GENERALIZED SYZ AND HOMOLOGICAL MIRROR SYMMETRY

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Abstract. Strominger-Yau-Zaslow proposed that mirror symmetry can be understood by T-duality. This survey gives a quick overview on SYZ mirror symmetry and Gross-Siebert program, and focuses on a generalized approach to SYZ construction based on deformation theory of immersed Lagrangians rather than smooth tori. In interesting examples it avoids complicated wall-crossing and constructs the mirrors which were not reached by the traditional SYZ approach. Moreover it naturally gives a geometric functor realizing homological mirror symmetry. An open mirror principle is also exhibited by the generalized SYZ approach.

1. j function and polygon countings for elliptic curves

Let me begin with two examples of elliptic curves with symmetries. We will see that the j-function has a miraculous relation with certain countings of polygons in elliptic curves, which can be explained by a generalized version of SYZ mirror symmetry in Section 3 and Section 4.

1.1. Elliptic curve with complex multiplication by cube root of unity.

Elliptic curve with complex multiplication by cube root of unity. Let E be the elliptic curve with complex multiplication by the cube root of unity. The action of $e^{2\pi \mathbf{i}/3}$ is given by rotation about the center of a minimal hexagon, which is labeled by a dot, in Figure 1. Let's equip E with the flat metric descended from the Euclidean plane. We denote its total area by t.

The quotient of E by $\mathbb{Z}_3$ is the orbifold projective line $E/\mathbb{Z}_3 = \mathbb{P}^1_{(3,3,3)}$, which has three orbifold points and each of the subscripts denotes the order of isotropy group of each orbifold points. Algebraically E can be described as the elliptic curve $\{x^3+y^3+z^3=0\} \subset \mathbb{P}^2$ quotient by the free action of $\{[\zeta_1 : \zeta_2 : \zeta_3] \in \mathbb{P}^2 : \zeta_1^3 = \zeta_2^3 = \zeta_3^3 = 1, \zeta_1\zeta_2\zeta_3 = 1\} \cong \mathbb{Z}_3$. Then $\mathbb{P}^1_{(3,3,3)}$ is the quotient of $\{x^3+y^3+z^3=0\} \subset \mathbb{P}^2$ by $\{[\zeta_1 : \zeta_2 : \zeta_3] \in \mathbb{P}^2 : \zeta_1^3 = \zeta_2^3 = \zeta_3^3 = 1\} \cong \mathbb{Z}_3^2$.

Now take a vertical line in the Euclidean plane, which descends to a circle in the elliptic curve E. We obtain two other circles by applying the action of $\mathbb{Z}_3$ on E. Let L be the union of these circles, whose lift to the universal cover of E is shown by the union of dotted lines in Figure 1. The position of the vertical line is taken such that L is invariant under reflection about a vertical axis passing through the three fixed points of the action by $\mathbb{Z}_3$. L is indeed the lift of an immersed curve in $E/\mathbb{Z}_3$ constructed by Seidel [Sei11], and we will get back to this point in Section 3.

We count the number of polygons in E bounded by L, where each corner of a polygon is required to be an angle of a minimal triangle. The requirement is to ensure 'weakly ubobstructedness' of the corresponding deformations. For instance, hexagon is not allowed

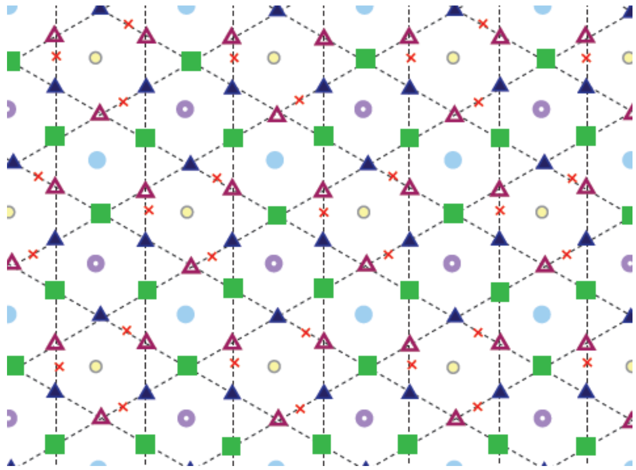

FIGURE 1. Polygon countings in the elliptic curve E. The dotted lines show the lift of L in the universal cover. The center of each minimal hexagon, which is labeled by a dot, is a fixed point of the $\mathbb{Z}_3$-action.

because its corners are not angles of any minimal triangles. Indeed in this case all such polygons are triangles.

Naively the count simply equals to ∞, because there are infinitely many triangles. In order to make sense of the counting we need to decorate each polygon by the following data:

(1) **Boundary marked point**. Fix a generic point p in L. It is marked by one of the small crosses in Figure 1. We always require the boundary of a polygon to be counted to pass through the point p.
(2) **Monomials.** We label the self-intersection points of L by either x, y or z as in Figure 1. In the figure they are marked as a tiny solid triangle, hollow triangle and solid square respectively. Then the vertices of each polygon are labelled by x, y or z, and hence each polygon β is attached with a monomial $z^{\partial\beta}$. In this case the monomial $z^{\partial\beta}$ attached to a polygon is either xyz, x^3, y^3 or z^3.
(3) **Areas.** Let $A(\beta)$ be the area of a polygon β, and we take $T^\beta := \exp(-A(\beta))$. Let α be the area of a minimal triangle. (Monomial attached to a minimal triangle is xyz.) The area of any other triangle is an integer multiple of α. In other words, for any triangle β we have $T^\beta = (T^\alpha)^k$ for some $k \in \mathbb{N}$. Moreover,
$$T := T^\alpha = \exp\left(\frac{-t}{24}\right)$$
where we recall t denotes the area of the elliptic curve.
(4) **Signs.** Fix a $\mathbb{Z}_3$-invariant orientation of L as shown in Figure 1. Moreover, we apply the $\mathbb{Z}_3$-action and Deck transformations on the generic chosen point $p \in L$, and obtain all the small red crosses shown in Figure 1. This fixes a $\mathbb{Z}_3$-invariant spin structure on L. Then each triangle β is attached with the sign $\operatorname{sign}(\beta) := (-1)^{r+s}$, where r is the number of edges whose orientations are reversed of the boundary counterclockwise orientation of β, and s is the number of red crosses on the boundary.

Define the generating function of polygon counting to be

$$W := \sum_{\beta} \operatorname{sign}(\beta) T^{\beta} z^{\partial\beta} \tag{1.1}$$

where the sum is over all the triangles whose boundaries pass through the generic point p. It takes the form

$$\begin{aligned} W &= -Txyz - T^9x^3 - T^9y^3 + T^9z^3 + \dots \\ &= \phi(T)(-x^3 - y^3 + z^3) - \psi(T)xyz \end{aligned}$$

where ϕ and ψ are explicit convergent series defined by

$$\begin{aligned} \phi(T) &= \sum_{k=0}^{\infty} (-1)^{3k+1}(2k+1)T^{3(12k^2+12k+3)}, \\ \psi(T) &= -T + \sum_{k=1}^{\infty} \left((-1)^{3k+1}(6k+1)T^{(6k+1)^2} + (-1)^{3k}(6k-1)T^{(6k-1)^2} \right). \end{aligned}$$

It is easy to perform a change of coordinates in (x, y, z) to arrange W to be

$$(x^3 + y^3 + z^3) - \frac{\psi(T)}{\phi(T)} xyz.$$

Now comes the miracle. Let

$$i_{333}(\sigma) := \frac{-\sigma^3(-216+\sigma^3)^3}{(27+\sigma^3)^3}. \tag{1.2}$$

Then one can directly check that

$$i_{333}\left(\frac{-\psi(T)}{\phi(T)}\right) = j(q) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots \tag{1.3}$$

where $q = \exp(-t) = T^{24}$ is the Kähler parameter of the elliptic curve E.

The function i_{333} is defined in Saito's theory for elliptic singularities. It has the defining property that

$$i_{333}(\sigma(Q)) = j(q = Q^3) \tag{1.4}$$

where $q(\sigma)$ is (exponential of) the flat coordinate on the complex moduli of the *mirror* elliptic curve

$$\check{E} = \{x^3 + y^3 + z^3 + \sigma xyz = 0\} \subset \mathbb{P}^2,$$

$Q(\sigma) = q^{1/3}(\sigma)$, and $\sigma(Q)$ is the inverse function of $Q(\sigma)$. $q(\sigma)$ is also known as the *mirror map*, which takes the form

$$q(\sigma) = \exp\left(2\pi\mathbf{i} \cdot \frac{\pi_B(\sigma)}{\pi_A(\sigma)}\right)$$

where π_A and π_B are certain periods of $\check{E}$, and they satisfy the Picard-Fuchs equation

$$u'' + \frac{3\sigma^2}{\sigma^3+27}u' + \frac{\sigma}{\sigma^3+27}u = 0.$$

The readers can find more details about the Frobenius structures in [MR, Section 6.2]. $\sigma(Q)$ is known as the inverse mirror map. Explicitly

$$\sigma(Q) = 3 + \left(\frac{\eta(Q)}{\eta(Q^8)}\right)^3 = \frac{1}{Q}(1 + 5Q^3 - 7Q^6 + \dots) \tag{1.5}$$

where $\eta(Q) = Q^{1/24}\prod_{n\geq 1}(1-Q^n)$ is the Dedekind eta function. $Q = q^{1/3}$ is understood as the Kähler parameter of the elliptic curve quotient $E/\mathbb{Z}_3$ (whose area is $1/3$ of that of E).

With the above understanding, Equation (1.3) can be reformulated as follows:

Theorem 1.1 (Theorem 1.6 of [CHL17]). *The following equality holds for the inverse mirror map and the counting functions of the elliptic curve E with a $\mathbb{Z}_3$ symmetry:*

$$\sigma(Q = T^8) = \frac{-\psi(T)}{\phi(T)}.$$

Theorem 1.1 gives an enumerative meaning of the inverse mirror map $\sigma(Q)$, which is expressed in terms of the Dedekind eta function by Equation (1.5), by counting triangles bounded by L. Here I want to emphasize that this is not just a coincidence. Indeed it gives a guiding principle: the mirror, the mirror map, and also the mirror functor realizing homological mirror symmetry can all be geometrically constructed using (immersed) Lagrangian deformation theory.

1.2. **Elliptic curve with complex multiplication by sixth root of unity.** Here is another example in order to convince you that Theorem 1.1 is an instance of a general principle, rather than just a coincidence.

Let E be the elliptic curve with complex multiplication by the cube root of unity as in the last section. This time we consider $\mathbb{Z}_6$ symmetry rather than $\mathbb{Z}_3$, and consider a specific immersed curve L in E invariant under $\mathbb{Z}_6$ whose lift in the universal cover $\mathbb{R}^2$ is shown by dotted curves in Figure 2. The action of $e^{2\pi\mathbf{i}/6}$ is given by rotation about any of the points labelled by tiny circles in the figure. Again we equip E with the flat metric descended from the Euclidean plane, and denote its total area by t. The quotient of E by $\mathbb{Z}_6$ is the orbifold projective line $E/\mathbb{Z}_6 = \mathbb{P}^1_{(2,3,6)}$. Similarly E and $E/\mathbb{Z}_6$ can be described algebraically by using the equation $x^2 + y^3 + z^6 = 0$.

Then we count polygons bounded by L in the same way as in the last section, and obtain a generating function of polygon counting W (see Equation (1.1)). A minimal triangle is shown in Figure 2 which is labelled by the monomial xyz. Again in the counting we exclude any polygon which has a corner not being an angle of a minimal triangle. The area of each minimal triangle is taken to be $t/48$, and we let $T = \mathbf{e}^{-t/48}$.

This time more types of polygons show up, namely triangles, trapezoids, parallelograms, pentagons and hexagons. Some of them are shown in Figure 2. W takes the form

$$W = T^6x^2 - Txyz + c_y(T)y^3 + c_z(T)z^6 + c_{yz2}(T)y^2z^2 + c_{yz4}(T)yz^4$$

where $c_y, c_z, c_{yz2}, c_{yz4}$ are certain series in T (see Theorem 9.1 of [CHKL17] for the explicit expression).

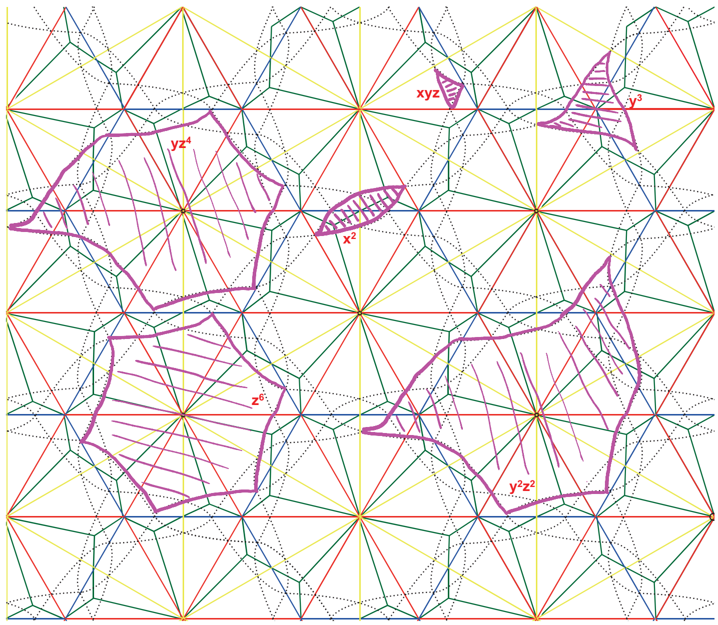

FIGURE 2. Polygon countings in the elliptic curve E. The dotted lines show the lift of L in the universal cover.

By a change of coordinates in (x, y, z), W can be arranged to be

$$x^2 + y^3 + z^6 - s(T)yz^4$$

for an explicit series $s(T)$. Let

$$i_{236}(\sigma) := 1728 \cdot \frac{4\sigma^3}{27 + 4\sigma^3}. \tag{1.6}$$

Then one can directly check that

$$i_{236}(s(T)) = j(q) \tag{1.7}$$

where $q = \exp(-t) = T^{48}$ is the Kähler parameter of the elliptic curve E.

The function i_{236} is defined in Saito's theory for elliptic singularities. It has the defining property that

$$i_{236}(\sigma(Q)) = j(q = Q^6) \tag{1.8}$$

where $q(\sigma)$ is (exponential of) the flat coordinate on the complex moduli of the *mirror* elliptic curve

$$\check{E} = \{x^2 + y^3 + z^6 + \sigma yz^4 = 0\} \subset \mathbb{P}^2(2, 3, 6)$$

around $\sigma = -3/2^{2/3}$, $Q(\sigma) = q^{1/6}(\sigma)$, and $\sigma(Q)$ is the inverse function of $Q(\sigma)$. $q(\sigma)$ is also known as the *mirror map*, which takes the form

$$q(\sigma) = \exp\left(2\pi\mathbf{i} \cdot \frac{\pi_B(\sigma)}{\pi_A(\sigma)}\right)$$

where π_A and π_B are certain periods of $\check{E}$. The readers can find more details about Frobenius structures in [MS16, Section 5.5]. $\sigma(Q)$ is known as the inverse mirror map. Explicitly

$$\sigma(Q) = \frac{-3}{2^{2/3}}\left(1 + 576Q^6 + 235008Q^{12} + 109880064Q^{18} + 53449592832Q^{24} + \dots\right) \tag{1.9}$$

$Q = q^{1/6}$ is understood as the Kähler parameter of the elliptic curve quotient $E/\mathbb{Z}_6$ (whose area is $1/6$ of that of E). One can check explicitly that $s(T)$ has the same expression as the inverse mirror map $\sigma(Q)$ (where $Q = T^8$).

The same procedure can be carried out for the elliptic curve with complex multiplication by fourth root of unity to express the corresponding inverse mirror map as polygon countings. The readers are referred to [CHKL17] for details.

The construction of W in Equation (1.1), the generating function of polygon countings, indeed work in a much more general context rather than just for elliptic curves with complex multiplications. We call this the generalized SYZ construction. Moreover in several interesting classes of geometries, W serves as the Landau-Ginzburg mirror, and there exists a canonical functor from the Fukaya category to the category of matrix factorizations of W, which realizes homological mirror symmetry. In the following we will give a short overview on SYZ mirror symmetry before introducing the generalized SYZ approach.

2. SYZ mirror construction

This section gives a quick review on the development of SYZ mirror symmetry in the last two decades.

2.1. **Symplectic approach to SYZ.** For a pair of mirror Calabi-Yau manifolds X and $\check{X}$, the Strominger-Yau-Zaslow (SYZ) conjecture [SYZ96] asserts that there exist special Lagrangian torus fibrations $\mu : X \to B$ and $\check{\mu} : \check{X} \to B$ which are fiberwise-dual to each other. In particular, this suggests an intrinsic construction of the mirror $\check{X}$ by fiberwise dualizing a special Lagrangian torus fibration on X. This process is called *T-duality*.

The SYZ program has been carried out successfully in the semi-flat case [KS01, LYZ00, Leu05], where the discriminant loci of special Lagrangian torus fibrations are empty (i.e. all fibers are regular) and the base B is a smooth integral affine manifold. On the other hand, mirror symmetry has been extended to non-Calabi-Yau settings, and the SYZ construction has been shown to work in the toric case [Aur07, CL10], where the discriminant locus appears as the boundary of the base B (so that B is an integral affine manifold with boundary). Topological evidences for SYZ mirror symmetry have been found by [Gro01b].

Beyond the semi-flat and toric cases, there are two main difficulties in order to carry out the SYZ construction. First, constructing Lagrangian fibrations is a rather difficult task. [Zha00] constructed topological torus fibrations for Calabi-Yau hypersurfaces in toric varieties. [Rua01, Rua02, CBM09] constructed Lagrangian fibrations for the quintic Calabi-Yau. Yet constructing Lagrangian fibrations for general Calabi-Yau manifolds (say, complete intersections in toric varieties) is still an open problem. Secondly, Lagrangian fibrations often have singular fibers. The semi-flat complex manifold obtained by fiberwise dualizing the special Lagrangian torus fibration away from discriminant loci is *not* the correct mirror. The speculation of [Fuk05] suggested that the correct mirror should be obtained by taking quantum corrections on the semi-flat complex manifold, which should be given by certain *open Gromov-Witten invariants*, which are roughly speaking countings of holomorphic discs emanated from the singular fibers.

Quantum corrections can be made rigorous in the symplectic approach for non-compact toric Calabi-Yau manifolds [CLL12]. They serve as local patches of compact Calabi-Yau varieties and give rise to local mirror symmetry, which has been intensively studied in a lot of literatures [CKYZ99, Tak01, KZ01, GZ02, Hos00, Hos06, FJ05, KM10, Sei10]. Moreover, Lagrangian fibrations for this class of manifolds were constructed by [Gol01, Gro01a]. Since they are toric, their open Gromov-Witten invariants were defined by [CO06, FOOO10]. In this case SYZ mirror construction can be carried out using symplectic geometry [CLL12]. The mirror constructed is expressed in terms of open Gromov-Witten invariants.

Wall-crossing of open Gromov-Witten invariants studied in [Aur07] plays the key role in the construction of [CLL12]. Roughly speaking it is the phenomenon that there is a chamber structure on the base of the Lagrangian fibration, such that open Gromov-Witten invariants remain the same within one chamber but change drastically across the chamber. See Figure 3 for an instance of wall-crossing phenomenon. For the case of toric Calabi-Yau manifolds of dimension n, there are two chambers in the base, and wall-crossing across the chamber is determined by a function $g(q, z_1, \ldots, z_{n-1})$. Then the SYZ mirror takes the form

$$uv = g(q, z_1, \ldots, z_{n-1}).$$

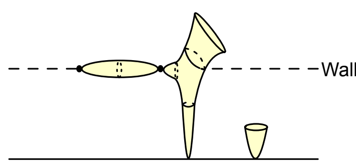

FIGURE 3. An example of wall-crossing. There is only one holomorphic disc below the wall, but above the wall another holomorphic disc shows up.

SYZ construction using symplectic geometry for blowups of toric varieties was carried out by [AAK16], which can be regarded as the reverse direction of [CLL12].

[Lau14] gave another class of geometries that SYZ can be carried out using symplectic geometry. The paper constructed the SYZ mirrors of smoothings of toric Gorenstein singularities, whose Lagrangian fibration was constructed in [Gro01a]. It was also proved that SYZ mirrors in conifold transitions are related by analytic continuation and change of coordinates on the Kähler moduli.

2.2. **Tropical approach to SYZ.** While the symplectic approach to SYZ works for interesting classes of geometries discussed above, it is still quite restrictive due to the two difficulties mentioned, namely construction of Lagrangian fibrations and quantum corrections.

Based on tropical geometry [Mik05, Mik06], affine geometry and wall-crossing [KS01, KS06], Gross-Siebert [GS11a] gave an algebraic version of SYZ construction *without* solving the above two problems. [GS11b] gave an excellent exposition to their construction. Instead of considering Lagrangian fibrations, they considered a toric degeneration and constructed an affine manifold from the intersection complex of the central fiber of degeneration.

Then they worked on walls attached with the so-called 'slab functions', which play the role of open Gromov-Witten invariants in this tropical setting. The key phenomenon is scattering when two walls hit each other. The effect of scattering can be computed order-by-order, and as a result a family of varieties is constructed order by order which should serve as the mirror. Roughly speaking, the variety is a bunch of toric charts corresponding to connected components of the complement of walls, glued to each other according to the slab functions. Log geometry plays an essential role to compute GW invariants in their theory of toric degeneration [GS06, GS10, GS13].

In the surface case [GHK11] constructed mirrors of log Calabi-Yau surfaces (which are basically pairs (X, D) for normal-crossing divisors $D \in |-K_X|$), by the further study of theta functions which should be mirror to Lagrangian intersections.

Remark 2.1. *In* [GS14] *the mirror of a toric Calabi-Yau manifold is also constructed by using the Gross-Siebert program, which is expressed in terms of slab functions.* [Lau15] *proved that the mirrors produced by these two different approaches actually equal to each other. The proof is based on the open mirror theorem of* [CCLT16] *and a tropical interpretation of hypergeometric functions.*

A correspondence between counting of certain rational curves and tropical curves in the toric case was established by [NS06]. *By similar technique countings of certain holomorphic discs and tropical discs in the toric case was established by* [Nis12].

The advantage of the tropical approach of Gross-Siebert is that toric degeneration is easy to construct, and so their scheme works for very general situations. On the other hand, the construction is rather complicated to implement in actual situations. Moreover homological mirror symmetry is rather unclear in the tropical setup. For instance, the Fermat quintic

$$X = \{x_0^5 + \ldots + x_4^5 = 0\} \subset \mathbb{P}^4$$

has mirror being its quotient by $(\mathbb{Z}/5)^3$. However quantum corrections involved in Gross-Siebert program for the Fermat quintic are overwhelmingly complicated and occur up to infinite order.

On the other hand, homological mirror symmetry for Fermat-type hypersurfaces was proved by Sheridan [She15, She11], based on the previous work of Seidel [Sei11] on homological mirror symmetry for genus-two curves. One key ingredient is an explicit Lagrangian immersion which generates the Fuakya category. This motivates the idea of generalized SYZ [CHL17] introduced in the next section, which avoids complicated wall-crossing and gives a natural functor realizing homological mirror symmetry in some important classes of geometries.

3. Generalized SYZ construction

The main idea is to choose a Lagrangian (immersion) $\mathbf{L}$ and use its deformation theory to construct a Landau-Ginzburg model W. The procedures are as follows:

(1) *Construct a suitable Lagrangian immersion* $\mathbf{L}$. This should be oriented and (relatively) spin. For simplicity it is assumed to have transverse self-intersections. For the purpose

of homological mirror symmetry we may want to require $\mathbf{L}$ split-generates the Fukaya category, although we do not need such an assumption in the construction. $\mathbf{L}$ plays the role of a Lagrangian fibration in the original SYZ approach.

(2) *Take a (weakly-unobstructed) deformation space V of $\mathbf{L}$ as the mirror space.* It plays the role of semi-flat mirror in the original SYZ approach. Note that deformations of an immersed Lagrangian not just include the usual Lagrangian deformations but also smoothings at immersed points.

(3) *The quantum corrections are given by countings of J-holomorphic polygons bounded by $\mathbf{L}$.* They form a generating function W defined by Equation (1.1). Then (V, W) forms a Landau-Ginzburg model, and we call this a generalized SYZ mirror.

One advantage of such a construction is that it avoids complicated scattering and gluing, and so the Landau-Ginzburg model (V, W) comes out in a direct and natural way. This also matches the general philosophy that Landau-Ginzburg model is easier to work with than Calabi-Yau model (and it is an important topic to study the correspondence between the two).

Elliptic curves with complex multiplications described in Section 1 fit into this setting. Consider the orbifold projective line $\mathbb{P}^1_{a,b,c}$ $(a, b, c \geq 1)$. The first example in Section 1 corresponds to $E/\mathbb{Z}_3 = \mathbb{P}^1_{(3,3,3)}$, and the second example corresponds to $E/\mathbb{Z}_6 = \mathbb{P}^1_{(2,3,6)}$. One has the following classification in relation with (affine) Dynkin diagram, where $\chi := \frac{1}{a} + \frac{1}{b} + \frac{1}{c} - 1$.

Spherical ($\chi > 0$)			Planar ($\chi = 0$)			Hyperbolic ($\chi < 0$)
Type A $(1, b, c)$	Type D $(2, 2, k), k \geq 2$	Type E $(2, 3, k), k = 3, 4, 5$	$\tilde{E}_6$ (3,3,3)	$\tilde{E}_7$ (2,4,4)	$\tilde{E}_8$ (2,3,6)	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$

Consider the Lagrangian immersion $L \subset X$ shown in Figure 4, which was constructed by [Sei11] for the purpose of proving homological mirror symmetry for genus-two Riemann surface (and used by [Efi12] for proving homological mirror symmetry for higher-genus Riemann surfaces). L is called to be the Seidel Lagrangian. It has three immersed points, and they give three independent directions of (weakly) unobstructed deformations labelled by x, y, z. Thus the mirror space in our construction is $V = \mathbb{C}^3$ with coordinates (x, y, z), and the generating function W is defined over $\mathbb{C}^3$. In Section 1, the curve L is the lift of the Seidel Lagrangian from $\mathbb{P}^1_{(3,3,3)}$ (or $\mathbb{P}^1_{(2,3,6)}$) to the elliptic curve.

In [CHL17] we formulated this mirror construction using a Lagrangian immersion, and applied it to construct the generalized SYZ mirror $(\mathbb{C}^3, W)$ of the orbifold sphere $\mathbb{P}^1_{a,b,c}$. The Seidel Lagrangian was used in the construction. One important thing is, the three independent directions of deformations labelled by x, y, z are weakly unobstructed, due to cancellations between holomorphic polygons and its reflection about the equator. Weakly unobstructedness is important to make sure that W is well-defined (independent of choice of the boundary marked point). In [CHL] we formulated a noncommutative version of the construction to cook up more solutions to the weakly unobstructedness condition and make it more flexible.

Let's compute the leading order terms of W. There is a minimal triangle bounded by L with x, y, z as vertices, which is the shaded region shown in Figure 4. The corresponding monomial is $-Txyz$, where $T = \exp(-A)$ and A is the area of the minimal triangle. We

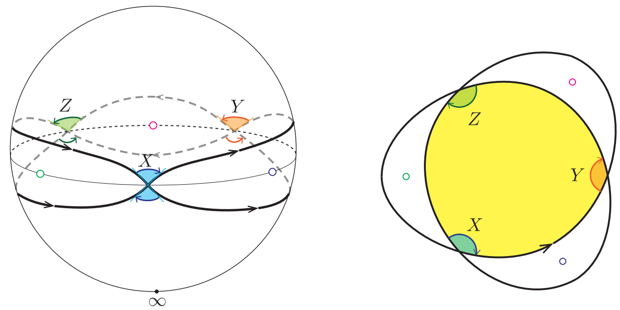

FIGURE 4. The Seidel Lagrangian. Two pictures show the same Lagrangian immersion from different viewpoints. The three dots on the equator are orbifold points of an orbifold sphere. The shaded triangle on the right contributes to the term xyz of the Landau-Ginzburg mirror.

may always take $Q = T^8$, where Q is the Kähler parameter of $\mathbb{P}^1_{(a,b,c)}$. Moreover L bounds an a-gon with x as vertices, a b-gon with y as vertices, and a c-gon with z as vertices. Hence W takes the form

$$W = -Txyz + T^{3a}x^a + T^{3b}y^b + T^{3c}z^c + \ldots$$

In [CHKL17] we give an algorithm to compute the open Gromov-Witten potential of $\mathbb{P}^1_{(a,b,c)}$ order by order. As a consequence, we derive the convergence of the open Gromov-Witten potential for all a, b, c. Notice that when $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$, $\mathbb{P}^1_{(a,b,c)}$ is of general type (and in particular its Riemann surface covers have genus bigger than one). This gives a first class of general-type manifolds whose open Gromov-Witten potentials can be computed and are convergent.

$\mathbb{P}^1_{a,b,c}$ can be written as a G-quotient of a Riemann surface Σ. Then the generalized SYZ mirror of the Riemann surface Σ is also given by the same superpotential W, but over the quotient $\mathbb{C}^3/\hat{G}$ where $\hat{G}$ is the group of characters of G (which is just isomorphic to G itself because G is Abelian). When $1/a + 1/b + 1/c \geq 1$, which corresponds to the case that Σ has genus less than or equal to one, the superpotential W has finitely many terms; when $1/a + 1/b + 1/c < 1$, which corresponds to the case that Σ has genus greater than one, it has infinitely many terms. Thus the construction gives the generalized SYZ mirror of a Riemann surface Σ.

One can also carry out generalized SYZ for (quotients of) Fermat-type hypersurfaces

$$\tilde{X} = \left\{[z_0 : \ldots : z_{n+1}] \in \mathbb{P}^{n+1} : z_0^{n+2} + \ldots + z_{n+1}^{n+2} = 0\right\}$$

using the Lagrangian immersion constructed by Sheridan [She11]. There are n degree-one independent deformation directions labelled by $x_1, \ldots, x_n$. Assuming that they are weakly unobstructed (which is not yet verified), we obtain a generalized SYZ mirror $(\mathbb{C}^{n+2}/\mathbb{Z}^{n+2}, W_n)$ of Fermat-type hypersurfaces where W_n has leading terms

$$\sum_{i=1}^{n} x_i^n + \sigma(q)x_1 \ldots x_n.$$

4. The open mirror principle

Whenever SYZ construction can be carried out (in either one of the three approaches introduced in the previous sections), one obtains a map from the Kähler moduli to the mirror complex moduli, which is called the *SYZ map*. On the other hand, mirror map, which goes from the mirror complex moduli to the Kähler moduli, plays a central role in mirror symmetry and was well studied in literatures. The mirror map is obtained by solving a PDE system called Picard-Fuchs equations. Its inverse, which also goes from the Kähler moduli to the mirror complex moduli, is called the inverse mirror map.

Open mirror principle. The SYZ map equals to the inverse mirror map.

Conjecture of this type was first proposed by Gross-Siebert [GS11a, Conjecture 0.2]. Namely, they conjectured that the formal deformation parameter in their reconstruction should be a 'canonical' coordinate of the mirror family. Moreover the slab functions involved in their reconstruction of mirror should have enumerative meaning in terms of counting certain holomorphic discs, and they found evidences in the case of $K\mathbb{P}^2$ based on the work of [CKYZ99] (although counting of holomorphic discs were not clear at that time).

In [CLL12] the principle was precisely formulated into a conjecture for toric Calabi-Yau manifolds using the symplectic approach to SYZ, where the SYZ map is written in terms of open Gromov-Witten invariants for discs of Maslov index two defined by [CO06, FOOO10, FOOO09a, FOOO09b]. In [Cha11], a relation between open and closed invariants for certain subclasses of toric Calabi-Yau manifolds was established, and and [LLW11] established strong evidences of the conjecture. [LLW12] proved the conjecture for toric Calabi-Yau surfaces, which are resolutions of A_n singularities, and [CLT13] proved the conjecture for the total space of the anticanonical line bundle of a toric Fano manifold. [CL14] proved the conjecture for every compact toric semi-Fano surfaces. Finally, the conjecture was proved in full generality for compact toric semi-Fano manifolds in [CLLT17], and for toric Calabi-Yau orbifolds by [CCLT16]. We call it to be open mirror theorem, which gives an effective way to compute the open Gromov-Witten invariants.

$K_{\mathbb{P}^2}$ is the best example to illustrate the open mirror principle for toric Calabi-Yau manifolds. In this case the SYZ mirror constructed in [CLL12] is the local Calabi-Yau defined by

$$\left\{(u,v,x,y)\in\mathbb{C}^2\times(\mathbb{C}^\times)^2 : uv=\left(\sum_{k=0}^{\infty} n_{\beta_0+kl}q^k\right)+x+y+qx^{-1}y^{-1}\right\}$$

where n_{β_0+kl} are one-pointed open Gromov-Witten invariants of the disc class β_0+kl, l is the line class of $\mathbb{P}^2\subset K_{\mathbb{P}^2}$ and q is the Kähler parameter of l. See Figure 5.

By the open mirror theorem of [CLT13, CCLT16], We have the equality

$$\sum_{k=0}^{\infty} n_{\beta_i+kl}q^k=\exp g(\check{q}(q))$$

where

$$g(\check{q})=\sum_{l>0}(-1)^{3l}\frac{(3l-1)!}{(l!)^3}\check{q}^l$$

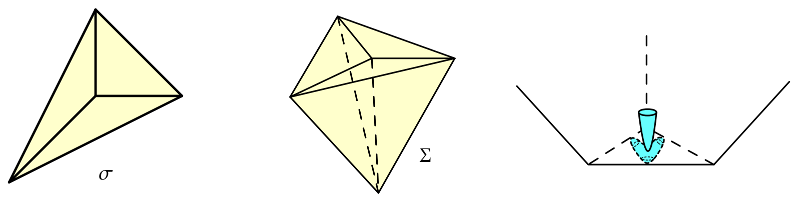

FIGURE 5. $K_{\mathbb{P}^2}$. The figure on the right shows a representative of the disc class $\beta_0 + l$.

and $q = \check{q}\exp(3g(\check{q}))$ is the mirror map (and $\check{q}(q)$ is the inverse mirror map). Explicitly

$$\sum_{k=0}^{\infty} n_{\beta_0+kl} q^k = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + 35870q^6 - \ldots.$$

Thus it gives an effective way to compute n_{β_0+kl}, which are all the non-trivial open Gromov-Witten invariants.

By a change of coordinates in (u, v, x, y), the mirror defining equation above can be rearranged as

$$uv = 1 + x + y + \check{q}(q)x^{-1}y^{-1}.$$

Hence we see that the SYZ map coincides precisely with the mirror map $\check{q}(q)$.

The open mirror principle was used to derive an open version of crepant resolution theorem for semi-Fano toric orbifolds in [CCLT14] and toric Calabi-Yau orbifolds [CCLT16]. Moreover, based on the open mirror theorem of [CCLT16] and a combinatorial understanding of hypergeometric functions (in relation with tropical geometry), [Lau15] verified the conjecture of Gross-Siebert mentioned above for toric Calabi-Yau manifolds.

Now consider the generalized SYZ construction explained in Section 3. Recall from Section 1 that the generalized SYZ mirror of $\mathbb{P}^1_{(3,3,3)} = E/\mathbb{Z}_3$ is

$$(x^3 + y^3 + z^3) - \frac{\psi(T)}{\phi(T)}xyz.$$

Thus the SYZ map in this case is by definition $-\frac{\psi(T=Q^{1/8})}{\phi(T=Q^{1/8})}$. By Theorem 1.1, the inverse mirror map $\sigma(Q)$ of the elliptic curve quotient $E/\mathbb{Z}_3$ equals to the SYZ map $\frac{-\psi(Q^{1/8})}{\phi(Q^{1/8})}$.

Similarly, the generalized SYZ mirror of $\mathbb{P}^1_{(2,3,6)} = E/\mathbb{Z}_6$ is

$$(x^2 + y^3 + z^6) - s(T)yz^4$$

where $s(T)$ is written in terms of polygon countings (see Figure 2). The generalized SYZ map is by definition $s(T)$. Both the inverse mirror map $\sigma(Q)$ and the generalized SYZ map $s(T)$ takes the form (where $Q = T^8$)

$$\frac{-3}{2^{2/3}}\left(1 + 576Q^6 + 235008Q^{12} + 109880064Q^{18} + 53449592832Q^{24} + \ldots\right)$$

Thus the miraculous relations between j function and polygon countings explained in Section 1 are instances of the big picture given by the open mirror principle. The generalized

SYZ appraoch gives an enumerative meaning of the inverse mirror map of (quotients of) elliptic curves.

5. Mirror functor

Homological mirror symmetry conjecture by Kontsevich [Kon95] asserts that for a pair of mirror manifolds $(X, \check{X})$, the derived Fukaya category of Lagrangian submanifolds of X is equivalent to the derived category of coherent sheaves of $\check{X}$.

More generally the mirror of X is a Landau-Ginzburg model W, which is a holomorphic function rather than a manifold, when X is not required to be Calabi-Yau. Homological mirror symmetry still makes sense by considering Fukaya-Seidel category of W [Sei08] (in place of Fukaya category) or category of matrix factorizations of W [Eis80, Orl04] (in place of derived category of coherent sheaves). In particular Lagrangian submanifolds of X should correspond to matrix factorizations of W.

The study of homological mirror symmetry leads to many new insights to Fukaya category and computational techniques for proving the conjecture in various cases, for instance see [Abo06, Abo09, AP01, Cha13, Sei11, FLTZ11, FLTZ12, She11, She15]. On the other hand, the main stream of the study of homological mirror symmetry is comparing generators and their relations (hom spaces) on both sides. This does not explain where homological mirror symmetry comes from geometrically.

Our generalized version of SYZ construction naturally gives an A_∞-functor $\mathcal{LM}^{\mathbf{L}}$ from the Fukaya category of X to the category of matrix factorization of W and hence explains the geometric origin of homological mirror symmetry:

Theorem 5.1 (Theorem 1.2 of [CHL17]). *Let W be the generalized SYZ mirror of X. We have an A_∞-functor*

$$\mathcal{LM}^{\mathbf{L}} : \mathcal{F}uk_{\lambda}(X) \to \mathcal{MF}(W - \lambda).$$

Here, $\mathcal{F}uk_{\lambda}(X)$ is the Fukaya category of X (as an A_∞-category) whose objects are weakly unobstructed Lagrangians with potential value λ, and $\mathcal{MF}(W-\lambda)$ is the dg category of matrix factorizations of $W - \lambda$.

The functor in the object level can be explained as follows. Recall that we have fixed a reference Lagrangian (immersion) $\mathbf{L}$ for the construction of W. To transform a (weakly unobstructed) Lagrangian brane L to a matrix factorization, we take the Lagrangian Floer 'complex' (H, δ) between $\mathbf{L}$ and L, where is basically the vector space formally spanned by the intersection points of $\mathbf{L}$ and L, and $\delta = m_1^{\mathbf{L},L}$ is the Floer differential obtained by counting strips between two intersection points. Actually H is not a complex: by the A_∞ relations the differential $m_1^{\mathbf{L},L}$ squares to $W - \lambda$ rather than 0, where λ is the potential value of L, that is $m_0^L = \lambda \cdot \mathbf{1}_L$. By definition (H, δ) gives a matrix factorization of W. It turns out that this corresopndence between objects determines an A_∞ functor, which is rather similar to the Hom functor in Yoneda embedding.

The work of Tu [Tu15] also constructed a functor for homological mirror symmetry based on Fourier-Mukai transform and Koszul duality. The main difference between [CHL17] and [Tu15] is that [CHL17] studied the formal deformations of a general Lagrangian (immersion), while [Tu15] studied Lagrangian torus fibrations. [CHL17] focused on the localized aspect of the functor (and 'globalize' the functor using group actions), while [Tu15] focused on sheaf theoretical aspect of the functor. In particular [Tu15] discussed the gluing of the functor over local patches away from singular fibers of the Lagrangian torus fibration. He applied his technique to toric Fano manifolds and obtain generating matrix factorizations when $\dim \leq 2$. In [CHL14] the localized mirror functor was applied to derive explicit generating matrix factorizations of a Laurent polynomial in all dimensions, which are described in the next section.

6. Homological mirror symmetry for orbifold spheres

By using the mirror functor introduced in the previous section, we deduce homological mirror symmetry for $X = \mathbb{P}^1_{a,b,c}$.

Theorem 6.1 (Theorem 1.5 of [CHL17])**.** *Let $X = \mathbb{P}^1_{a,b,c}$, $\mathbf{L}$ the Seidel Lagrangian and W its generalized SYZ mirror. Assume $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 1$. The A_∞-functor $\mathcal{LM}^{\mathbf{L}}$ in Theorem 5.1 derives an equivalence of triangulated categories*

$$D^\pi(\mathrm{Fuk}(\mathbb{P}^1_{a,b,c})) \cong D^\pi(\mathcal{MF}(W)).$$

The reason for the restriction $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq 1$ is to ensure that W has an isolated singularity at the origin. In the Fano case $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} > 1$, W gets 'Morsified', meaning that its critical points get dispersed from origin. The technique of proving homological mirror symmetry will be different (although the principle of using the mirror functor basically is the same).

One key ingredient is computation of the matrix factorization $\mathcal{LM}^{\mathbf{L}}(\mathbf{L})$. It turns out that it equals to

$$\left(\bigwedge^* \langle X,Y,Z\rangle, xX\wedge(\cdot) + yY\wedge(\cdot) + zZ\wedge(\cdot) + w_x\,\iota_X + w_y\,\iota_Y + w_z\,\iota_Z\right), \tag{6.1}$$

up to some non-trivial change of coordinates, where $xw_x + yw_y + zw_z = W$. For instance, when $(a,b,c) = (3,3,3)$,

$$\begin{aligned}
w_x =& x^2\sum_{k=0}^{\infty}(-1)^{k+1}(2k+1)T^{(3(2k+1))^2} \\
&+ yz\left(-T + \sum_{k=1}^{\infty}(-1)^{k+1}\left((2k+1)T^{(6k+1)^2}(T) - (2k-1)T^{(6k-1)^2}(T)\right)\right), \\
w_y =& -y^2\sum_{k=0}^{\infty}(-1)^{k}(2k+1)T^{(3(2k+1))^2} - xz\sum_{k=1}^{\infty}(-1)^{k+1}\left(2kT^{(6k+1)^2}(T) - 2kT^{(6k-1)^2}(T)\right), \\
w_z =& z^2\sum_{k=0}^{\infty}(-1)^{k+1}(2k+1)T^{(3(2k+1))^2} + xy\sum_{k=1}^{\infty}(-1)^{k+1}\left(2kT^{(6k+1)^2}(T) - 2kT^{(6k-1)^2}(T)\right).
\end{aligned}$$

This type of matrix factorizations was proved to split-generates the derived category of matrix factorizations by Dyckerhoff [Dyc11]. On the other hand $\mathbf{L}$ split generates the derived Fukaya category, and hence homological mirror symmetry follows.

Another half of homological mirror symmetry, which is the equivalence between the derived category of $\mathbb{P}^1_{(a,b,c)}$ and the Fukaya-Seidel category of its Landau-Ginzburg mirror $x^a + y^b + z^c - \sigma xyz$, is formulated by Takahashi [Tak10] and studied by [Ued06, Kea15].

7. Matrix factorizations for toric mirrors

The Lagrangian $\mathbf{L}$ that we start with in the construction of genearlized SYZ mirror and mirror functor can be taken to be *smooth*. In particular, we can take $\mathbf{L}$ to be a smooth torus, which is closer to the original setting of SYZ. This simple choice of $\mathbf{L}$ already gives some interesting results. In [CHL14], we applied our functor to moment-map fibers of toric manifolds and constructed generators of the category of matrix facotorizations of a Laurent polynomial

$$W = \sum_{i=1}^{m} c_i z^{v_i}$$

which are mirror to critical moment-map fibers of the mirror toric manifold (we have omitted several technical assumptions here; see [CHL14] for the precise statements). Let $\underline{z}$ be the critical points of W, and we assume their values of W are pairwise distinct. The generators are one-to-one corresponding to the critical points, and they take the explicit following expression (for a fixed $\underline{z}$):

$$R^{(l)} = \left[\left(\sum_{i=1}^{n} (z_i - \underline{z}_i) e_i \wedge \right) + \left(\sum_{i=1}^{m} c_i \sum_{j=1}^{n} \alpha^i_j \iota_{e_j} \right) \right]_{\underline{z} = \underline{z}^{(l)}} \tag{7.1}$$

where $\alpha^i_j = 0$ when $v_{i,j} = 0$,

$$\begin{aligned} \alpha^i_j =& z_1^{v_{i,1}} \ldots z_n^{v_{i,n}} z_j^{-1} \left(\prod_{l \neq j} \underline{z}_l^{-\delta(-1, s_{i,l})} \right) \left(\prod_{l \neq j} z_l^{-\delta(1, s_{i,l})} \right) \left(\left(\prod_{l > j} \underline{z}_l^{|s_{i,l}|} \right) \left(\prod_{l < j} \underline{z}_l^{\delta(s_{i,l}, -1)} z_l^{\delta(s_{i,l}, 1)} \right) \right. \\ & \left. + \left(\prod_{l \neq j} \underline{z}_l^{|s_{i,l}|} \right) \sum_{p=1}^{v_{i,j}-1} \left(\frac{\underline{z}_j}{z_j} \right)^p \prod_{l \neq j} \left(\frac{\underline{z}_l}{z_l} \right)^{s_{i,l} \left\lfloor p \left| \frac{v_{i,l}}{v_{i,j}} \right| \right\rfloor} \right) \end{aligned}$$

when $s_{i,j} = 1$, and

$$\begin{aligned} \alpha^i_j =& \underline{z}_1^{v_{i,1}} \ldots \underline{z}_n^{v_{i,n}} z_j^{-1} \left(\prod_{l \neq j} z_l^{-\delta(-1, s_{i,l})} \right) \left(\prod_{l \neq j} \underline{z}_l^{-\delta(1, s_{i,l})} \right) \left(\left(\prod_{l > j} \underline{z}_l^{|s_{i,l}|} \right) \left(\prod_{l < j} \underline{z}_l^{\delta(s_{i,l}, 1)} z_l^{\delta(s_{i,l}, -1)} \right) \right. \\ & \left. + \left(\prod_{l \neq j} \underline{z}_l^{|s_{i,l}|} \right) \sum_{p=1}^{|v_{i,j}|-1} \left(\frac{\underline{z}_j}{z_j} \right)^p \prod_{l \neq j} \left(\frac{z_l}{\underline{z}_l} \right)^{s_{i,l} \left\lfloor p \left| \frac{v_{i,l}}{v_{i,j}} \right| \right\rfloor} \right) \end{aligned}$$

when $s_{i,j} = -1$.

Chan-Leung [CL12] and Cho-Hong-Lee [CHL12] derived the matrix factorization mirror to the Clifford torus of $\mathbb{P}^2$ and of $\mathbb{P}^1 \times \mathbb{P}^1$ respectively from SYZ arguments. The general expression here agrees with their results (through some simple change of coordinates).

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