GENERALIZED SYZ MIRROR TRANSFORMATION

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Abstract. Strominger-Yau-Zaslow proposed that mirror symmetry can be understood by torus duality. In this article we explain how it fits into a bigger framework, where tori are replaced by general Lagrangian immersions. The generalized construction is applicable to a wider class of geometries. We also give a brief introduction to our ongoing work on gluing local mirrors into global geometries.

1. SYZ mirror symmetry and why to generalize it

Let (X,ω) be a symplectic manifold, which is a smooth manifold X of dimension $2n$ equipped with a non-degenerate closed two-form ω. A fundamental non-compact example is the cotangent bundle T^*M of a manifold M, which gives a geometric formulation of Hamiltonian mechanics. The most important geometric objects in (X,ω) are Lagrangian submanifolds, which are submanifolds L of dimension n with $\omega|_L = 0 \in \Omega^2(L)$. For instance, the zero section $M \subset T^*M$ is a Lagrangian submanifold. Fukaya-Oh-Ohta-Ono [FOOO09] defined an A_∞ category $\mathrm{Fuk}(X)$ whose objects are Lagrangian submanifolds.

A mirror of (X,ω) is a variety $\check{X}$ whose complex geometry reflects the symplectic geometry of (X,ω). In terms of homological mirror symmetry formulated by Kontsevich [Kon95], $\check{X}$ is called to be a mirror of X if

$$D^\pi \mathrm{Fuk}(X) \cong D^b(\check{X})$$

where $D^b(\check{X})$ denotes the derived category of $\check{X}$. Complex geometry is a beautiful and well-developed branch in mathematics which behaves much neater than real geometry. Mirror symmetry transforms symplectic geometry, which models Hamiltonian dynamics and its quantizations, to complex geometry, which is more well-understood and computable.

One fundamental question is how to construct mirrors. Motivated from string theorists, Batyrev-Borisov [BB96] gave a combinatorial recipe to write down the mirrors of complete intersections in toric varieties. We would like to find a universal construction for symplectic manifolds.

Strominger-Yau-Zaslow [SYZ96] proposed to construct mirrors by using Lagrangian fibrations. Namely, suppose X has a Lagrangian torus fibration.

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They suggested that the dual torus fibration gives a mirror of X. Moreover it asserted that there should be a real version of *Fourier-Mukai transform* [LYZ00] which realizes homological mirror symmetry. The SYZ program leads to a lot of important developments in mirror symmetry, including Gross-Siebert program [GS11] and family Floer theory [Fuk02, Tu14, Tu15, Abo, Abo17].

The reason for using a Lagrangian torus fibration is the following. The key idea is to construct a mirror $\check{X}$ as the (complexified) moduli space of certain Lagrangian submanifolds of X. Suppose $\check{X}$ is a smooth complex manifold. Tautologically $\check{X}$ is the moduli space of points $p \in \check{X}$. The endomorphism space is $\mathrm{Ext}^*(\mathcal{O}_p, \mathcal{O}_p) \cong \bigwedge^* \mathbb{C}^n \cong H^*(T, \mathbb{C})$, the cohomology of a torus T! It suggests that $\mathcal{O}_p$ should be mirror to a Lagrangian torus in X (whose Floer cohomology equals to the usual cohomology). A nice assumption is that deformations of a certain Lagrangian torus produce a Lagrangian fibration, and the moduli space of such tori forms the base of the fibration.

However, recent developments of mirror symmetry found a lot of situations where $\check{X}$ is NOT a smooth complex manifold. One important scenario is that $\check{X}$ is given as the critical locus (equipped with the sheaf of vanishing cycles) of a holomorphic function $W : Y \to \mathbb{C}$, which is called a Landau-Ginzburg model. Another scenario is that $\check{X}$ is a non-commutative geometry where we cannot even talk about points. In such cases the above argument about endomorphism space does not apply, and we don't expect such mirrors come from taking dual of Lagrangian torus fibrations!

Rigid Calabi-Yau manifolds (whose deformation space $H^1(T_X)$ is zero) provide an excellent class of Calabi-Yau manifolds indicating that the SYZ program needs to be generalized. Their mirrors are given by Landau-Ginzburg models $W : Y \to \mathbb{C}$, whose critical locus has higher dimension than X. Due to the above reason, we do not expect Y is constructed from a Lagrangian torus fibration on X, and indeed we do not expect there exists a Lagrangian fibration on X. We shall construct (Y, W) as a moduli space of certain Lagrangian *non-tori*.

We propose to construct mirrors by using (families of) immersed Lagrangians in place of Lagrangian torus fibration. The work of Seidel [Sei11] and Sheridan [She11, She15] showed that the deformation theory of a specific immersed Lagrangian is already rich enough to prove homological mirror symmetry for Fermat-type hypersurfaces. The immersed Lagrangians they constructed have Floer cohomology of a torus of dimension $n+2$. This is one important reason why we allow immersed Lagrangians rather than smooth Lagrangians. Another important reason is that immersed Lagrangians have well-defined Floer theory by the work of Akaho-Joyce [AJ10]. It behaves very much like smooth Lagrangians.

Our approach can be understood as developing Floer theory for a family of immersed Lagrangian branes. *Family Lagrangian Floer theory* was suggested by Fukaya [Fuk02] to deduce homological mirror symmetry [Kon95] from Lagrangian torus fibrations. Tu [Tu14, Tu15] and Abouzaid [Abo, Abo17] further developed foundations of the theory away from singular torus fibers. The theory around singular fibers is currently under a lot of investigations.

The construction is divided into two parts, namely construction of local mirrors and gluing of the local pieces. Our previous works [CHL17, CHLa, CHLb] focused on the first part. We developed a program of constructing (noncommutative) mirrors using one Lagrangian immersion. It is family Floer theory of an immersed Lagrangian over a formal polydisc. Even the construction is localized to one Lagrangian, it suffices to construct the mirrors and functors in many important situations. We applied it to derive mirror symmetry for elliptic and hyperbolic orbifolds, punctured Riemann surfaces and local Calabi-Yau threefolds associated to Hitchin systems. It also has applications to deformation quantizations [Kon03]. There is no wall-crossing for the construction over a formal disc. As a result the construction can be made very clean and explicit.

The second part is currently an ongoing project. We consider a family of immersed Lagrangians over a polyhedral complex. Each immersed Lagrangian produce a local piece of mirror, and we shall glue them together by certain maps which identifies the Floer theories of the various immersed Lagrangians. The difference from the case of a torus fibration is that, for torus fibrations all non-singular fibers have the same topology, and their Floer theories are linked by isotopies of diffeomorphisms. In our case the immersed Lagrangians in different polyhedral strata has different topologies, and so there is no diffeomorphism between them. We need to relate Floer theory of an immersion with that of its (partial) smoothings.

The construction is related to Kontsevich's program of understanding symplectic geometry by cosheaves of categories and noncommutative geometry. He proposed that given a Lagrangian skeleton, each local part of the skeleton corresponds to a derived category, and the Fukaya category can be glued from these local charts of derived categories. Sibilla-Treumann-Zaslow [STZ14] and Dyckerhoff-Kapranov [DK] realized this construction on punctured Riemann surfaces. In our construction we consider gluing quiver algebras with relations rather than derived categories, which is considerably simpler and much more explicit. As a result we obtain the underlying 'mirror space' rather than an 'algebraic definition' for the Fukaya category. The Fukaya category is captured by the derived category of modules over the mirror space via the mirror functor.

Precisely the mirrors that we construct are rigid analytic spaces defined over the Novikov ring. In this article (especially in Section 3) we have skipped this important technical issue. See [HL] for a more precise formulation.

2. Local construction

Let $\mathbf{L} = \{L_1, \ldots, L_N\}$ be a collection of spin oriented connected compact Lagrangian immersions in X which intersect each other transversely. In

[CHLb], we used the deformation and obstruction theory of $\mathbf{L}$ to construct a non-commutative Landau-Ginzburg model $(\mathcal{A}, W)$, together with a non-trivial functor from $\mathrm{Fuk}(X)$ to the category of matrix factorizations (W-twisted complexes) of $(\mathcal{A}, W)$. The functor is automatically injective on the morphism space $\mathrm{HF}(\mathbf{L}, \mathbf{L})$. In particular if $\mathbf{L}$ and its image under the functor are both generators, and their endomorphism spaces have the same dimension, then the functor derives homological mirror symmetry.

The formal deformation space of $\mathbf{L}$ are given by degree one endomorphisms. (If we just have $\mathbb{Z}_2$ grading, 'degree one' means 'odd degree'.) They are described by a directed graph Q (so-called a quiver). The path algebra ΛQ is regarded as the noncommutative space of formal deformations of $\mathbf{L}$. Each edge e of Q corresponds to an odd-degree Floer generator X_e and a formal dual variable $x_e \in \Lambda Q$. We take the formal deformation $b = \sum_e x_e X_e$, where x_e is taken as an element in the path algebra ΛQ.

Obstructions of $\mathbf{L}$ is defined by counting of holomorphic polygons bounded by $\mathbf{L}$. Roughly speaking, it is captured by the superpotential W, which is defined as a weighted count of holomorphic polygons passing through a marked point:

$$W = \sum_{\beta} n_\beta q^\beta x^{\partial\beta} \tag{2.1}$$

where n_β is the counting of holomorphic polygons in class β passing through a marked point, q^β records the symplectic area of β, and $x^{\partial\beta}$ records the self-intersection points hit by the corners of the polygon. Note that $x^{\partial\beta}$ is an element in the path algebra ΛQ; in particular we also record the order of the corners of each polygon.

However such a counting is not well-defined in general, since it depends on the position of the marked point. We need to consider *weakly-unobstructed* deformations, and W is only well-defined for such deformations. *Weakly unobstructedness* was introduced by Fukaya-Oh-Ohta-Ono [FOOO09]. It is an analog of the exponential map, which integrates infinitesimal deformations to actual small deformations.

We extend the notion of weakly unobstructedness by Fukaya-Oh-Ohta-Ono [FOOO09] to the current noncommutative setting. The corresponding Maurer-Cartan equation is

$$m_0^b := \sum_{k=0}^{\infty} m_k(b, \ldots, b) = \sum_{i=1}^{k} W_i(b) \mathbf{1}_{L_i}$$

where m_k is the A_∞ operations defined by counting holomorphic polygons bounded by L, and $\mathbf{1}_{L_i}$ is the Floer-theoretical unit corresponding to the fundamental class of L_i. Essentially the equation means that the counting W is well-defined and in particular does not depend on the position of the marked point.

The solution space is given by the quiver algebra with relations $\mathcal{A} = \Lambda Q / R$ where R is the two-sided ideal generated by weakly unobstructed relations.

The relations are coefficients of m_0^b in all generators other than $\mathbf{1}_{L_i}$. As a result, we obtain a noncommutative Landau-Ginzburg model

$$\left(\mathcal{A}, W = \sum_i W_i\right).$$

It is a 'generalized mirror' of X in the following sense.

Theorem 2.1 ([CHLb]). *There exists an A_∞-functor $\mathcal{F}^{\mathbf{L}} : \mathrm{Fuk}(X) \to \mathrm{MF}(\mathcal{A}, W)$, which is injective on $H^\bullet(\mathrm{Hom}(\mathbf{L}, U))$ for any object U in $\mathrm{Fuk}(X)$.*

Another important feature is that the Landau-Ginzburg superpotential W constructed in this way is automatically a central element in $\mathcal{A}$. In particular we can make sense of $\mathcal{A}/\langle W\rangle$ as a hypersurface singularity defined by 'the zero set' of W.

Theorem 2.2 ([CHLb]). *$W \in \mathcal{A}$ is a central element.*

In summary the construction goes as follows.

(1) Start with a set of Lagrangian immersions $\mathbf{L} = \{L_1, \ldots, L_N\}$.
(2) Take the path algebra of odd-degree deformations of $\mathbf{L}$, which is non-commutative.
(3) Solve the Maurer-Cartan Equation and thereby construct a non-commutative Landau-Ginzburg model $(\mathcal{A}, W)$ and an A_∞-functor $\mathcal{F}^{\mathbf{L}} : \mathrm{Fuk}(X) \to \mathrm{MF}(\mathcal{A}, W)$.
(4) If $\mathbf{L}$ and its image under the functor are both split-generators of the derived categories, and their endomorphism spaces have the same dimension, then the functor derives homological mirror symmetry. Thus $(\mathcal{A}, W)$ is a mirror of X in the sense of homological mirror symmetry.

In the following we shall illustrate the construction by several examples.

2.1. **Pair-of-pants.** A pair-of-pants is $(\mathbb{C}^\times)^n - H$ where $H = \{z_1 + \ldots + z_n = 1\}$ where $z_1, \ldots, z_n$ are the standard coordinates. By Seidel [Sei11] and Sheridan [She11], a pair-of-pants is homologically mirror to the Landau-Ginzburg model $(\mathbb{C}^{n+2}, W)$ where $W = z_1 \ldots z_{n+2}$. In this section we focus on $n = 1$ and illustrate how to obtain this Landau-Ginzburg mirror from the generalized SYZ construction.

Recall the immersed Lagrangian constructed by Seidel shown in Figure 1 (extracted from [CHL17][Figure 1]). There is a natural generalization of this Lagrangian to zig-zag Lagrangians for dimer models in Riemann surfaces, which was used by Bocklandt [Boc16] to prove homological mirror symmetry for punctured Riemann surfaces.

The collection of immersed Lagrangians we consider here is the singleton $\mathbf{L}$, the Seidel Lagrangian (with a chosen orientation). It has three self-intersection points. Each immersed point gives two independent deformation directions by smoothing, see Figure 2. One direction is of odd degree, while the other is even. As in a typical deformation theory, the odd-degree part

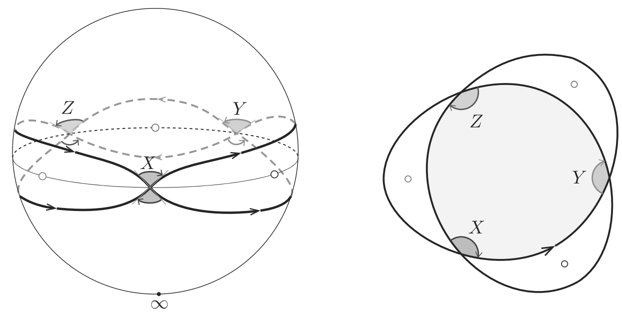

FIGURE 1. The immersed Lagrangian constructed by Seidel in the pair-of-pants.

governs deformations while the even-degree part governs obstructions. The three immersed points give three odd-degree deformation directions denoted as X, Y, Z, and three even-degree deformation directions denoted as $\bar{X}, \bar{Y}, \bar{Z}$.

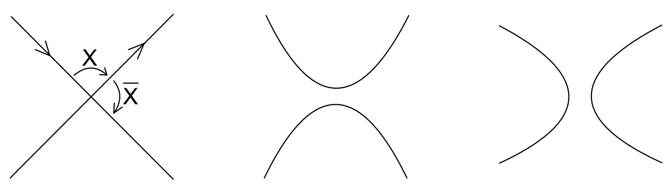

FIGURE 2. Two ways of smoothing at an immersed point. The figure in the middle shows an odd-degree deformation, while the figure on the right shows an even-degree deformation.

Since $\mathbf{L}$ is just a singleton, the corresponding quiver Q has only one vertex. It has three arrows labeled by x, y, z corresponding to the three odd-degree deformation directions. The path algebra ΛQ is the free algebra generated by x, y, z. Let $b = xX + yY + zZ$ be the formal deformations.

The obstruction is m_0^b which takes the form $W \cdot \mathbf{1} + h_1\bar{X} + h_2\bar{Y} + h_3\bar{Z}$. In this example the only holomorphic polygons bounded by $\mathbf{L}$ are the two triangles. They contribute two monomials, which are $q_{\text{front}}zy$ and $q_{\text{back}}yz$, to $\bar{X}$, where q_{front} and q_{back} are exponential of the areas of the two triangles. The contributions to $\bar{Y}$ and $\bar{Z}$ are similar. They also contribute two monomials, namely $q_{\text{front}}zyx$ and $q_{\text{back}}xyz$, to the fundamental class $\mathbf{1}$. By choosing a suitable non-trivial spin structure on $\mathbf{L}$ (to fix the signs of the contributions), we have

$$\begin{aligned} m_0^b =& (q_{\text{front}}zyx + q_{\text{back}}xyz) \cdot \mathbf{1} + (q_{\text{front}}zy - q_{\text{back}}yz)\bar{X} + (q_{\text{front}}xz - q_{\text{back}}zx)\bar{Y} \\ &+ (q_{\text{front}}yx - q_{\text{back}}xy)\bar{Z} \end{aligned}$$

As a result, we obtain the non-commutative Landau-Ginzburg model $(\mathcal{A}, W)$, where $\mathcal{A}$ is a quotient of the free algebra generated by x, y, z by the relations $q_{\text{front}} zy = q_{\text{back}} yz, q_{\text{front}} xz = q_{\text{back}} zx, q_{\text{front}} yx = q_{\text{back}} xy$, and $W = q_{\text{front}} zyx + q_{\text{back}} xyz$. One can easily check that W is a central element.

Now suppose we choose the Seidel Lagrangian such that it is symmetric about the equator. Then the areas of the two triangles are the same: $q_{\text{front}} = q_{\text{back}}$. Then R simply reduces to the polynomial ring in three variables. This reduces to the mirror $(\mathbb{C}^3, W = xyz)$ which was used by Seidel for proving homological mirror symmetry.

2.2. **Elliptic orbifolds.** The above consideration can be readily carried out for orbifold compactifications of the pair-of-pants. In this section we focus on introducing the results for the elliptic orbifold $E/\mathbb{Z}_3 = \mathbb{P}^1_{3,3,3}$, the compactification of the pair-of-pants by three orbifold points which are locally $\mathbb{C}/\mathbb{Z}_3$.

After the compactification, we have infinitely many holomorphic polygons. The Seidel Lagrangian in $E/\mathbb{Z}_3$ lifted to E gives a union of three circles, see Figure 3. We see that there are infinitely many triangles bounded by $\mathbf{L}$.

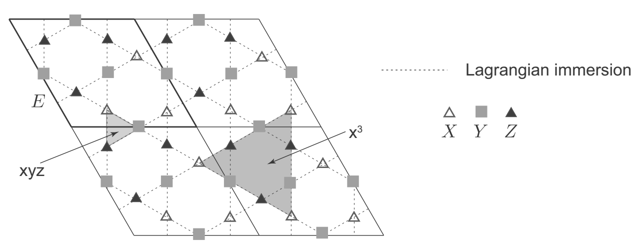

FIGURE 3. Seidel Lagrangian in $E/\mathbb{Z}_3$ lifted to an elliptic curve E.

For the moment let's assume that the Seidel Lagrangian is symmetric about the equator. In this case, we showed that the formal deformations $b = xX + yY + zZ$, where $x, y, z \in \mathbb{C}$, are weakly unobstructed [CHL17], namely $m_0^b = W \cdot \mathbf{1_L}$. Thus we obtain a Landau-Ginzburg model $(\mathbb{C}^3, W)$. Moreover we computed the disc potential

$$W = \phi(q)(x^3 + y^3 + z^3) + \psi(q)xyz \tag{2.2}$$

where $\phi(q)$ and $\psi(q)$ are generating series counting triangles with vertices at x, x, x and at x, y, z respectively, and verified that it exactly coincides with the mirror map. We deduced the following.

Theorem 2.3 ([CHL17])**.** *The generalized SYZ mirror constructed above equals to the one given by the mirror map obtained by solving Picard-Fuchs equation. Moreover the mirror functor* $\mathcal{F}^{\mathbf{L}} : \mathrm{Fuk}(X) \to \mathrm{MF}(\mathcal{A}, W)$ *derives homological mirror symmetry.*

Now consider the generic case that the Seidel Lagrangian is NOT symmetric about the equator. Then as in the case of the pair-of-pants, we cannot take $x, y, z \in \mathbb{C}$ since they are no longer weakly unobstructed. We need to take the quotient $\mathcal{A}$ of the free algebra generated by x, y, z by the weakly unobstructed relations, which become much more complicated than that for the pair-of-pants due to the additional polygons. Indeed we obtained the so called Sklyanin algebra, which can be understood as a non-commutative projective plane and was used by Auroux-Katzarkov-Orlov [AKO08] as a non-commutative mirror of the Landau-Ginzburg model $W = x + y + 1/xy$.

Theorem 2.4 ([CHLb]). *The generalized SYZ mirror constructed above is* $(\mathcal{A}, \tilde{W})$*, where* $\mathcal{A}$ *is a Sklyanin algebra of the form*

$$\mathcal{A} = \frac{\Lambda < x, y, z >}{\left(axy + byx + cz^2, ayz + bzy + cx^2, azx + bxz + cy^2\right)} \tag{2.3}$$

and $\tilde{W}$ *is a central element in* $\mathcal{A}$*. Moreover, by taking a family of Seidel Lagrangians whose central fiber is symmetric about the equator, we obtain a family of non-commutative algebras* $\mathcal{A}/\langle\tilde{W}\rangle$ *which are deformation quantizations of the affine del Pezzo surface* $\{W(x, y, z) = 0\} \subset \mathbb{C}^3$ *where* W *is given in Equation* (2.2)*. Furthermore, the mirror functor* $\mathcal{F} : \mathrm{Fuk}(\mathbb{P}^1_{3,3,3}) \to \mathrm{MF}(\mathcal{A}, W)$ *derives homological mirror symmetry.*

2.3. **Rigid Calabi-Yau manifolds.** A Calabi-Yau manifold is said to be *rigid* if it admits no deformation of complex structure, namely $H^1(T) = 0$. In particular it does not admit any degeneration, and the method of toric degeneration does not apply.

Mirror symmetry for rigid Calabi-Yau manifolds was studied by several physics groups [Sch93, CDP93] and Batyrev-Borisov [BB96]. It appeared to be puzzling because the mirror cannot have any Kähler structure, and in particular not a Calabi-Yau manifold. Otherwise it would have at least one-dimensional Kähler cone, which is mirror to the moduli space of complex structures on the rigid CY, contradicting that it is 'rigid'. Currently I am working with Cheol-Hyun Cho, Hansol Hong and Lino Amorim to construct mirrors of rigid Calabi-Yau manifolds. We shall see that the mirror is a Landau-Ginzburg mirror whose derived category could be thought as a 'noncommutative Calabi-Yau'.

An important example of a rigid Calabi-Yau manifold studied by Leung [Leu04] is the following. Let $E = \mathbb{C}/\mathbb{Z}\langle 1, \tau\rangle$ be the elliptic curve with complex multiplication by the cube root of unity $\tau = e^{2\pi i/3}$. Take $X_0 = E^3/\mathbb{Z}_3$ where $\mathbb{Z}_3$ acts diagonally on the product E^3. The holomorphic volume form $\mathrm{d}z_1 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_3$ of E^3 is preserved by the action and hence descends to the quotient. By taking a crepant resolution of X_0 at the 27 isolated orbifold points, we obtain a rigid Calabi-Yau manifold X.

We fix the collection of immersed Lagrangians as follows. There is a canonical quotient map $X_0 \to (E/\mathbb{Z}_3)^3$. The Seidel Lagrangian $\bar{L}$ in the elliptic orbifold $E/\mathbb{Z}_3$ illustrated in the last section lifts to three Lagrangian circles L_1, L_2, L_3 in the elliptic curve E. (We take the Seidel Lagrangian symmetric about the equator for simplicity.) Then $L_{i_1} \times L_{i_2} \times L_{i_3}$ ($i_1, i_2, i_3 \in \{1, 2, 3\}$) gives 27 Lagrangian tori in E^3 intersecting with each other cleanly. Taking quotient by $\mathbb{Z}_3$, we obtain 9 immersed Lagrangians (with transverse self intersections) $L_{[i_1,i_2,i_3]}$in $E^3/\mathbb{Z}_3$ where $L_{[i_1,i_2,i_3]} = L_{[i_1+k,i_2+k,i_3+k]}$. Since these Lagrangians never intersect the orbifold points, they can be lifted to the rigid Calabi-Yau manifold.

The union $\bigcup_{i_1,i_2,i_3} L_{[i_1,i_2,i_3]}$ is invariant under the residual $(\mathbb{Z}_3^3)/\mathbb{Z}_3$-action on $E^3/\mathbb{Z}_3$. We consider the $(\mathbb{Z}_3^3)/\mathbb{Z}_3$-equivariant formal deformations of $\bigcup_{i_1,i_2,i_3} L_{[i_1,i_2,i_3]}$, which are identified as formal deformations of $\bar{L}^3 \subset (E/\mathbb{Z}_3)^3$. They are given by tensor products of the formal deformations of each factor $\bar{L} \subset E/\mathbb{Z}_3$. Let X, Y, Z be the three odd-degree formal deformations of $\bar{L}$. Let $X_1 = X \otimes \mathbf{1}_{\bar{L}} \otimes \mathbf{1}_{\bar{L}}, X_2 = \mathbf{1}_{\bar{L}} \otimes X \otimes \mathbf{1}_{\bar{L}}, X_3 = \mathbf{1}_{\bar{L}} \otimes \mathbf{1}_{\bar{L}} \otimes X$, and similar for Y_i and Z_i. (Recall that $\mathbf{1}_{\bar{L}}$ denotes the fundamental class of $\bar{L}$.) Take the equivariant formal deformations $b = \sum_{i=1}^3 (x_i X_i + y_i Y_i + z_i Z_i)$ of the collection $\mathbf{L} = \{L_{[i_1,i_2,i_3]} : i_1, i_2, i_3 \in 1, 2, 3\}$, where $x_i, y_i, z_i \in \mathbb{C}$. Using the symmetry about the equator as in the case of pair-of-pants, we shall prove the following.

Lemma 2.5. *The formal deformations $b = \sum_{i=1}^3 (x_i X_i + y_i Y_i + z_i Z_i)$ are weakly unobstructed.*

Then we compute $m_0^b = W(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) \cdot \mathbf{1}_{\bigcup_{i_1,i_2,i_3} L_{[i_1,i_2,i_3]}}$. First it is easy compute it for $\bar{L}^3 \subset (E/\mathbb{Z}_3)^3$ (without any orbifold insertion) by dimension argument. Namely, $W(x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3) = \sum_{i=1}^3 W_{\bar{L}}(x_i, y_i, z_i)$ where $W_{\bar{L}} = \phi(q)\,(x^3 + y^3 + z^3) + \psi(q)\,xyz$ is the disc potential for $\bar{L} \subset E/\mathbb{Z}_3$ known in our previous work [CHL17]. Then by symmetry we obtain the following diagram.

$$\begin{tikzcd}
E^3/\mathbb{Z}_3 \arrow[r, leftrightarrow, "\text{mirror}"] \arrow[d, "/\mathbb{Z}_3^2"] & \left((\mathbb{C}^3)^3/\mathbb{Z}_3^2, W_0 = \sum_{i=1}^3 \left(\phi(q)(x_i^3 + y_i^3 + z_i^3) + \psi(q)x_i y_i z_i\right)\right) \\
(E/\mathbb{Z}_3)^3 \arrow[r, leftrightarrow, "\text{mirror}"] & \left((\mathbb{C}^3)^3, W_0 = \sum_{i=1}^3 \left(\phi(q)(x_i^3 + y_i^3 + z_i^3) + \psi(q)x_i y_i z_i\right)\right) \arrow[u, "/\mathbb{Z}_3^2"]
\end{tikzcd}$$

where $\mathbb{Z}_3^2 \subset \mathbb{Z}_3^3$ appeared on the right hand side is the subgroup consisting of elements (ρ_1, ρ_2, ρ_3) such that $\rho_1\rho_2\rho_3 = 1$, and $\psi(q), \phi(q)$ are explicit series in the Kähler parameter q of the elliptic curve E obtained by counting holomorphic polygons.

In our ongoing work, we shall show that the mirror of the resolution X is a (big) deformation of W_0; it roughly takes the form

$$W = W_0 + \sum c_{\xi,\eta,\zeta}(q)\xi_1 \eta_2 \zeta_3$$

where ξ, η, ζ are any one of the variables x, y, z.[1] The coefficients $c_{\xi,\eta,\zeta}(q)$ are series counting stable discs emanated from the exceptional divisors in the resolution. Due to our experience in modularity of open Gromov-Witten invariants [LZ15, KL], we expect that $c_{\xi,\eta,\zeta}(q)$ are nice (quasi-)modular forms related to the affine Dynkin diagram $\tilde{E}_6$. It would be very interesting to number theory.

While we focus on dimension three to illustrate, this mirror picture is interesting even in dimension two. Namely, we have an analogous mirror diagram for the K3 surface X resolved from $E^2/\mathbb{Z}_3$. From the diagram the mirror is a cubic polynomial W (deformed from W_0) defined on $\mathbb{C}^6/\mathbb{Z}^3$ whose resolution is the total space $\mathscr{O}_{\mathbb{P}^5}(-3)$. It defines a cubic fourfold, and by Orlov's LG/CY correspondence [Orl09] the Calabi-Yau category $D^b\mathrm{MF}(W)$ sits in the derived category of the cubic fourfold. Thus we may interpret the mirror of X as a noncommutative K3 embedded in the cubic fourfold.

3. Gluing in an example

Gluing local mirrors to a global mirror is our ongoing project. Here we illustrate gluing of local mirrors with a key example. Seidel [Sei12] speculated the use of pair-of-pants decomposition to understand Fukaya category. Here we study a construction of mirrors and mirror functors using pair-of-pants decomposition.

More precisely we need to work over the Novikov ring, and distinguish between isomorphisms and 'pseudo-isomorphisms'. At this stage we ignore the areas and technical issues. See [HL] for more precise formulation.

Consider the pair-of-pants decomposition of a 4-punctured sphere X shown in Figure 4. In each pair-of-pants, we have an immersed Lagrangian constructed by Seidel. We would like to have a family of immersed Lagrangians in X in which the Seidel Lagrangian in each pair-of-pants is a member. To do this, we smooth out one of the immersed points in one of the Seidel Lagrangian (labeled by X_1 in Figure 4). The immersed Lagrangian becomes a union of two circles after the smoothing, and we move it to a neighboring pair-of-pants, degenerating it to a Seidel Lagrangian by forming another immersed point (labeled by X_2). As a result a pair-of-pants decomposition induces a family of immersed Lagrangians over a graph.

As illustrated in Section 2.1, each Seidel Lagrangian in a pair-of-pants gives $\mathbb{C}^3$ as the deformation space (with obstruction given by a disc potential W). We need to give a rule to glue them together to form a global geometry.

[1] More precisely W_0 and W should be related by analytic continuation. We have to be careful about the Kähler parameters in the above expression.

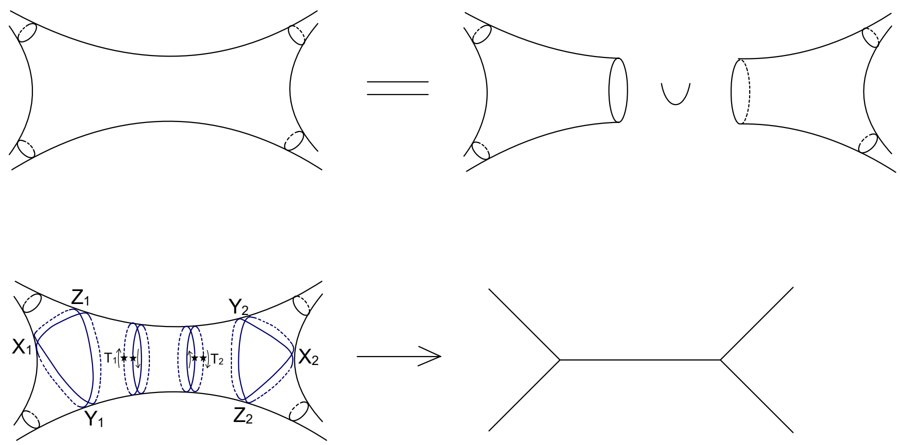

FIGURE 4. A pair-of-pants decomposition induces a family of immersed Lagrangians over a polyhedral complex.

For this, we shall analyze two important transitions, namely smoothing and gauge change.

3.1. **Smoothing.** First consider the effect of smoothing out one immersed point X at odd degree in a Seidel Lagrangian L. The immersed point X corresponds to a vanishing sphere in the immersed Lagrangian $\tilde{L}$, which is $\mathbb{S}^0 = 2$ points. ($\tilde{L}$ is a union of two circles as shown in Figure 4). The deformation space of $\tilde{L}$ is $\operatorname{Spec}\mathbb{C}[e^t, e^{-t}, y, z] = \mathbb{C}^\times \times \mathbb{C}^2$, where t is the coordinate corresponding to the vanishing cycle. $e^t \in \mathbb{C}^\times$ can also be understood as a flat $\mathbb{C}^\times$-connection on $\tilde{L}$ with holonomy concentrated at the vanishing $\mathbb{S}^0$.

Then a triangle bounded by L whose corners are X, Y, Z corresponds to a bi-gon bounded by $\tilde{L}$, with corners Y, Z and passing through the vanishing $\mathbb{S}^0$. From this observation, we shall prove that the Floer theory of L can be identified with that of its smoothing $\tilde{L}$ via the coordinate change $x = e^t$.

For the current example the coordinate change is very simple. However it becomes much more complicated in general since there are holomorphic discs bounded by L which are not bounded by $\tilde{L}$, and vice versa. See Section 3.4 for a bit more complicated example of coordinate change. In this project, we shall analyze the change in the moduli spaces of stable discs under smoothing, and find a coordinate change to identify the Floer theories.

To conclude, in this example *the deformation space* $\mathbb{C}^3$ *of the Seidel Lagrangian is glued with the deformation space* $\mathbb{C}^\times \times \mathbb{C}^2$ *after smoothing via* $x = e^t$. Other variables y and z remain unchanged.

3.2. **Gauge change.** Now let's compare the two Lagrangians (which are unions of two circles) smoothed out from the Seidel Lagrangians on the left and right hand side respectively in Figure 4.

A key observation is that the positions of the vanishing cycles are different. We have to choose a way to move the vanishing cycle marked by T_1 to the one marked by T_2 in order to identify them. This amounts to a gauge change of the corresponding flat $\mathbb{C}^\times$-connections.

When a cycle moves across an immersed point of the Lagrangian, there is a non-trivial change of coordinates in the Floer theory, since the collection of holomorphic polygons which pass through the cycle changes. We assert that in general (no matter in what dimensions) the Floer theories before and after moving a cycle across an immersed point marked by Y can be identified via the coordinate change $y \mapsto ye^t$.

There are different choices of moving a vanishing cycle to another; they *result in different mirror models.* For instance, we can move T_1 to T_2 in a way such that the immersed points marked by Y_1 and Z_1 are passed through exactly once. Using the above change of coordinates, the gluing of the deformation spaces is

$$\mathbb{C}[e^{t_1}, e^{-t_1}, y_1, z_1] \to \mathbb{C}[e^{t_2}, e^{-t_2}, y_2, z_2], t_1 = -t_2, y_1 = z_2 e^{t_2}, z_1 = y_2 e^{t_2}.$$

Combining with the gluing for smoothing, it is easy to check that we obtain the resolved conifold $\check{X} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ after gluing the four deformation spaces together, see Figure 5.

We can also take other choices. For instance, we can move T_1 to T_2 in a way such that the immersed point Y_1 is passed through twice, while Z_1 is never passed through. Then the gluing is

$$\operatorname{Spec} \mathbb{C}[e^{t_1}, e^{-t_1}, y_1, z_1] \to \operatorname{Spec} \mathbb{C}[e^{t_2}, e^{-t_2}, y_2, z_2], t_1 = -t_2, y_1 = z_2 e^{2t_2}, z_1 = y_2. \tag{3.1}$$

It results in $\check{X} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$ rather than $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. We see an explicit dependence the mirror space on the choice of gauge.

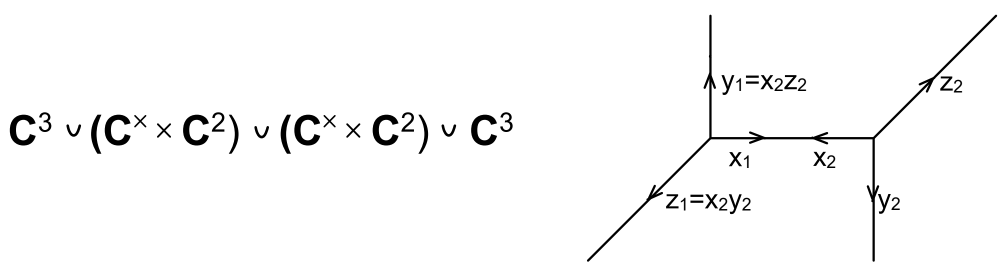

FIGURE 5. The resulting mirror space by gluing the deformation spaces of the immersed Lagrangians is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The right hand side is its toric diagram.

We also have a superpotential W on the glued space from stable discs passing through one generic marked point. In this example it is

$$W = z_1 y_1 x_1 = z_1 y_1 e^{t_1} = y_2 e^{t_2} z_2 = y_2 x_2 z_2$$

which corresponds to a polygon bounded by each Lagrangian in the family. Thus the mirror is a Landau-Ginzburg model $(\check{X}, W)$. While different choices of gauge changes result in different underlying spaces $\check{X}$, we shall prove that the resulting Landau-Ginzburg models are equivalent.

3.3. **Non-commutative deformations.** Each pair-of-pants produces a local piece of the mirror space. As we have explained in Section 2.1, the mirror space is not $\mathbb{C}^3$ in general if the Seidel Lagrangian is not symmetric. Instead it is a quotient of the free algebra $\langle x, y, z\rangle$ by the weakly unobstructed relations

$$q_{\text{front}}\, yx = q_{\text{back}}\, xy, q_{\text{front}}\, zy = q_{\text{back}}\, yz, q_{\text{front}}\, xz = q_{\text{back}}\, zx.$$

These relations come from the even-degree outputs of m_0^b.

We shall show that the local pieces described by quiver algebras with relations can be consistently glued together to produce a global noncommutative geometry. The procedure is similar to the one given previously, namely we shall consider smoothing and gauge change. The key difference is that in the more general setup, the coordinate changes are noncommutative, namely we need to keep track of the order of the variables in every step and make sure that they are compatible with each other. Achieving this would give us a rich family of noncommutative geometries glued from quiver algebras with relations serving as mirror models.

In the current example, we have two copies of the noncommutative $\mathcal{A}$ of $\mathbb{C}^3$ described above, which correspond to the two pieces of pair-of-pants in the decomposition. There is a restriction morphism from $\mathcal{A}$ to the noncommutative deformation $\mathcal{A}'$ of $\mathbb{C}^\times \times \mathbb{C}^2$ constructed in a similar manner. Then the two copies of $\mathcal{A}'$ are glued together according to the rule for gauge change described in Section 3.2. As a result we produce a noncommutative deformation of the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. It would be interesting to compare this deformation with the one in our previous work [CHLb].

3.4. **Compactification.** In general the coordinate changes receive much more complicated quantum corrections coming from holomorphic discs in non-exact situations. In the example above the Floer theory of the Lagrangians involved (which are exact) is simple, but it becomes more complicated in the compact case.

For instance, consider the compactification $\bar{X} = \mathbb{S}^2$ by filling in the four punctures, which still contains the family of immersed Lagrangians depicted in Figure 4. Before compactification, we only have triangles and bigons bounded by the family of Lagrangians, and from their Floer theory we obtain the previous coordinate change.

New holomorphic discs are formed in the compactification. For instance, the superpotential counting holomorphic discs bounded by a Seidel Lagrangian L equals to

$$W_L = xyz - y + x + z$$

where the terms x, y, z correspond to new one-gons formed from compactification. We shall prove that the new coordinate change is the following:

$$x = e^t, y = \tilde{y} - e^{-t}, z = \tilde{z} + e^{-t}.$$

The new term e^{-t} appeared in the above change of y comes from counting discs emanated from the punctures with an input $\bar{X}$ and the output Y; it is similar for that of z. Applying the above coordinate change to W_L gives

$$W_{\tilde{L}} = e^t \tilde{y}\tilde{z} + e^t + e^{-t}$$

which exactly agrees with the disc potential of the smoothing $\tilde{L}$.

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