ is SYZ mirror symmetric to the deformed*

orbifolded conifold $\widetilde{O}_{k,l}$:

$$\begin{array}{ccccc}
 \widetilde{G}_{k,l} & \leftarrow \text{wavy} & G_{k,l} & \leftarrow \text{dotted} & \widehat{G}_{k,l} \\
 \uparrow \text{SYZ} & & \uparrow \text{MS} & & \uparrow \text{SYZ} \\
 \widetilde{O}_{k,l} & \text{dotted} \rightarrow & O_{k,l} & \text{wavy} \rightarrow & \widehat{O}_{k,l}
 \end{array}$$

This mathematically confirms the gauge-theoretic assertion of the string theorists Aganagic–Karch–Lüst–Miemiec [AKLM] and also provides a supportive evidence to Morrison’s conjecture [Mor] from the view point of SYZ mirror symmetry.

3.1. SYZ construction with quantum corrections. In this subsection we review the SYZ construction with quantum corrections given in [CLL]. We add a clarification that we only use transversal disc classes (Definition 3.5) in the definition of the mirror space.

Let $\pi : X \rightarrow B$ be a proper Lagrangian torus fibration of a Kähler manifold (X, ω) such that the base B is a compact manifold with corners, and the preimage of each codimension-one facet of B is a smooth irreducible divisor denoted as D_i for $1 \leq i \leq m$. We assume that the regular Lagrangian fibers of π are special with respect to a nowhere-vanishing meromorphic volume form Ω on X whose pole divisor is the boundary divisor $D := \sum_{i=1}^m D_i$ (and hence D is an anti-canonical divisor). We denote by $B_0 \subset B$ the complement of the discriminant locus of π , and we assume that B_0 is connected⁴. We always denote by F_b a fiber of π at $b \in B_0$.

Lemma 3.2 (Maslov index of disc classes [Aur, Lemma 3.1]). *For a disc class $\beta \in \pi_2(X, F_b)$ where $b \in B_0$, the Maslov index of β is $\mu(\beta) = 2D \cdot \beta$.*

Definition 3.3 (Wall [CLL]). *The wall H of a Lagrangian fibration $\pi : X \rightarrow B$ is the set of point $b \in B_0$ such that $F_b := \pi^{-1}(b)$ bounds non-constant holomorphic disks with Maslov index 0.*

The complement of $H \subset B_0$ consists of several connected components, which we call chambers. Over different chambers the Lagrangian fibers behave differently in a Floer-theoretic sense. Away from the wall H , the one-point open Gromov–Witten invariants are well-defined using the machinery of Fukaya–Oh–Ohta–Ono [FOOO].

Definition 3.4 (Open Gromov–Witten invariants [FOOO]). *For $b \in B_0 \setminus H$ and $\beta \in \pi_2(X, F_b)$, let $\mathfrak{M}_1(\beta)$ be the moduli space of stable discs with one boundary marked point representing β , and $[\mathfrak{M}_1(\beta)]^{\text{virt}}$ be the virtual fundamental class of $\mathfrak{M}_1(\beta)$. The open Gromov–Witten invariant associated to β is $n_\beta := \int_{[\mathfrak{M}_1(\beta)]^{\text{virt}}} \text{ev}^*[\text{pt}]$, where $\text{ev} : \mathfrak{M}_1(\beta) \rightarrow F_b$ is the evaluation map at the boundary marked point and $[\text{pt}]$ is the Poincaré dual of the point class of F_b .*

We will restrict to disc classes which are transversal to the boundary divisor D when we construct the mirror space (while for the mirror superpotential we need to consider all disc classes).

Definition 3.5 (Transversal disc class). *A disc class $\beta \in \pi_2(X, F_b)$ for $b \in B_0$ is said to be transversal to the boundary divisor D , which is denoted as $\beta \pitchfork D$, if all stable discs in $\mathfrak{M}_1(\beta)$ intersect transversely with the boundary divisor D .*

⁴When the discriminant locus has codimension-two, B_0 is automatically connected. Although the Lagrangian fibrations of $\widetilde{G}_{k,l}$ we study have codimension-one discriminant loci, B_0 is still connected.

Due to dimension reason, the open Gromov–Witten invariant n_β is non-zero only when the Maslov index $\mu(\beta) = 2$. When β is transversal to D or when X is semi-Fano, namely $c_1(\alpha) = D \cdot \alpha \geq 0$ for all holomorphic sphere classes α , the number n_β is invariant under small deformation of complex structure and under Lagrangian isotopy in which all Lagrangian submanifolds in the isotopy do not intersect D nor bound non-constant holomorphic disc with Maslov index $\mu(\beta) < 2$.

The paper [CLL] proposed a procedure which realizes the SYZ program based on symplectic geometry as follows:

- (1) Construct the semi-flat mirror X_0^\vee of $X_0 := \pi^{-1}(B_0)$ as the space of pairs (b, ∇) where $b \in B_0$ and ∇ is a flat $U(1)$ -connection on the trivial complex line bundle over F_b up to gauge. There is a natural map $\pi^\vee : X_0^\vee \rightarrow B_0$ given by forgetting the second coordinate. The semi-flat mirror X_0^\vee has a canonical complex structure [Leu] and the functions $e^{-\int_\beta \omega} \text{Hol}_\nabla(\partial\beta)$ on X_0^\vee for disc classes $\beta \in \pi_2(X, F_b)$ are called semi-flat complex coordinates. Here $\text{Hol}_\nabla(\partial\beta)$ denotes the holonomy of the flat $U(1)$ -connection ∇ along the path $\partial\beta \in \pi_1(F_b)$.
- (2) Define the generating functions of open Gromov–Witten invariants for $1 \leq i \leq m$

$$(2) \quad Z_i(b, \nabla) := \sum_{\substack{\beta \in \pi_2(X, F_b) \\ \beta \cdot D_i = 1, \beta \nabla D}} n_\beta e^{-\int_\beta \omega} \text{Hol}_\nabla(\partial\beta),$$

for $(b, \nabla) \in (\pi^\vee)^{-1}(B_0 \setminus H)$, which serve as quantum corrected complex coordinates. The function Z_i can be written in terms of the semi-flat complex coordinates, and hence they generate a subring $\mathbb{C}[Z_1, \dots, Z_m]$ in the function ring⁵ over $(\pi^\vee)^{-1}(B_0 \setminus H)$.

- (3) Define the SYZ mirror of X with respect to the Lagrangian torus fibration π to be the pair (X^\vee, W) where $X^\vee := \text{Spec}(\mathbb{C}[Z_1, \dots, Z_m])$ and

$$W = \sum_{\beta \in \pi_2(X, F_b)} n_\beta e^{-\int_\beta \omega} \text{Hol}_\nabla(\partial\beta).$$

Moreover, X^\vee is defined as the SYZ mirror of a non-compact Calabi–Yau manifold Y if it is obtained from the above construction for a compactification of Y . It is expected that different compactifications would result in the same SYZ mirror. In this paper we fix one compactification as an initial data for the SYZ construction.

In the following sections we will apply the above recipe to the generalized conifolds and orbifolded conifolds. We will carry out in detail the SYZ construction from $\widetilde{G}_{k,l}$ to $\widetilde{O}_{k,l}$ which is the most interesting case (Section 3.2), in which we construct a doubled version of the Gross fibration [Gol, Gro] and compute the open Gromov–Witten invariants. The other cases, namely the SYZ constructions from $\widetilde{O}_{k,l}$ to $\widetilde{G}_{k,l}$, from $\widetilde{O}_{k,l}$ to $\widetilde{G}_{k,l}$, and from $\widetilde{G}_{k,l}$ to $\widetilde{O}_{k,l}$, are essentially obtained by applying the techniques developed in [Lau1, CLL, AAK], and so we will be brief. In fact $G_{k,l}^\sharp$ and $O_{k,l}^\sharp$ are useful testing grounds for the SYZ program and we shall illustrate how these various important ideas fit together by examining them.

⁵In general we need to use the Novikov ring instead of \mathbb{C} since Z_i could be a formal Laurent series. In the cases that we study later, Z_i are Laurent polynomials whose coefficients are convergent, and hence the Novikov ring is not necessary.

3.2. **SYZ from $\widehat{G}_{k,l}$ to $\widetilde{O}_{k,l}$.** We first construct the SYZ mirror of the resolved generalized conifold $\widehat{G}_{k,l}$. While the resolved generalized conifold $\widehat{G}_{k,l}^\sharp$ is a toric Calabi–Yau threefold, we will *not* use the Gross fibration [Gol, Gro] because it is not compatible with the chosen anti-canonical divisor D_G and hence do not produce the resolved orbifolded conifold $\widetilde{O}_{k,l}$ as the mirror. We will instead use a *doubled* version of the Gross fibration explained below.

The fan of $\widehat{G}_{k,l}^\sharp$ is given by the cone over a triangulation depicted in Figure 1(b). We label the divisors corresponding to the rays generated by $(i, 0, 1)$ to be D_i for $0 \leq i \leq k$, and those corresponding to the rays generated by $(j, 1, 1)$ to be D_{k+1+j} for $0 \leq j \leq l$. Each divisor D_i corresponds to a basic disc class $\beta_i \in \pi_2(X, L)$ where L denotes a moment-map fiber [CLL].

Let us first compactify the resolved generalized conifold $\widehat{G}_{k,l}$ as follows. We add the rays generated by $(0, 0, -1)$, $(0, -1, -1)$, $(1, 0, 0)$ and $(-1, 0, 0)$, and the corresponding cones, to the fan of $\widehat{G}_{k,l}^\sharp$. Let us denote the resulting toric variety by $\widehat{G}_{k,l}^*$ (and we fix a toric Kähler form on it). Let $\mathcal{D}_{z=\infty}$, $\mathcal{D}_{w=\infty}$, $\mathcal{D}_{\xi=0}$, and $\mathcal{D}_{\xi=\infty}$ be the corresponding additional toric prime divisors and $\beta_{z=\infty}$, $\beta_{w=\infty}$, $\beta_{\xi=0}$ and $\beta_{\xi=\infty}$ be the additional basic disc classes respectively.

Note that $\widehat{G}_{k,l}^*$ is in general not semi-Fano since there could be holomorphic spheres with the Chern class $c_1 < 0$ supported in the newly added divisors (or in other words the fan polytope of $\widehat{G}_{k,l}^*$ may contain a interior lattice point). However since we only need to consider transversal disc classes (Definition 3.5) in the definition of X^\vee , these holomorphic spheres do not enter into our constructions.

We now construct a special Lagrangian fibration and apply the SYZ construction on $\widehat{G}_{k,l}^*$. Consider the Hamiltonian T^1 -action on $\widehat{G}_{k,l}^*$ corresponding to the vector $(1, 0, 0)$ in the vector space which supports the fan. Denote by $\pi_{T^1} : \widehat{G}_{k,l}^* \rightarrow \mathbb{R}$ the moment map associated to this Hamiltonian T^1 -action, whose image is a closed interval I . Let θ be the angular coordinate corresponding to the Hamiltonian T^1 -action. Recall that x, y, z, w are toric functions corresponding to the lattice points $\nu_1, \nu_3, \nu_2, \nu_4$ (in the vector space which supports the moment polytope) defined by Equation (1) respectively. Note that $z = 0$ on the toric divisors D_0, \dots, D_k , while $w = 0$ on the toric divisors $D_{k+1}, \dots, D_{k+l+1}$. Moreover the pole divisors of z and w are $\mathcal{D}_{z=\infty}$ and $\mathcal{D}_{w=\infty}$ respectively.

The toric Kähler form on $\widehat{G}_{k,l}^*$ can be written as

$$\omega := d\pi_{T^1} \wedge d\theta + \frac{\sqrt{-1}}{c_1(1+|z|^2)^2} dz \wedge d\bar{z} + \frac{\sqrt{-1}}{c_2(1+|w|^2)^2} dw \wedge d\bar{w}$$

for some $c_1, c_2 \in \mathbb{R}_{>0}$. We define a T^3 -fibration $\pi : \widehat{G}_{k,l}^* \rightarrow B := [-\infty, \infty]^2 \times I$ by

$$\pi(x, y, z, w) = (b_1, b_2, b_3) = (\log|z-1|, \log|w-1|, \pi_{T^1}(x, y, z, w)).$$

We also define a nowhere-vanishing meromorphic volume form by

$$\Omega := d \log x \wedge d \log(z-1) \wedge d \log(w-1).$$

The pole divisor D of Ω is given by the union

$$D = \mathcal{D}_{z=1} \cup \mathcal{D}_{w=1} \cup \mathcal{D}_{z=\infty} \cup \mathcal{D}_{w=\infty} \cup \mathcal{D}_{\xi=0} \cup \mathcal{D}_{\xi=\infty}$$

whose image under π is the boundary of B ($\mathcal{D}_{z=1}$ and $\mathcal{D}_{w=1}$ denotes the divisors $\{z = 1\}$ and $\{w = 1\}$ respectively). Using the method of symplectic reductions [Gol], we obtain the following.

Proposition 3.6. *The T^3 -fibration π defined above is a special Lagrangian fibration with respect to ω and Ω .*

Proof. Consider the symplectic quotient of the Hamiltonian T^1 -action: $\widetilde{M} := \pi_{T^1}^{-1}(\{b_3\})/T^1$ for certain $b_3 \in \mathbb{R}$. Since the toric coordinates z and w are invariant under the T^1 -action, they descend to the quotient \widetilde{M} . This gives an identification of \widetilde{M} with $\mathbb{P}^1 \times \mathbb{P}^1$. The induced symplectic form on the quotient \widetilde{M} is given by

$$\begin{aligned} \tilde{\omega} &= \frac{\sqrt{-1}}{c_1(1+|z|^2)^2} dz \wedge d\bar{z} + \frac{\sqrt{-1}}{c_2(1+|w|^2)^2} dw \wedge d\bar{w} \\ &= \frac{\sqrt{-1}|z-1|^2}{c_1(1+|z|^2)^2} d \log(z-1) \wedge \overline{d \log(z-1)} + \frac{\sqrt{-1}|w-1|^2}{c_2(1+|w|^2)^2} d \log(w-1) \wedge \overline{d \log(w-1)}. \end{aligned}$$

The induced holomorphic volume form $\tilde{\Omega}$, which is the contraction of Ω by the vector field induced from the T^1 -action, equals to

$$\tilde{\Omega} = d \log(z-1) \wedge d \log(w-1).$$

It is clear that that $\tilde{\omega}$ and $\text{Re}(\tilde{\Omega})$ restricted on each fiber of the fibration $(|z-1|, |w-1|)$ are both zero. Hence the fibers of the map $(|z-1|, |w-1|)$ are special Lagrangian in \widetilde{M} . By [Gol, Lemma 2], we therefore conclude that the fibers of π are special Lagrangian. \square

We may think of this fibration as the combination of a conic bundle (Figure 3) and the moment map π_{T^1} associated to the lift of (x, y) -coordinates (Figure

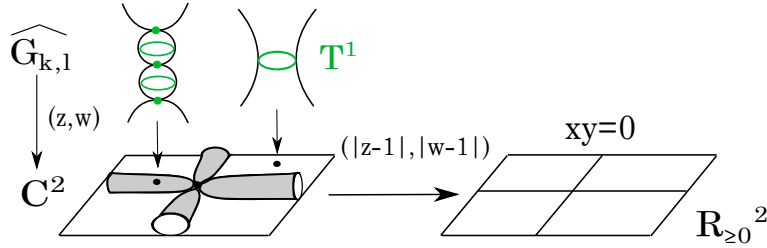


FIGURE 3. Conic fibration after resolution

4(a)). The latter measures the volumes of the exceptional curves \mathbb{P}^1 of the crepant

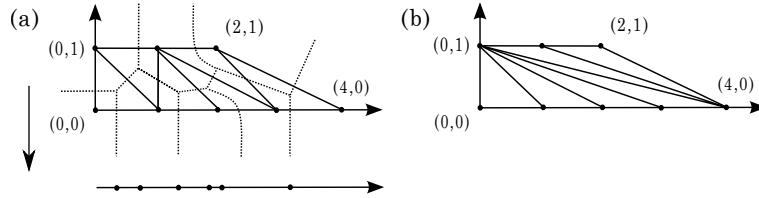


FIGURE 4. (a) Moment map π_{T^1} , (b) Crepant resolution

resolution $\widehat{G}_{k,l} \rightarrow G_{k,l}$.

Proposition 3.7. *The discriminant locus of the fibration π is the union of the boundary ∂B together with the lines $\{b_1 = 0, b_3 = s_i\}_{i=1}^k \cup \{b_2 = 0, b_3 = t_j\}_{j=1}^l \subset B$ for $s_i, t_j \in \mathbb{R}$ with $\text{Crit}(\pi_{T^1}) = \{s_1, \dots, s_k, t_1, \dots, t_l\}$.*

Proof. The first and second coordinates of π are $b_1 = \log |z - 1|$ and $b_2 = \log |w - 1|$ respectively, which degenerates over the boundaries $b_1 = \log |z - 1| = \pm\infty$ or $b_2 = \log |w - 1| = \pm\infty$. The third coordinate π_{T^1} degenerates at those codimension-two toric strata whose corresponding 2-dimensional cones in the fan contain the vector $(1, 0, 0)$. These cones are either $[i - 1, i] \times \{0\} \times \mathbb{R}$ for $1 \leq i \leq k$ or $[j - 1, j] \times \{1\} \times \mathbb{R}$ for $1 \leq j \leq l$. The corresponding images under π_{T^1} are isolated points s_1, \dots, s_k or t_1, \dots, t_l respectively. Moreover $z = 0$ on a toric strata corresponding to a cone $[i - 1, i] \times \{0\} \times \mathbb{R}$, while $w = 0$ on a toric strata corresponding to a cone $[j - 1, j] \times \{1\} \times \mathbb{R}$. Hence $b_1 = 1$ or $b_2 = 1$ respectively, and the discriminant locus is as stated above. \square

Proposition 3.8. *The wall H of the fibration π is given by the union of two vertical planes given by $b_1 = 0$ and $b_2 = 0$.*

Proof. Suppose a fiber F_r bounds a non-constant holomorphic disc u of Maslov index 0. By the Maslov index formula in Lemma 3.2, the disc does not intersect the boundary divisors $\{z = 0\}$ nor $\{w = 0\}$. Thus the functions $(z - 1) \circ u$ and $(w - 1) \circ u$ can only be constants. If both the numbers $z \circ u$ and $w \circ u$ are non-zero, the fiber of (z, w) is just a cylinder, and a fiber of b_3 defines a non-contractible circle in this cylinder, which topologically does not bound any non-trivial disc. Thus either $z = 0$ or $w = 0$ on the disc, which implies that $b_1 = \log |z - 1| = 0$ or $b_2 = \log |w - 1| = 0$. In these cases F_r intersects a toric divisor and bounds holomorphic discs in the toric divisor. \square

Figure 5(a) illustrates the wall stated in the above proposition.

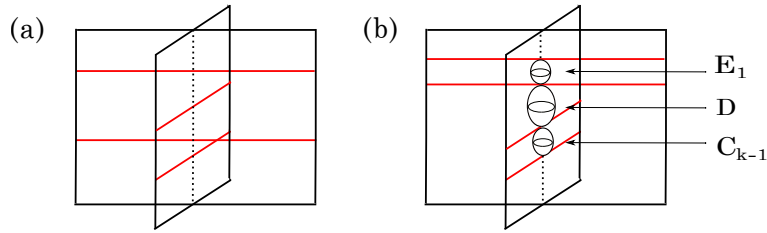


FIGURE 5. (a) Walls, (b) Holomorphic spheres

From now on we fix the unique crepant resolution of $G_{k,l}$ such that $s_1 < \dots < s_k < t_1 < \dots < t_l$ holds (Figure 4(b)). For other crepant resolution the construction is similar (while the SYZ mirrors have different coefficients, namely the mirror maps are different). Such a choice is just for simplifying the notations and is not really necessary (see also Remark 3.13).

We then fix the basis

$$(3) \quad \{C_i\}_{i=1}^{k-1} \cup \{C_0\} \cup \{E_i\}_{i=1}^{l-1}$$

of $H_2(\widehat{G_{k,l}})$ (Figure 5(b)), where C_i for $1 \leq i \leq k-1$ is the holomorphic sphere class represented by the toric 1-stratum corresponding to the 2-cone by $\{(0, 1), (i, 0)\}$; C_0 corresponds to the 2-cone generated by $\{(0, 1), (k, 0)\}$; E_i corresponds to the 2-cone generated by $\{(i, 1), (k, 0)\}$. The image of a holomorphic sphere in C_i under the fibration map lies in $\{0\} \times \mathbb{R} \times [s_i, s_{i+1}]$; the image of C_0 lies in $\{0\} \times \{0\} \times [s_k, t_1]$, and the image of E_i lies in $\mathbb{R} \times \{0\} \times [t_i, t_{i+1}]$.

Fix the contractible open subset

$$U := B_0 \setminus \{(b_1, b_2, b_3) \mid b_1 = 0 \text{ or } b_2 = 0, b_3 \in [s_1, +\infty)\} \subset B_0$$

over which the Lagrangian fibration π trivializes. For $b = (b_1, b_2, b_3)$ with $b_1 > 0$ and $b_2 > 0$, we use the Lagrangian isotopy

$$(4) \quad L^t = \{\log |z - t| = b_1, \log |w - t| = b_2, \pi_T(x, y, z, w) = b_3\}$$

for $t \in [0, 1]$ to link a moment-map fiber (when $t = 0$) with a Lagrangian torus fiber F_b of π (when $t = 1$). Then for a general base point $b' \in U$, we can link the fibers F_b and $F_{b'}$ by a Lagrangian isotopy induced by a path joining b and b' in the contractible set U (and the isotopy is independent of choice of the path). Through the isotopy disc classes bounded by a moment-map fiber L can be identified with those bounded by F_b , that is, $\pi_2(\widehat{G_{k,l}}^*, F_b) \cong \pi_2(\widehat{G_{k,l}}^*, L)$. Note that this identification depends on choice of trivialization, and henceforth we fix such a choice. The two vertical walls $\{b_1 = 0\}$ and $\{b_2 = 0\}$ divides the base $B = [-\infty, \infty]^2 \times I$ into four chambers. Lagrangian torus fibers over different chambers have different open Gromov–Witten invariants.

Theorem 3.9. *Denote by L a moment-map fiber of $\widehat{G_{k,l}}^*$ and by F_b a Lagrangian torus fiber of π at $b \in B_0$. Let $\beta \in \pi_2(\widehat{G_{k,l}}^*, F_b)$ with $\beta \pitchfork D$.*

- (1) *Over the chamber $C_{++} := \{b_1 > 0, b_2 > 0\}$, we have $n_{\beta}^{F_b} = n_{\beta}^L$.*
- (2) *Over the chamber $C_{+-} := \{b_1 > 0, b_2 < 0\}$, we have $n_{\beta}^{F_b} = 0$ unless $\beta = \beta_{k+1}, \beta_{\xi=0}, \beta_{\xi=\infty}, \beta_{z=\infty}, \beta_{w=\infty} + (\beta_j - \beta_{k+1}) + \alpha$ for $k+1 \leq j \leq k+l+1$ and $\alpha \in H_2^{c_1=0}$ being a class of rational curves which intersect the open toric orbit of the toric divisor D_j , or $\beta = \beta_i + \alpha$ for $0 \leq i \leq k$ and $\alpha \in H_2^{c_1=0}$ being a class of rational curves which intersect the open toric orbit of the toric divisor D_i . Moreover*

$$n_{\beta_{k+1}}^{F_b} = n_{\beta_{\xi=0}}^{F_b} = n_{\beta_{\xi=\infty}}^{F_b} = n_{\beta_{z=\infty}}^{F_b} = 1, \quad n_{\beta_{w=\infty} + (\beta_j - \beta_{k+1}) + \alpha}^{F_b} = n_{\beta_j + \alpha}^L$$

for $k+1 \leq j \leq k+l+1$, and

$$n_{\beta_i + \alpha}^{F_b} = n_{\beta_i + \alpha}^L$$

for $0 \leq i \leq k$.

- (3) *Over the chamber $C_{-+} := \{b_1 < 0, b_2 > 0\}$, we have $n_{\beta}^{F_b} = 0$ unless $\beta = \beta_0, \beta_{\xi=0}, \beta_{\xi=\infty}, \beta_{w=\infty}, \beta_{z=\infty} + (\beta_i - \beta_0) + \alpha$ for $0 \leq i \leq k$ and $\alpha \in H_2^{c_1=0}$ being a class of rational curves which intersect the open toric orbit of the toric divisor D_i , or $\beta = \beta_j + \alpha$ for $k+1 \leq j \leq k+l+1$ and α being a class of rational curves which intersect the open toric orbit of the toric divisor D_j . Moreover*

$$n_{\beta_0}^{F_b} = n_{\beta_{\xi=0}}^{F_b} = n_{\beta_{\xi=\infty}}^{F_b} = n_{\beta_{w=\infty}}^{F_b} = 1, \quad n_{\beta_{z=\infty} + (\beta_i - \beta_0) + \alpha}^{F_b} = n_{\beta_i + \alpha}^L$$

for $0 \leq i \leq k$, and

$$n_{\beta_j+\alpha}^{F_b} = n_{\beta_j+\alpha}^L$$

for $k+1 \leq j \leq k+l+1$.

- (4) Over the chamber $C_{--} := \{b_1 < 0, b_2 < 0\}$, we have $n_{\beta}^{F_b} = 0$ unless $\beta = \beta_0, \beta_{k+1}, \beta_{\xi=0}, \beta_{\xi=\infty}, \beta_{z=\infty} + (\beta_i - \beta_0) + \alpha$ for $0 \leq i \leq k$ and $\alpha \in H_2^{c_1=0}$ being a class of rational curves which intersect the open toric orbit of the toric divisor D_i , or $\beta_{w=\infty} + (\beta_j - \beta_{k+1}) + \alpha$ for $k+1 \leq j \leq k+l+1$ and $\alpha \in H_2^{c_1=0}$ being a class of rational curves which intersect the open toric orbit of the toric divisor D_j . Moreover

$$n_{\beta_0}^{F_b} = n_{\beta_{k+1}}^{F_b} = n_{\beta_{\xi=0}}^{F_b} = n_{\beta_{\xi=\infty}}^{F_b} = 1, \quad n_{\beta_{z=\infty}+(\beta_i-\beta_0)+\alpha}^{F_b} = n_{\beta_i+\alpha}^L$$

for $0 \leq i \leq k$, and

$$n_{\beta_{w=\infty}+(\beta_j-\beta_{k+1})+\alpha}^{F_b} = n_{\beta_j+\alpha}^L$$

for $k+1 \leq j \leq k+l+1$.

Proof. The open Gromov–Witten invariants n_{β} is non-zero only when β has Maslov index 2, and so we can focus on $\mu(\beta) = 2$ with $\beta \pitchfork D$.

For a fiber $F_{(b_1, b_2, b_3)}$ with $b_1 > 0$ and $b_2 > 0$, $F_{(b_1, b_2, b_3)}$ is Lagrangian isotopic to a moment-map fiber L through L_t defined by Equation (4). Moreover each L_t does not bound any holomorphic disc of Maslov index 0 because for every $t \in [0, 1]$, the circles $|z-t| = b_1$ and $|w-t| = b_2$ never pass through $z = 0$ and $w = 0$ respectively. Thus the open Gromov–Witten invariants of L and that of $F_{(b_1, b_2, b_3)}$ are the same.

Now consider the chamber C_{+-} . First we use the Lagrangian isotopy

$$L^{1,t} = \{\log |z-1| = b_1, \log |w-t| = b_2, \pi_T(x, y, z, w) = b_3\}$$

for $t \in [1, R]$ which identifies $F_{(b_1, b_2, b_3)}$ with $L^{1,R}$ for $R \gg 0$. $L^{1,t}$ never bounds any holomorphic disc of Maslov index 0, since for every $t \in [1, R]$, the circles $|z-1| = b_1$ and $|w-t| = b_2$ never pass through $z = 0$ and $w = 0$ respectively.

Then we take the involution $\iota : \widehat{G}_{k,l}^* \rightarrow \widehat{G}_{k,l}^*$ defined as identity on z, π_{T^1}, θ and mapping $w \mapsto \frac{Rw}{w-R}$. This involution maps the fiber $L^{1,R}$ to the Lagrangian

$$L' = \{\log |z-1| = b_1, \log |w-R| = 2(\log R) - b_2, \pi_T(x, y, z, w) = b_3\}$$

which can again be identified with the fiber $F_{(b_1, 2(\log R) - b_2, b_3)}$ with $b_1 > 0$ and $2(\log R) - b_2 > 0$. Also ι tends to the negative identity map as R tends to infinity. Hence for $R \gg 0$, the pulled-back complex structure by ι is a small deformation of the original complex structure, and hence the open Gromov–Witten invariants of $F_{(b_1, b_2, b_3)}$ remain invariant. Now using Case 1 the open Gromov–Witten invariants of $F_{(b_1, 2(\log R) - b_2, b_3)}$ can be identified with a moment-map fiber L .

By considering the intersection numbers of the disc classes and the divisors, one can check that the disc classes β_i for $0 \leq i \leq k$, $\beta_{k+1}, \beta_{\xi=0}, \beta_{\xi=\infty}, \beta_{z=\infty}, (\beta_j - \beta_{k+1})$ for $k+1 \leq j \leq k+l+1$, and all rational curve classes α are invariant under the involution. Moreover β_{k+1} and $\beta_{w=\infty}$ are switched under the involution. Putting all together, we obtain a complete relation between open Gromov–Witten invariants of $F_{(b_1, b_2, b_3)}$ and that of L , and this gives the formulae in Case 2.

Cases 3 and 4 are similar. For Case 3 we use the Lagrangian isotopy

$$L^{t,1} = \{\log |z-t| = b_1, \log |w-1| = b_2, \pi_T(x, y, z, w) = b_3\}$$

for $t \in [1, R]$ and the involution defined as identity on w, π_{T^1}, θ and mapping $z \mapsto \frac{Rz}{z-R}$. For Case 4 we use the Lagrangian isotopy

$$L^t = \{\log |z - t| = b_1, \log |w - t| = b_2, \pi_T(x, y, z, w) = b_3\}$$

for $t \in [1, R]$ and the involution defined as identity on π_{T^1}, θ and mapping $z \mapsto \frac{Rz}{z-R}$, $w \mapsto \frac{Rw}{w-R}$. \square

We can compute the the open Gromov–Witten invariants of the moment-map fiber, using the open mirror theorem [CCLT, Theorem 1.4 (1)]. The result is essentially the same as the one in [LLW, Theorem 4.2] for the minimal resolution of A_n -singularities as follows.

Theorem 3.10. *Let L be a regular moment-map fiber of $\widehat{G}_{k,l}$, and consider a disc class $\beta \in \pi_2(X, L)$. The open Gromov–Witten invariant n_β^L equals to 1 if $\beta = \beta_p + \alpha$ for $0 \leq p \leq k + l + 1$, where α is a class of rational curves which takes the form*

$$\alpha = \begin{cases} \sum_{i=1}^{k-1} s_i C_i & \text{when } 1 \leq p \leq k-1; \\ \sum_{i=1}^{l-1} s_i E_i & \text{when } k+2 \leq p \leq k+l; \\ 0 & \text{when } p = 0, k, k+1 \text{ or } k+l+1, \end{cases}$$

and $\{s_i\}_{i=1}^{m-1}$ (where m equals to k in the first case and l in the second case) is an admissible sequence with center p in the first case, which means that

- (1) $s_i \geq 0$ for all i and $s_1, s_{m-1} \leq 1$;
- (2) $s_i \leq s_{i+1} \leq s_i + 1$ when $i < p$, and $s_i \geq s_{i+1} \geq s_i - 1$ when $i \geq p$,

and with center $p - k - 1$ in the second case. For any other β , $n_\beta^L = 0$.

Proof. We will prove the assertion by using the open mirror theorem. Recall the curve classes C_i, C_0 and E_j introduced in Equation (3) for $1 \leq i \leq k-1$ and $1 \leq j \leq l-1$. C_i and E_i are $(-2, 0)$ -curves, while C_0 is a $(-1, -1)$ -curve. The intersection numbers with the toric prime divisors D_j are as follows:

- (1) $C_i \cdot D_{i-1} = C_i \cdot D_{i+1} = 1$; $C_i \cdot D_i = -2$; and $C_i \cdot D_j = 0$ for all $j \neq i-1, i, i+1$;
- (2) $C_0 \cdot D_k = C_0 \cdot D_{k+1} = -1$; $C_0 \cdot D_{k-1} = C_0 \cdot D_{k+2} = 1$; and $C_0 \cdot D_j = 0$ for all $j \neq k-1, k, k+1, k+2$;
- (3) $E_i \cdot D_{k+i-1} = E_i \cdot D_{k+i+1} = 1$; $E_i \cdot D_{k+i} = -2$; and $E_i \cdot D_j = 0$ for all $j \neq k+i-1, k+i, k+i+1$.

Let $q^{C_i}, q^{C_0}, q^{E_i}$ be the corresponding Kähler parameters and $\check{q}^{C_i}, \check{q}^{C_0}, \check{q}^{E_i}$ be the corresponding complex parameters. They are related by the mirror map:

$$q^{C_i} = \check{q}^{C_i} \exp\left(-\sum_{j=0}^{k+1+l} (C_i \cdot D_j) g_j(\check{q})\right) = \check{q}^{C_i} \exp(-(g_{i-1}(\check{q}) + g_{i+1}(\check{q}) - 2g_i(\check{q}))),$$

$$q^{C_0} = \check{q}^{C_0} \exp\left(-\sum_{j=0}^{k+1+l} (C_0 \cdot D_j) g_j(\check{q})\right) = \check{q}^{C_0} \exp(-(g_{k-1}(\check{q}) + g_{k+1}(\check{q}))),$$

and

$$q^{E_i} = \check{q}^{E_i} \exp\left(-\sum_{j=0}^{k+1+l} (E_i \cdot D_j) g_j(\check{q})\right) = \check{q}^{E_i} \exp(-(g_{k+i-1}(\check{q}) + g_{k+i+1}(\check{q}) - 2g_{k+i}(\check{q}))).$$

The functions $g_i(\check{q})$ are attached to the toric prime divisor D_i for $0 \leq i \leq k+l+1$. We have $g_0 = g_k = g_{k+1} = g_{k+l+1} = 0$. Moreover for $1 \leq i \leq k-1$, g_i only depends

on the variables \check{q}^{C_r} for $1 \leq r \leq k-1$; for $k+2 \leq i \leq k+l$, the function g_i only depends on the variables \check{q}^{E_r} for $1 \leq r \leq l-1$. Explicitly g_i is written in terms of hypergeometric series:

$$g_i(\check{q}) := \sum_{\substack{d \cdot D_i < 0 \\ d \cdot D_r \geq 0 \\ \text{for all } r \neq i}} \frac{(-1)^{(D_i \cdot d)} (- (D_i \cdot d) - 1)!}{\prod_{p \neq i} (D_p \cdot d)!} \check{q}^d$$

where for $1 \leq i \leq k-1$, the summation is over $d = \sum_{r=1}^{k-1} n_j C_j$ ($n_j \in \mathbb{Z}_{\geq 0}$) with $d \cdot D_i < 0$ and $d \cdot D_p \geq 0$ for all $p \neq i$; for $k+2 \leq i \leq k+l$, the summation is over $d = \sum_{r=1}^{l-1} n_j E_j$ ($n_j \in \mathbb{Z}_{\geq 0}$) with d satisfying the same condition. Then the open mirror theorem [CCLT, Theorem 1.4 (1)] states that

$$\sum_{\alpha} n_{\beta_i + \alpha} q^{\alpha}(\check{q}) = \exp g_i(\check{q}).$$

Note that for $1 \leq i \leq k-1$, the function g_i takes exactly the same expression as that in the toric resolution of A_{k-1} -singularity; for $1 \leq i \leq l-1$, the function g_{i+k+1} takes exactly the same expression as that in the toric resolution of A_{l-1} -singularity. Thus the mirror maps for q^{C_i} and q^{E_i} coincide with that for A_{k-1} -resolution and A_{l-1} -resolution respectively. Moreover the above generating function of open Gromov–Witten invariants coincide. Then result follows from the formula for open Gromov–Witten invariants in A_n -resolution given in [LLW, Theorem 4.2]. \square

Remark 3.11. Theorem 3.10 can also be proved by comparing the disc moduli for $\widehat{G}_{k,l}$ and resolution of A_k - and A_l -singularities, which involves details of obstruction theory of disc moduli space. Here take the more combinatorial approach using open mirror theorems.

Theorem 3.12. *The SYZ mirror of the resolved generalized conifold $\widehat{G}_{k,l}$ is given by the deformed orbifolded conifold in $\mathbb{C}^4 \times \mathbb{C}^\times$ defined by the equations $(U_1, U_2, V_1, V_2 \in \mathbb{C}$ and $Z \in \mathbb{C}^\times$)*

$$\begin{aligned} U_1 V_1 &= (1 + Z)(1 + q_1 Z) \dots (1 + q_1 \dots q_{k-1} Z), \\ U_2 V_2 &= (1 + cZ)(1 + q'_1 cZ) \dots (1 + q'_1 \dots q'_{l-1} cZ), \end{aligned}$$

where $q_i = e^{-\int_{C_i} \omega}$, $q'_j = e^{-\int_{E_j} \omega}$, and $c = q_1 \dots q_{k-1} e^{-\int_{C_0} \omega}$.

Proof. Let $\tilde{Z}_\beta = e^{-\int_\beta \omega} \text{Hol}_\nabla(\partial\beta)$ be the semi-flat mirror complex coordinates corresponding to each disc class $\beta \in \pi_2(\widehat{G}_{k,l}^*, F_b)$ for $b \in B_0 \setminus H$. For simplicity we denote $\tilde{Z}_{\beta_{\xi=0}}$ by \tilde{Z} , \tilde{Z}_{β_0} by \tilde{U}_1 , $\tilde{Z}_{\beta_{k+1}}$ by \tilde{U}_2 , $\tilde{Z}_{\beta_{z=\infty}}$ by \tilde{V}_1 , and $\tilde{Z}_{\beta_{w=\infty}}$ by \tilde{V}_2 . We have $\tilde{Z} \tilde{Z}_{\beta_{\xi=\infty}} = q^{\beta_{\xi=0} + \beta_{\xi=\infty}}$ is a constant (since $\beta_{\xi=0} + \beta_{\xi=\infty} \in H_2(\widehat{G}_{k,l}^*)$), and for simplicity we set the constant to be 1. Thus $\tilde{Z}_{\beta_{\xi=\infty}} = \tilde{Z}^{-1}$.

Let $Z_{\mathcal{D}}$ be the generating function of open Gromov–Witten invariants corresponding to a boundary divisor \mathcal{D} (Equation (2)). By Theorem 3.9 there is no wall-crossing for the disc classes $\beta_{\xi=0}$ and $\beta_{\xi=\infty}$, and they are the only disc classes of Maslov index 2 and intersecting $\mathcal{D}_{\xi=0}$ and $\mathcal{D}_{\xi=\infty}$ exactly once respectively. Hence $Z_{\mathcal{D}_{\xi=0}} = \tilde{Z}$ and $Z_{\mathcal{D}_{\xi=\infty}} = \tilde{Z}^{-1}$. For simplicity we denote $Z_{\mathcal{D}_{\xi=0}}$ by Z , and hence $Z = \tilde{Z}$ (meaning that the coordinate Z does not need quantum corrections).

Theorem 3.10 gives the open Gromov–Witten invariants of moment-map tori, which then gives the open Gromov–Witten invariants of fibers of our Lagrangian fibration by Theorem 3.9. Then by some nice combinatorics which also appears in [LLW, Proof of Corollary 4.3], the generating functions factorizes as follows:

- (1) $Z_{\mathcal{D}_{z=1}} = \tilde{U}_1$ and $Z_{\mathcal{D}_{w=1}} = \tilde{U}_2$ over C_{--} ,
- (2) $Z_{\mathcal{D}_{z=1}} = \tilde{U}_1(1+Z)(1+q_1Z)\dots(1+q_1\dots q_{k-1}Z)$ and $Z_{\mathcal{D}_{w=1}} = \tilde{U}_2$ over C_{+-} ,
- (3) $Z_{\mathcal{D}_{z=1}} = \tilde{U}_1$ and $Z_{\mathcal{D}_{w=1}} = \tilde{U}_2(1+cZ)(1+q'_1cZ)\dots(1+q'_1\dots q'_{l-1}cZ)$ over C_{-+} ,
- (4) $Z_{\mathcal{D}_{z=1}} = \tilde{U}_1(1+Z)(1+q_1Z)\dots(1+q_1\dots q_{k-1}Z)$ and $Z_{\mathcal{D}_{w=1}} = \tilde{U}_2(1+cZ)(1+q'_1cZ)\dots(1+q'_1\dots q'_{l-1}cZ)$ over C_{++} .
- (5) $Z_{\mathcal{D}_{z=\infty}} = \tilde{U}_1^{-1}$ over $C_{++} \cup C_{+-}$,
- (6) $Z_{\mathcal{D}_{z=\infty}} = \tilde{U}_1^{-1}(1+Z)(1+q_1Z)\dots(1+q_1\dots q_{k-1}Z)$ over $C_{-+} \cup C_{--}$,
- (7) $Z_{\mathcal{D}_{w=\infty}} = \tilde{U}_2^{-1}$ over $C_{++} \cup C_{-+}$,
- (8) $Z_{\mathcal{D}_{w=\infty}} = \tilde{U}_2^{-1}(1+cZ)(1+q'_1cZ)\dots(1+q'_1\dots q'_{l-1}cZ)$ over $C_{+-} \cup C_{--}$.

Therefore we conclude that the ring generated by the functions $Z = Z_{\mathcal{D}_{\xi=0}}, Z_{\mathcal{D}_{\xi=\infty}} = Z^{-1}$, $U_1 := Z_{\mathcal{D}_{z=1}}, V_1 := Z_{\mathcal{D}_{z=\infty}}, U_2 := Z_{\mathcal{D}_{w=1}}, V_2 := Z_{\mathcal{D}_{w=\infty}}$ is the polynomial ring $\mathbb{C}[U_1, U_2, V_1, V_2, Z, Z^{-1}]$ mod out by the relations

$$\begin{aligned} U_1V_1 &= (1+Z)(1+q_1Z)\dots(1+q_1\dots q_{k-1}Z), \\ U_2V_2 &= (1+cZ)(1+q'_1cZ)\dots(1+q'_1\dots q'_{l-1}cZ). \end{aligned}$$

This completes the proof of the theorem. \square

Remark 3.13. If a different crepant resolution is taken, the mirror takes the same form as above while the coefficients of the polynomials on the right hand side are different functions of the Kähler parameters q_i and q'_i . They correspond to different choices of limit points (and hence different flat coordinates) over the complex moduli.

3.3. SYZ from $\widetilde{G}_{k,l}$ to $\widehat{O}_{k,l}$. The deformed generalized conifold $\widetilde{G}_{k,l}$ is given by

$$\left\{ (x, y, z, w) \in \mathbb{C}^2 \times (\mathbb{C} \setminus \{1\})^2 \mid xy - \sum_{i=0}^k \sum_{j=0}^l a_{i,j} z^i w^j = 0 \right\}$$

for generic coefficients $a_{i,j} \in \mathbb{C}$. It is a conic fibration over the second factor $(\mathbb{C} \setminus \{1\})^2$ with discriminant locus being the Riemann surface $\Sigma_{k,l} \subset (\mathbb{C} \setminus \{1\})^2$ defined by the equation $\sum_{i=0}^k \sum_{j=0}^l a_{i,j} z^i w^j = 0$ which has genus kl and $(k+l)$ punctures. The SYZ construction for such a conic fibration follows from [AAK]. Here we just give a brief description. We will use the standard symplectic form on $\mathbb{C}^2 \times (\mathbb{C} \setminus \{1\})^2$ restricted to the hypersurface $\widetilde{G}_{k,l}$. First, $\widetilde{G}_{k,l}$ is naturally compactified in $\mathbb{P}^2 \times (\mathbb{P}^1)^2$ as a symplectic manifold, and we denote the compactification by $\widetilde{G}_{k,l}^*$. There is a natural Hamiltonian T^1 -action on $\widetilde{G}_{k,l}^*$ given by, for $t \in T^1 \subset \mathbb{C}$

$$t \cdot (x, y, z, w) := (tx, t^{-1}y, z, w).$$

By carefully analyzing the symplectic reduction of this T^1 -action, a Lagrangian torus fibration $\pi : \widehat{G}_{k,l}^* \rightarrow B := [-\infty, \infty]^3$ was constructed in [AAK, Section 4].

Topologically the fibration is the homeomorphic to the naive one given by

$$(x, y, z, w) \mapsto (b_1, b_2, b_3) = \left(\log |z|, \log |w|, \frac{1}{2}(|x|^2 - |y|^2) \right).$$

However since the symplectic form induced on the symplectic quotient is not the standard one on \mathbb{P}^2 , it has to be deformed to give a Lagrangian fibration.

The discriminant locus of this fibration consists of the boundary of B and $\overline{A_{k,l}} \times \{0\}$, where $\overline{A_{k,l}} \subset [-\infty, \infty]^2$ is the compactification of the amoeba $A_{k,l} \subset \mathbb{R}_{\geq 0}^2$ of the Riemann surface $\Sigma_{k,l}$ (Figure 6), namely the image of $\Sigma_{k,l}$ under the map $(z, w) \mapsto (\log |z|, \log |w|)$. The wall for open Gromov–Witten invariants is given by

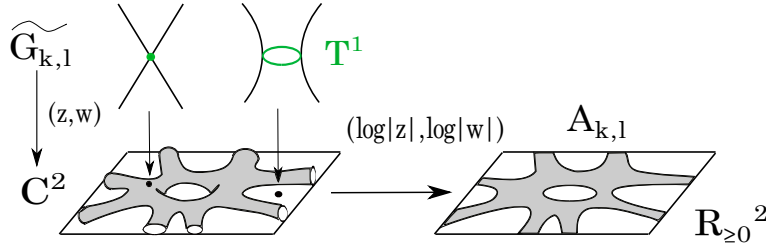


FIGURE 6. Conic fibration and amoeba $A_{k,l}$

$H = \overline{A_{k,l}} \times [-\infty, \infty]$ [AAK, Proposition 5.1]. The complement $B \setminus H$ consists of $(k+1)(l+1)$ chambers. In this specific case, we have a nice degeneration as follows. Let us consider a special point on the the complex moduli space of $\widetilde{G_{k,l}}^*$ where the defining equation of the Riemann surface $\Sigma_{k,l}$ factorizes as

$$\sum_{i=0}^k \sum_{j=0}^l a_{i,j} z^i w^j = f(z)g(w)$$

for polynomials $f(z)$ and $g(w)$ of degree k and l respectively (and we assume that their roots are all distinct and non-zero). At this point, $\widetilde{G_{k,l}}^*$ acquires kl conifold singularities. The wall becomes the union of vertical hyperplanes $\{b_2 = \log |r_i|\}_{i=1}^k \cup \{b_3 = \log |s_j|\}_{j=1}^l \subset \mathbb{R}^3$, where r_i and s_j are the roots of $f(z)$ and $g(w)$ respectively (Figure 7). These hyperplanes divide the base into $(k+1)(l+1)$ chambers. We

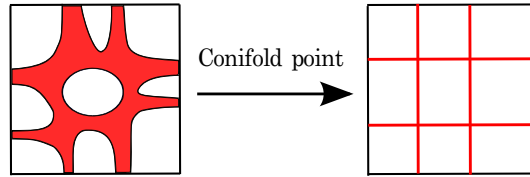


FIGURE 7. Amoeba around conifold locus

label the chambers by $C_{i,j}$ for $i = 0, \dots, k$ and $j = 0, \dots, l$ from left to right and from bottom to up.

Theorem 3.14. *The SYZ mirror of the deformed generalized conifold $\widehat{G}_{k,l}$ is given by, for $U_i, V_i \in \mathbb{C}$ and $Z \in \mathbb{C}^\times$,*

$$U_1 V_1 = (1 + Z)^k, \quad U_2 V_2 = (1 + Z)^l,$$

which is the punctured orbifolded conifold $O_{k,l}$.

Proof. The wall-crossing of open Gromov–Witten invariants was deduced in [AAK, Lemma 5.4], and we just sketch the result here. For $p = 1, 2$, let U_p be the generating function of open Gromov–Witten invariants for disc classes intersecting the boundary divisor $\pi^{-1}(\{b_p = 0\})$ once. Denote the semi-flat coordinates corresponding to the basic disc classes emanated from $\pi^{-1}(\{b_p = 0\})$ by \tilde{Z}_i , and denote by Z the semi-flat coordinate corresponding to the b_3 -direction (which admits no quantum corrections). Then U_1 restricted to the chamber $C_{i,j}$ equals to the polynomial $\tilde{Z}_1(1 + Z)^i$, and U_2 restricted to the chamber $C_{i,j}$ equals to $\tilde{Z}_2(1 + Z)^j$. By gluing the various chambers together using the above wall-crossing factor $1 + Z$, we obtain the SYZ mirror as claimed. \square

Remark 3.15. The equation in Theorem 3.14 defines a singular variety $O_{k,l}$. This is a typical feature of our SYZ construction, which produces a complex variety out of a symplectic manifold: we may obtain a singular variety as the SYZ mirror, and we need to take a crepant resolution to get a smooth mirror. Since we concern about the complex geometry of this variety, $O_{k,l}$ is not distinguishable from its crepant resolution $\widehat{O}_{k,l}$.

3.4. SYZ from $\widehat{O}_{k,l}$ to $\widehat{G}_{k,l}$. The partial compactification $\widehat{O}_{k,l}^\sharp$ of $\widehat{O}_{k,l}$ is a toric Calabi–Yau threefold and its SYZ mirror was constructed in [CLL]. In this section we quote the relevant results, omitting the details. First, a crepant resolution $\widehat{O}_{k,l}$ of $\widehat{O}_{k,l}$ corresponds to a maximal triangulation of $\square_{k,l}$ (Figure 2(b)). We have the Lagrangian torus fibration $\pi : \widehat{O}_{k,l} \rightarrow B := \mathbb{R}^2 \times \mathbb{R}_{\geq 0}$ constructed in [Gol, Gro], whose discriminant locus consists of two components. One is the boundary ∂B , and the other is topologically given by the dual graph of the maximal triangulation lying in the hyperplane $\{b_3 = 1\} \subset B$, where we denote the coordinates of B by $b = (b_1, b_2, b_3)$.

The wall is exactly the hyperplane $\{b_3 = 1\}$ containing one component of the discriminant locus. It is associated with a wall-crossing factor, which is a polynomial whose coefficients encode the information coming from holomorphic discs with Maslov index 0. The explicit formula for the coefficients were computed in [CCLT]. Applying these results, we obtain the following:

Theorem 3.16. *The SYZ mirror of the resolved orbifolded conifold $\widehat{O}_{k,l}$ is a deformed generalized conifold $\widehat{G}_{k,l}$ given as*

$$\left\{ (U, V, Z, W) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^2 \mid UV = \sum_{i=0}^k \sum_{j=0}^l q^{C_{ij}} (1 + \delta_{ij}(\tilde{q})) Z^i W^j \right\}.$$

The notations are explained as follows. Let β_{ij} be the basic disc class corresponding to the toric divisor $D_{ij} \subset \widehat{O}_{k,l}$. Then C_{ij} denotes the curve class $\beta_{ij} - i(\beta_{10} - \beta_{00}) - j(\beta_{01} - \beta_{00}) - \beta_{00}$. The coefficients $1 + \delta_{ij}(\tilde{q})$ is given by

$\exp(g_{ij}(\tilde{q}))$ where

$$g_{ij}(\tilde{q}) := \sum_d \frac{(-1)^{(D_{ij} \cdot d)} (- (D_{ij} \cdot d) - 1)!}{\prod_{(a,b) \neq (i,j)} (D_{ab} \cdot d)!} \tilde{q}^d,$$

and the summation is over all effective curve classes $d \in H_2^{\text{eff}}(O_{k,l})$ satisfying $D_{ij} \cdot d < 0$ and $D_p \cdot d \geq 0$ for all $p \neq (i, j)$. Lastly q and \tilde{q} are related by the mirror map:

$$q^C = \tilde{q}^C \exp\left(-\sum_{i,j} (D_{ij} \cdot C) g_{ij}(\tilde{q})\right).$$

It is worth noting that the above SYZ mirror manifold can be identified with the Hori–Iqbal–Vafa mirror manifold [HIV]. The former has the advantage that it is intrinsically expressed in terms of flat coordinates and contains the information about certain open Gromov–Witten invariants.

3.5. SYZ from $\widetilde{O}_{k,l}$ to $\widehat{G}_{k,l}$. Recall that the fan polytope of the orbifolded conifold $O_{k,l}^\sharp$ is the cone over a rectangle $[0, k] \times [0, l]$. Smoothings of $\widetilde{O}_{k,l}$ correspond to the Minkowski decompositions of $[0, k] \times [0, l]$ into k copies of $[0, 1] \times \{0\}$ and l copies of $\{0\} \times [0, 1]$ [Alt]. The SYZ mirrors for such smoothings were constructed in [Lau1]. Here we can write down the Lagrangian fibration more explicitly by realizing $\widetilde{O}_{k,l}$ as a double conic fibration. Recall that the deformed orbifolded conifold $\widetilde{O}_{k,l}$ is defined as

$$\widetilde{O}_{k,l} = \{(u_1, u_2, v_1, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times \mid u_1 v_1 = f(z), u_2 v_2 = g(z)\}$$

where $f(z)$ and $g(z)$ are generic polynomials of degree k and l respectively. We assume that all roots r_i and s_j of $f(z)$ and $g(z)$ respectively are distinct and non-zero. Moreover, we can naturally compactly $\widetilde{O}_{k,l}$ in $(\mathbb{P}^2)^2 \times \mathbb{P}^1$ to obtain $\widetilde{O}_{k,l}^* \subset (\mathbb{P}^2)^2 \times \mathbb{P}^1$ (where (u_1, u_2) and (v_1, v_2) above become inhomogeneous coordinates of the two \mathbb{P}^2 factors.). There is also a natural Hamiltonian T^2 -action on $\widetilde{O}_{k,l}^*$ given by, for $(s, t) \in T^2 \subset \mathbb{C}^2$

$$(s, t) \cdot (u_1, v_1, u_2, v_2, z) := (su_1, s^{-1}v_1, tu_2, t^{-1}v_2, z).$$

On the other hand, $\widetilde{O}_{k,l}^*$ admits a double conic fibration $\pi_z : \widetilde{O}_{k,l}^* \rightarrow \mathbb{P}^1$ by the projection to the z -coordinate (Figure 8). In this situation, the base \mathbb{P}^1 of the

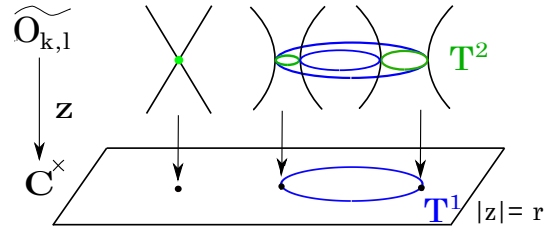


FIGURE 8. Double conic fibration

fibration can be identified as the symplectic reduction of $\widetilde{O}_{k,l}^*$ by the Hamiltonian

T^2 -action. As is discussed in [Gro], the Lagrangian fibration $|z| : \mathbb{P}^1 \rightarrow [0, \infty]$ gives rise to the Lagrangian torus fibration $\pi : \widetilde{O}_{k,l}^* \rightarrow B := [-\infty, \infty]^2 \times [0, \infty]$ given by

$$\pi(u_1, v_1, u_2, v_2, z) = \left(\frac{1}{2}(|u_1|^2 - |v_1|^2), \frac{1}{2}(|u_2|^2 - |v_2|^2), |z| \right).$$

The map to the first two coordinates is the moment map of the Hamiltonian T^2 -action. We denote the coordinates of B by $b = (b_1, b_2, b_3)$.

Proposition 3.17. (1) *The discriminant locus of the fibration π is given by the disjoint union $\partial B \cup \left(\bigcup_{i=1}^k \{b_1 = 0, b_3 = |r_i|\} \right) \cup \left(\bigcup_{j=1}^l \{b_2 = 0, b_3 = |s_j|\} \right) \subset B$.*

(2) *The fibration π is special with respect to the nowhere-vanishing meromorphic volume form $\Omega := du_1 \wedge du_2 \wedge d \log z$ on $\widetilde{O}_{k,l}^*$.*

Proof. The fibration has tori T^3 as generic fibers. Over ∂B where $z = 0$, the fibers degenerate to T^2 . Thus ∂B is a component of the discriminant locus. Away from $z = 0$, the map $z \rightarrow |z|$ is a submersion. Hence the discriminant locus of the fibration π comes from that of the moment map of the Hamiltonian T^2 -action. This action has non-trivial stabilizers at $u_1 = v_1 = 0$ or $u_2 = v_2 = 0$, which implies $f(z) = 0$ or $g(z) = 0$ respectively. Their images under π are $\{b_1 = 0, b_3 = |r_i|\}$ or $\{b_2 = 0, b_3 = |s_j|\}$ respectively. \square

Proposition 3.18. *A regular fiber of the fibration π bounds a holomorphic disc of Maslov index 0 only when $b_3 = |r_i|$ or $b_3 = |s_j|$. Thus the wall H of the fibration π is $H = \left(\bigcup_{i=1}^k \{b_3 = |r_i|\} \right) \cup \left(\bigcup_{j=1}^l \{b_3 = |s_j|\} \right) \subset B$.*

Proof. A singular fiber of the double conic fibration $\pi_z : \widetilde{O}_{k,l}^* \rightarrow \mathbb{P}^1$ bounds a holomorphic disc, which has Maslov index 0 by Lemma 3.2, and this happened only when $b_3 = |r_i|$ or $b_3 = |s_j|$. \square

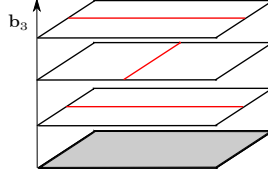


FIGURE 9. Walls $(k, l) = (2, 1)$

The wall components $\{b_3 = |r_i|\}$ and $\{b_3 = |s_j|\}$ correspond to the pieces $[0, 1] \times \{0\}$ and $\{0\} \times [0, 1]$ of the Minkowski decomposition respectively. Wall-crossing of open Gromov–Witten invariants in this case has essentially been studied in [Lau1] in details, and we will not repeat the details here. The key result is that each wall component contributes a linear factor: each component $\{b_3 = |r_i|\}$ contributes $1 + X$, and each component $\{b_3 = |s_j|\}$ contributes $1 + Y$. The SYZ mirror is essentially the product of all these factors, namely, we obtain the following.

Theorem 3.19. *The SYZ mirror of $\widetilde{O}_{k,l}$ is given by*

$$UV = (1 + X)^k (1 + Y)^l$$

for $U, V \in \mathbb{C}$ and $X, Y \in \mathbb{C}^\times$, which is the punctured generalized conifold $G_{k,l}$.

An almost the same remark as Remark 3.15 applies to Theorem 3.19 and thus we confirm the SYZ construction in this case.

4. GLOBAL GEOMETRIC TRANSITIONS: SOME DISCUSSION

We are now in position to turn to the global case. Let \widehat{X} and \widetilde{X} be compact Calabi–Yau threefolds. We call a geometric transition $\widehat{X} \dashrightarrow X \rightsquigarrow \widetilde{X}$ a generalized conifold transition if X has only generalized conifolds and orbifolded conifolds. The birational contraction appearing in the geometric transition of compact Calabi–Yau threefolds can be factorized into a sequence of primitive contractions of type I, type II and type III [Ros]. In general, type I and type III appear in the geometric transition of $G_{k,l}$ and all types appear in the geometric transition of $O_{k,l}$.

Motivated by the local case, we are tempted to propose that *generalized conifold transitions are reversed under mirror symmetry*. However, this naive conjecture does not hold because some global conifold transitions are mirror to hyperconifold transitions, which are not generalized conifold transitions [Dav]. We expect that a generalized conifold transition is mirror to a reversed generalized conifold transition if the Calabi–Yau threefold has a Lagrangian torus fibration and the transition is locally modeled by those given in Section 3.

Example 4.1 (Schoen’s CY threefold and its mirror [Sch, Lau2]). Using the methods of Castano-Bernard and Matessi [CnBM09, CnBM14] in the Gross–Siebert program, generalized conifold transitions and their mirrors are studied for the Schoen’s Calabi–Yau threefold in [Lau2]. The threefold is a resolution of the fiber product of two rational elliptic surfaces over the base \mathbb{P}^1 [Sch]. It gives a global manifestation that orbifolded conifolds and generalized conifolds are mirror to each other.

For each pair of reflexive polygons (P_1, P_2) (where P_1 and P_2 are not necessarily dual to each other), we have an orbifolded conifold degeneration $O^{(P_1, P_2)}$ of a Schoen’s Calabi–Yau threefold, and a generalized conifold degeneration $G^{(P_1, P_2)}$ of its mirror (in the sense of Legendre transform in Gross–Siebert program [GS]). A resolution of $O^{(P_1, P_2)}$ is mirror to a smoothing of $G^{(\check{P}_1, \check{P}_2)}$, and vice versa. \check{P} denotes the dual polygon of P (see Figure 10 for an example). We refer the reader to the paper [Lau2] for more details.

Example 4.2 (Quintic threefold and its mirror). We set $(k, l) = (4, 1)$ or $(3, 2)$ in the following. Let $X \subset \mathbb{P}^4$ be the singular quintic threefold defined by

$$x_0 f(x_0, \dots, x_4) + x_1^k x_2^l = 0,$$

where $f(x)$ is a generic homogeneous polynomial of degree 4. The singular locus of X consists of 2 curves

$$\{x_0 = x_1 = f(x) = 0\} \cup \{x_0 = x_2 = f(x) = 0\} \subset X$$

of genus 3 intersecting at 4 points. The quintic threefold X has $G_{k,l}$ around the each intersection point. Successively blowing up X along the two curves followed by the blow-up along the divisor $\{x_0 = x_1 = 0\}$, we obtain a projective crepant resolution \widehat{X} of X . Thus a quintic threefold admits a generalized conifold transition.

On the other hand, the mirror quintic of a quintic threefold is defined as a crepant resolution of the orbifold

$$Y_\phi := \left\{ \sum_{i=0}^4 x_i^5 + \phi \prod_{i=0}^4 x_i = 0 \right\} / G, \quad G := \left\{ (a_i) \in (\mathbb{Z}_5)^5 \mid \sum_{i=0}^4 a_i = 0 \right\} / \mathbb{Z}_5$$

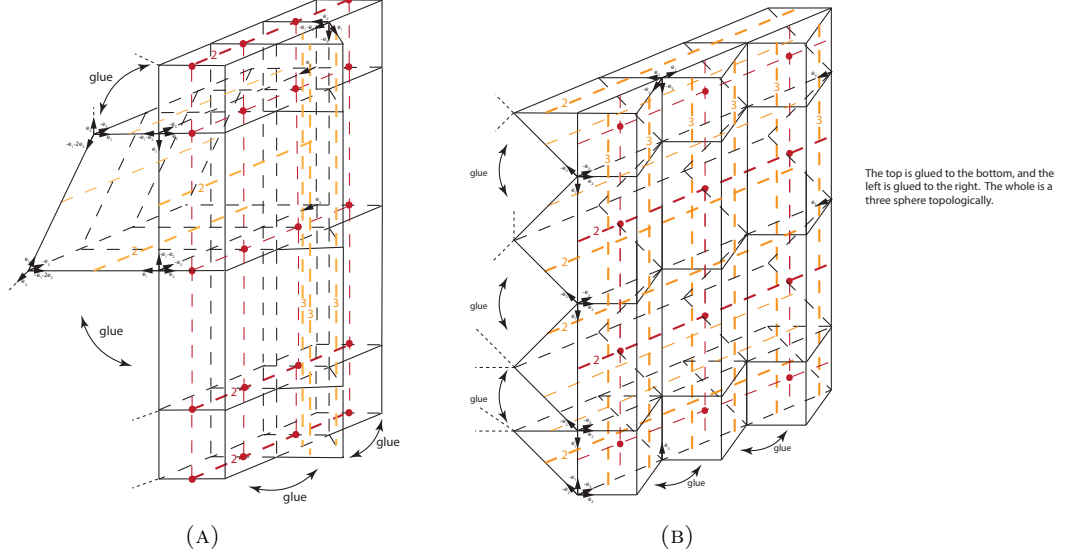


FIGURE 10. Base of Lagrangian fibrations on degenerations of a Schoen's Calabi-Yau threefold. Topologically they are \mathbb{S}^3 and the figures show polyhedral decompositions which are useful to describe the affine structures. The left shows an orbifolded conifold degeneration of Schoen's Calabi-Yau threefold. The right shows its mirror. Each thick dot represents an orbifolded conifold singularity on the left, and a generalized conifold singularity on the right. There are 24 orbifold singularities counted with multiplicities.

for $\phi \in \mathbb{C}$. The orbifold Y_ϕ has A_4 -singularities along 10 curves $C_{ij} = \{x_i = x_j = 0\}/G \cong \mathbb{P}^1$, $(0 \leq i < j \leq 4)$. We can partially resolve Y_ϕ to obtain Y whose singular locus consists of A_k -singularities along C_{01} and A_l -singularities along C_{02} such that Y has $O_{k,l}$ around $C_{01} \cap C_{02}$ (Figure 11). It is interesting to ask whether

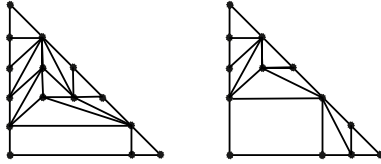


FIGURE 11. 2-dimensional faces of the polytope for $O_{4,1}$ and $O_{3,2}$

or not Y admits any smoothing.

Although X and Y lie in the boundaries of the complex moduli space of the quintic and the Kähler moduli space of the mirror quintic respectively, we do not know whether or not they correspond each other under the mirror correspondence. This may be seen by the monomial-divisor correspondence in toric geometry, but it is possible that the mirror of X is a non-toric blow-down of the mirror quintic.

It is straightforward to generalize this type of constructions to Calabi-Yau hypersurfaces in 4-dimensional weighted projective spaces.

We may also consider existence of generalized conifold transitions for compact Calabi–Yau geometries. Let \widehat{X} be a smooth threefold and C_1, \dots, C_n be $(-1, -1)$ -curves in \widehat{X} . Let X be their contraction and \widetilde{X} be a smoothing of X . Small resolutions and deformations always exist topologically, but there are obstructions if we wish to preserve either the complex or symplectic structure:

Theorem 4.3 (Friedman [Fri], Tian [Tia]). *Assume that X satisfies the $\partial\bar{\partial}$ -lemma (for example Kähler). Then a smoothing \widetilde{X} to exist if and only if there is a relation $\sum_{i=1}^n \lambda_i [C_i] = 0$ ($\lambda_i \neq 0 \forall i$) in $H_2(X, \mathbb{Q})$.*

Theorem 4.4 (Smith–Thomas–Yau [STY]). *Let \widetilde{Y} be a symplectic sixfold with embedded Lagrangian S^3 's, say L_1, \dots, L_n . Then there is a relation $\sum_{i=1}^n \lambda_i [L_i] = 0$ ($\lambda_i \neq 0 \forall i$) in $H_3(\widetilde{Y}, \mathbb{Q})$ if and only if there is a symplectic structure on one of 2^n choices of (reversed) conifold transitions of \widetilde{Y} in the Lagrangians L_1, \dots, L_n , such that the resulting exceptional \mathbb{P}^1 's are symplectic.*

In our case, the contractions collapse 4-cycles as well as 2-cycles. On the other hand, smoothing $G_{k,l} \rightsquigarrow \widetilde{G}_{k,l}$ produces $(k+1)(l+1) - 3$ vanishing S^3 's and smoothing $O_{k,l} \rightsquigarrow \widetilde{O}_{k,l}$ produces $k+l-2$ vanishing $S^1 \times S^2$'s and one vanishing S^3 . The generators of these cycles can be found by considering the standard double Riemann surface fibrations [FHKV](Figure 12). It is interesting to investigate the obstruc-

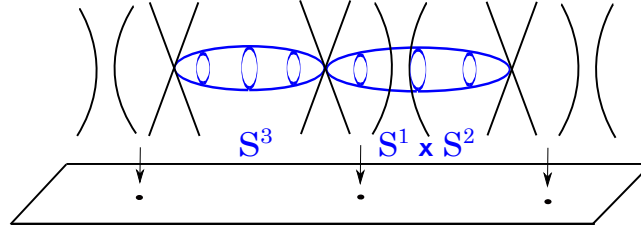


FIGURE 12. A double Riemann surface fibration

tions to the deformations/resolutions of the generalized and orbifolded conifolds in terms of these cycles. We hope to come back to these questions in future work.

REFERENCES

[AAK] M. Abouzaid, D. Auroux, and L. Katzarkov, Lagrangian fibrations on blowups of toric varieties and mirror symmetry for hypersurfaces, Publ. Math. de l’IHÉS, 23, Issue 1 (2016) 199–282.
 [AKLM] M. Aganagic, A. Karch, D. Lüst and A. Miemiec, Mirror symmetries for brane configurations and branes at singularities, Nuclear Phys. B 569 (2000), no. 1-3, 277–302.
 [Alt] K. Altmann, The versal deformation of an isolated toric Gorenstein singularity, Invent. Math. 128 (1997), no. 3, 443–479.
 [Aur] D. Auroux, Mirror symmetry and T-duality in the complement of an anticanonical divisor, J. Gökova Geom. Topol. 1 (2007), 51–91.
 [CnBM09] R. Castaño Bernard and D. Matessi, Lagrangian 3-torus fibrations, J. Differential Geom. 81 (2009), no. 3, 483–573.
 [CnBM14] R. Castaño Bernard and D. Matessi, Conifold transitions via affine geometry and mirror symmetry, Geom. Topol. 18 (2014), no. 3, 1769–1863.
 [CCLT] K. Chan, C.-H. Cho, S.-C. Lau and H.-H. Tseng, Gross fibration, SYZ mirror symmetry, and open Gromov–Witten invariants for toric Calabi–Yau orbifolds, J. Diff. Geom. 103 (2016), no.2, 207–288.

- [CPU] K. Chan, D. Pomerleano and K. Ueda, Lagrangian torus fibrations and homological mirror symmetry for the conifold, *Comm. in Math. Physics*, 341 (2016), Issue 1, 135–178.
- [CLL] K. Chan, S.-C. Lau and N.-C. Leung, SYZ mirror symmetry for toric Calabi–Yau manifolds, *J. Diff. Geom.* 90 (2012) 177–250.
- [CC] C.-H. Cho and Y.-G. Oh, Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds, *Asian J. Math.* 10 (2006) 773–814.
- [Dav] R. Davies, Hyperconifold transitions, mirror symmetry, and string theory, *Nuclear Phys. B* 850 (2011), no. 1, 214–231.
- [FHKV] B. Feng, Y.-H. He, K. Kennaway and C. Vafa, Dimer models from mirror symmetry and quivering amoebae, *Adv. Theor. Math. Phys.* 12 (2008), no. 3, 489–545.
- [Fri] R. Friedman, Simultaneous resolution of threefold double points, *Math. Ann.* 274 (1986), 671–689.
- [FOOO] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, Lagrangian intersection Floer theory: anomaly and obstruction. Part I/II, *AMS/IP Studies in Advanced Mathematics*, vol. 46, American Mathematical Society, Providence, RI, 2009.
- [Gol] E. Goldstein, Calibrated fibrations on noncompact manifolds via group actions, *Duke Math. J.* 110 (2001), no. 2, 309–343.
- [Gro] M. Gross, Examples of special Lagrangian fibrations, *Symplectic geometry and mirror symmetry (Seoul, 2000)* (World Scientific Publishing, River Edge, NJ, 2001) 81–109.
- [GS] M. Gross and B. Siebert, From affine geometry to complex geometry, *Ann. of Math. (2)* 174 (2011), no. 3, 1301–1428.
- [HIV] K. Hori, A. Iqbal, and C. Vafa, D-branes and mirror symmetry, arXiv:hep-th/0005247.
- [KKV] S. Katz, A. Klemm and C. Vafa, Geometric engineering of quantum field theories, *Nuclear Phys. B* 497 (1997), no. 1-2, 173–195.
- [KZ] V. Kaibel and G.M. Ziegler, Counting unimodular lattice triangulations, in C.D. Wensley (Ed.), *British Combinatorial Surveys*, Cambridge University Press, 2003, pp. 277–307.
- [Lau2] S.-C. Lau, Affine elliptic surfaces with type-A singularities and orbi-conifolds, arXiv:1802.08891.
- [Lau1] S.-C. Lau, Open Gromov–Witten invariants and SYZ under local conifold transitions, *J. London Math. Soc.* 90(2) (2014), 413–435.
- [LLW] S.-C. Lau, N. C. Leung and B. Wu, Mirror maps equal SYZ maps for toric Calabi–Yau surfaces, *Bull. London Math. Soc.* 44 (2012), no.2, 255–270.
- [Leu] N.-C. Leung, Mirror symmetry without corrections, *Comm. Anal. Geom.* 13 (2005), no. 2, 287–331.
- [Mie] A. Miemiec, Branes between geometry and gauge theory, *Fortschr. Phys.* 48 (2000), no. 12, 1143–1227.
- [Mor] D. Morrison, Through the looking glass, *Mirror Symmetry III* (D. H. Phong, L. Vinet, and S.-T. Yau, eds.), American Mathematical Society and International Press, 1999, 263–277.
- [Ros] M. Rossi, Geometric transitions, *J. Geom. Phys.* 56 (2006), no. 9, 1940–1983.
- [Sch] C. Schoen, On fiber products of rational elliptic surfaces with section, *Math. Z.* 197,(1988), 177-199.
- [STY] I. Smith, R. Thomas and S.-T. Yau, Symplectic conifold transitions, *J. Diff. Geom.* Vol. 62, No. 2 (2002), 209–242.
- [SYZ] A. Strominger, S.-T. Yau and E. Zaslow, Mirror symmetry is T-duality, *Nuclear Phys. B* 479 (1996), no. 1-2, 243–259.
- [Sze] B. Szendrői, Contractions and monodromy in homological mirror symmetry, in: *Strings and geometry* (Douglas, Gauntlett and Gross ed.), AMS, 2004.
- [Tia] G. Tian, Smoothing 3-folds with trivial canonical bundle and ordinary double points, *Essays on mirror manifolds* (1992) 458–479, Int. Press, Hong Kong.

DEPARTMENT OF MATHEMATICS, KYOTO UNIVERSITY
 KITASHIRAKAWA-OIWAKE, SAKYO, KYOTO, 606-8502, JAPAN
 akanazawa@math.kyoto-u.ac.jp

DEPARTMENT OF MATHEMATICS AND STATISTICS, BOSTON UNIVERSITY
 111 CUMMINGTON MALL, BOSTON MA 02215 USA
 lau@math.bu.edu