EQUIVALENT SYZ MIRROR SYMMETRY

SIU-CHEONG LAU

ABSTRACT. In this talk, I explain a Morse model for the equivariant Lagrangian Floer theory, and apply it to SYZ fibers to construct an equivariant SYZ mirror. This method computes equivariant disc potentials for immersed SYZ fibers of toric Calabi-Yau manifolds, which are closely related to the potentials of Aganagic-Vafa branes.

1. INTRODUCTION

Since the discovery of mirror symmetry in the 90's, Gromov-Witten theory has made tremendous developments. Via mirror symmetry, it gives an enumerative meaning for important hypergeometric functions. Mirror symmetry and Gromov-Witten theory have stimulated the rapid growth of many fundamental subjects, including moduli theory, PDE in gauge theory, and symplectic geometry.

Strominger-Yau-Zaslow [SYZ96] proposed that mirror symmetry can be understood via T-duality. Namely, it asserts that a mirror pair of Calabi-Yau manifolds admits dual special Lagrangian torus fibrations. The reconstruction problem in mirror symmetry was studied by Kontsevich-Soibelman [KS01, KS06] and Gross-Siebert [GS06, GS10, GS11]. They have motivated a lot of important developments in symplectic and algebraic geometry, including [Aur07, Aur09, CLL12, FOOO10, AAK16, CHL17, GK15, Abo17, PT].

Current literature in the SYZ program has mostly focus on non-equivariant mirrors. On the other hand, equivariant torus action plays a key role in the formulation of Givental [Giv98] and Hori-Vafa [HV], and the mirror theorem for Gromov-Witten invariants in [LLY97, LLY99a, LLY99b, LLY00]. In the early days, localization by $\mathbb{S}^1$-actions was the most important tool in the computation of Gromov-Witten invariants and the discovery of mirror symmetry.

The mirrors of equivariant Gromov-Witten invariants are oscillatory integrals of an ***equivariant superpotential*** $W(\vec{\lambda},\vec{x})$, where $\vec{\lambda}$ is the equivariant parameter for a torus action[1]. We may regard W as a family of superpotentials over the equivariant parameter space. In view of this, Teleman [Tel14] proposed that for a Kähler manifold X with a Hamiltonian G-action, its G-equivariant mirror is a holomorphic fibration. Moreover for a symplectic quotient $X//G$, its mirror is given by a fiber of this holomorphic fibration. It gives a mathematical formulation of the gauged linear sigma model of Hori-Vafa [HV], and also suggests a relation with Langlands duality.

[1] It is denoted by $\hbar$ or z in the literature.

The aim of this talk is to understand the equivariant mirrors from the SYZ perspective. By [CO06, Aur07, CL10, FOOO10, CLLT17, CHL17, CHKL17, CHLb], the *non-equivariant* Landau-Ginzburg superpotential can be understood as a generating function of one-pointed open Gromov-Witten invariants. We would like to have a similar understanding for the equivariant superpotential.

The setting is the following. Given a symplectic manifold (X, ω), suppose there is a compact Lie group G preserving ω. Let's say we have constructed a collection of Lagrangians in the same deformation class, which usually come from fibers of a Lagrangian torus fibration, and assume that they are G-invariant.

We shall construct a family of Landau-Ginzburg models over equivariant parameters, together with a functor from the equivariant Fukaya category to the category of matrix factorizations over the family. The Landau-Ginzburg superpotential will be glued from the non-trivial curved terms m_0^G of the G-equivariant Lagrangians that we begin with.

Equivariant Lagrangian Floer theory is the main tool for the construction. Different versions of such a theory were developed by [SS10, HLS16, Woo11, HLS, DFb, DFc] in various unobstructed situations. In order to work in the weakly unobstructed setting, and to treat the curved term m_0^G in a more explicit way, we have developed a Morse model formulation [KLZ]. We will introduce this model in the next section. Then we apply this model for the equivariant mirror construction and computing equivariant open Gromov-Witten invariants in the later two sections. In this talk we will focus on the case $G = T$ is a torus.

G-equivariance has also been studied in the B-side. For instance, the work of [FLTZ11] studied T-equivariant coherent sheaves on toric varieties and their mirror objects. Moreover, Lian-Yau [LY13] has used G-equivariance to study Picard-Fuchs equations for period integrals.

On the other hand, the curved term m_0^G is a distinguished object on the A-side that has a lot of important applications. In Section 2, we introduce a Morse model for the equivariant Lagrangian Floer theory. In Section 3, we explain an equivariant version of the gluing between the (quantum-corrected) deformation spaces of G-invariant Lagrangians. In Section 4, we apply to toric Calabi-Yau manifolds to formulate and compute its equivariant open Gromov-Witten invariants.

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2. Equivariant disc potential for a smooth SYZ fiber

2.1. **Floer theory for a smooth SYZ fiber.** An essential ingredient of the SYZ program is quantum correction, which is contributed from holomorphic discs emanated from singular fibers of the Lagrangian fibration. Fukaya-Oh-Ohta-Ono [FOOO09b] defines the deformations and obstructions of Lagrangian submanifolds by a detailed study of the Floer theory of pseudoholomorphic disc moduli. The family Floer theory of Lagrangian torus fibers [Fuk02, Tu14, Abo17] was extensively studied to understand homological mirror symmetry[Kon95] in the SYZ setting.

We briefly review as follows. The A_∞ algebra $(C^*(L), m_k)$ encodes the deformation and obstruction theory of a Lagrangian L. $C^*(L)$ denotes the space of chains in L, which can be taken to be singular chains, differential forms, or linear combinations of critical points of a Morse function on L. The virtual fundamental classes of moduli spaces of stable discs constructed by [FOOO09b] is the main technical tool. $m_k : (C^*(L))^{\otimes k} \to C^*(L)$ is defined from the moduli of stable discs with several input marked points and one output marked point. The precise definition depends on which model of $C^*(L)$ one takes.

The space of (degree-one) weak bounding cochains b, modulo gauge equivalence, gives the local deformation space of L. Given $b \in C^1(L)$, one has a deformed A_∞ algebra $(C^*(L), m_k^b)$. b is a weak bounding cochain if

$$m_0^b = W(b) \cdot \mathbf{1}_L \tag{2.1}$$

for some $W(b)$, where $\mathbf{1}_L$ denotes the unit (which is essentially the fundamental class of L). This gives a Landau-Ginzburg model, namely the disc potential $W(b)$ defined over the space of weak bounding cochains. Such a formulation was used in [FOOO10, FOOO11, FOOO16] to derive mirror symmetry for compact toric manifolds.

In [CHL17], we construct local mirrors from Lagrangian immersions formally deformed by weak bounding cochains, and an A_∞ functor

$$\mathrm{Fuk}(X) \to \mathrm{MF}(W)$$

where $\mathrm{MF}(W)$ denotes the dg category of matrix factorizations [Eis80, Orl09].

More specifically we studied the immersed Lagrangian curve found by Seidel [Sei11] (see Figure 2a), classified its weak bounding cochains, and used them to construct homological mirror symmetry for the orbifold spheres $\mathbb{P}^1_{a,b,c}$. The orbifold spheres have a very interesting Gromov-Witten theory (see for instance [ST11, ET13, MT08, MS16, SZ18]).

2.2. **Equivariant Floer theory of a smooth SYZ fiber.** We develop a similar formulation to construct (local) equivariant mirrors out of a Lagrangian L preserved by a symplectic G-action on X. In a joint work with Yoosik Kim and Xiao Zheng [KLZ], we formulated a G-equivariant Lagrangian Floer theory. In the case of a torus action, we computed the equivariant disc potential. The result can be stated as follows.

Theorem 2.1. *Suppose T^k acts on (X, ω, J). Let D be an anti-canonical divisor preserved by T, and $L \subset X - D$ a graded Lagrangian preserved by T. Suppose the T-action restricted to L is free. Then for degree-one boundary deformations*

$$b = x_1 X_1 + \ldots + x_k X_k$$

where $X_1, \ldots, X_k$ are degree-one cycles dual to the orbit loops of the T^k-action, the equivariant obstruction term for $L \subset X$ equals to

$$m_0^{T, b\otimes \mathbf{1}_{BT}} = m_0^b \otimes \mathbf{1}_{BT} + (x_1 + h_1(x))\mathbf{1}_L \otimes \lambda_1 + \ldots + (x_k + h_k(x))\mathbf{1}_L \otimes \lambda_k \tag{2.2}$$

where λ_i are the equivariant parameters, and $h_i(x)$ are contributed by non-constant Maslov-zero holomorphic discs.

In particular if L is weakly unobstructed (meaning $m_0^b = W(x) \cdot \mathbf{1}_L$ for some W), then $m_0^{T, b\otimes \mathbf{1}_{BT}} = W^T \cdot \mathbf{1}_{L^T}$, where the equivariant disc potential equals to

$$W^T = W(x) + (x_1 + h_1(x))\lambda_1 + \ldots + (x_k + h_k(x))\lambda_k. \tag{2.3}$$

The above theorem gives an efficient way to construct a (local) mirror for a symplectic quotient. Namely, the fiber

$$Y_0 := \{x_1 + h_1(x) = \ldots = x_k + h_k(x) = 0\} \tag{2.4}$$

together with the superpotential $W|_{Y_0}$ can be taken to be a (local) Landau-Ginzburg mirror of $X /\!\!/ G$ probed by the Lagrangian L/G. Daemi-Fukaya [DFa] asserted that in the unobstructed case, by using Lagrangian correspondence, there exists a homotopy equivalence between the G-equivariant Fukaya category of X and that of (a bulk deformation of) the symplectic quotient $X /\!\!/ G$. Woodward-Xu [WX] have formulated a gauged Floer theory for (L, G) and constructed an equivalence with the Floer theory of $L \subset X /\!\!/ G$.

For instance, consider the simplest example $\mathbb{P}^1$, the symplectic quotient of $X = \mathbb{C}^2$ by $G = U(1)$. For a suitable moment-map torus $L \subset \mathbb{C}^2$, its disc potential can be written as $\mathbf{T}^A(z_1 + z_2 z_1^{-1})$ for certain choice of parametrization (z_1, z_2) for the flat $\mathbb{C}^\times$-connections on L. By [KLZ], the equivariant disc potential equals to

$$W^{\text{equiv}} = \mathbf{T}^A \left(z_1 + z_2 z_1^{-1}\right) + \lambda \cdot x_2$$

where $z_2 = e^{x_2}$. In other words, the obstruction term equals to $h_\lambda = x_2$. Setting $x_2 = 0$, we obtain

$$\mathbf{T}^A \left(z_1 + z_1^{-1}\right)$$

which is exactly the disc potential of $L/G \subset \mathbb{P}^1$, the quotient. This easily generalizes to other toric manifolds.

Now let's consider general compact toric semi-Fano manifolds. The (non-equivariant) disc potential W in this case has been computed in terms of the mirror map [CLLT17, GI]. Combining, we obtain the following corollary.

Corollary 2.2. *Let X^n be a compact semi-Fano toric manifold. The T^n-equivariant disc potential equals to*

$$W^T = \sum_{l=1}^{m} (\exp(g_l(\check{q}(q)))) \cdot \mathbf{T}^{\int_{\beta_l} \omega} e^{(x, v_l)} + \sum_{j=1}^{n} x_j \lambda_j$$

where $v_1, \ldots, v_m$ are the primitive generators of the rays of the fan, q is the Kähler parameter (which can be expressed in terms of the Novikov parameter $\mathbf{T}$), $\check{q} = \check{q}(q)$ is the inverse of the mirror map $q(\check{q})$, and

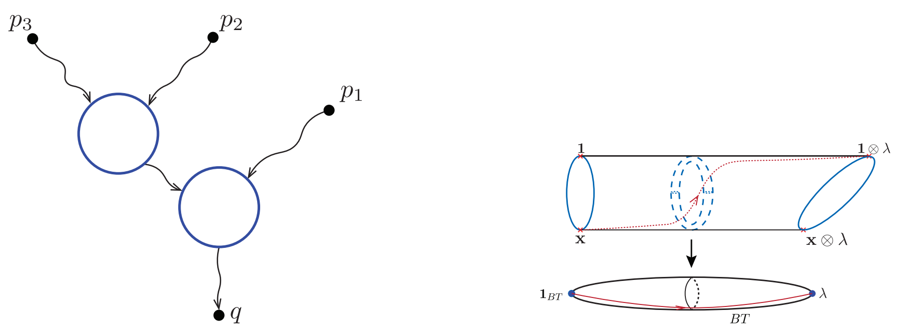
$$g_l(\check{q}) := \sum_d \frac{(-1)^{(D_l \cdot d)}(-(D_l \cdot d) - 1)!}{\prod_{p \neq l}(D_p \cdot d)!} \check{q}^d, \tag{2.5}$$

in which the summation is over all effective curve classes $d \in H_2^{\text{eff}}(X)$ satisfying

$$-K_X \cdot d = 0, D_l \cdot d < 0 \text{ and } D_p \cdot d \geq 0 \text{ for all } p \neq l.$$

Below we briefly describe our Morse model for the equivariant Floer theory. We consider a family Morse theory [Hut08] on the fiber bundle $L \hookrightarrow L_G \to B_G$, where $B_G = EG/G$ is the classifying space of G, and $L_G := (L \times EG)/G$ is the standard Borel construction. Let's take a Morse function f on L and a perfect Morse function f_{BG} on BG, and cook up a function on L_G using a partition of unity (with respect to a bundle trivialization), such that the function restricted to a fiber over a critical point of f_{BG} equals to f up to adding a constant. Then the Morse generators are of the form $X \otimes \lambda$, where X is a critical point of f and λ is a critical point of f_{BG}.[2] $C^*_G(L)$ is taken to be the formal span of these generators $X \otimes \lambda$.

The Morse model for L_G counts pearl trajectories to define the A_∞ operations $m^G_k : C^*_G(L)^{\otimes k} \to C^*_G(L)$. Pearl trajectories were formulated by Oh [Oh05, OZ11] and Biran-Cornea [BC12]. In this situation, a pearl trajectory is a (connected) union of negative gradient flow line segments of the family Morse model on L_G, together with stable discs contained in fibers of $L_G \to B_G$. See Figure 1a.



(A) (B)

FIGURE 1. The left shows a pearl trajectory for the definition of a Morse model of Lagrangian Floer theory. The right shows a Morse flow on the equivariant space $L^{\mathbb{S}^1}$ the free $\mathbb{S}^1$-action on $L \cong \mathbb{S}^1$.

Below are some technical points. First, the moduli spaces of pearl trajectories have highly non-trivial Kuranishi structures. In the work of [FOOO09a], a chain homotopy between the classical Morse and singular complexes is used, so that the definition of m_k reduces to the singular chain model [FOOO09b]. This formulation requires to include degenerate singular simplices, and we have defined such a version of the chain homotopy in Section 2.2 of [KLZ].

Another important ingredient is the unit. The unit $\mathbf{1}_{LG} = \mathbf{1}_L \otimes \mathbf{1}_{BG}$ is important for defining weak bounding cochains (2.1). Strictly speaking, the fundamental class, or correspondingly the maximum point $\mathbf{1}^{\blacktriangledown}_{LG}$ of the Morse function, are not exactly the unit, since the Kuranishi structures chosen inductively to satisfy transversality may not be compatible with the forgetful map for marked points. [FOOO09b] has defined the notion of a

[2]More precisely we take a finite dimensional approximation $L_G^{(N)} \to B_G^{(N)}$, consider a family Morse theory for each N respecting the inclusions $L_G^{(N)} \subset L_G^{(N+1)}$ and $B_G^{(N)} \subset B_G^{(N+1)}$, and let $N \to \infty$.

homotopy unit in the singular chain model. We use the chain homotopy mentioned in the last paragraph to construct a homotopy unit $\mathbf{1}^{\blacktriangledown}_{L^G}$ in the Morse model. The deg $= -1$ element $\mathbf{1}^{\blacktriangledown}_{L^G}$ and the (degree zero) unit $\mathbf{1}_{L^G}$ are generators added to the space of Morse chains, and they satisfy

$$m_1^G(\mathbf{1}^{\blacktriangledown}_{L^G}) = \mathbf{1}_{L^G} - \mathbf{1}^{\blacktriangledown}_{L^G} + h$$

where h is contributed from pearl trajectories consisting of Maslov-zero stable discs appeared in the homotopy. Such a homotopy unit was constructed by Charest-Woodward [CW] in another setting.

For the purpose of equivariant Floer theory, we also construct a 'partial unit' $\lambda_{L^G} = \mathbf{1}_L \otimes \lambda_{BG}$ for every $\lambda \in H^*(BG)$. They have the property that

$$\begin{aligned} m_1^G(\lambda_{L^G}) &= 0; \\ m_2^G(\lambda_{L^G}, X) &= \lambda_{L^G} \cdot X \text{ and } m_2^G(X, \lambda_{L^G}) = X \cdot \lambda_{L^G} \;\; \forall X \in C^*_G(L); \\ m_k^G(\ldots, \lambda_{L^G}, \ldots) &= 0 \;\; \forall k \geq 3. \end{aligned}$$

Using this, we can derive the module structure of the equivariant Floer theory over equivariant parameters in $H^*(BG)$.

Theorem 2.3 (Theorem 2.14 of [KLZ])**.**

$$m_k^G(X_1, \ldots, \lambda \cdot X_i, \ldots, X_k) = \lambda \cdot m_k^G(X_1, \ldots, X_k).$$

So far we have focused on a local equivariant mirror from one single Lagrangian. In the next section, we explain the gluing construction in the equivariant setting.

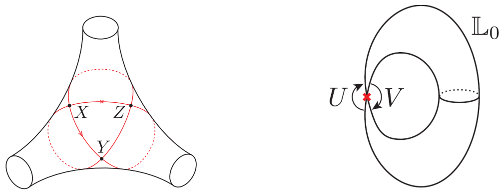
3. Equivariantly gluing for singular SYZ fibers

3.1. **Gluing in the non-equivariant setting.** First, let us recall our gluing construction [CHLa, HL18, HKL] in the non-equivariant setting. In the SYZ reconstruction program, past literature has mostly focused on smooth Lagrangian torus fibers. We need to (partially) compactify by gluing in deformation spaces of singular fibers.

The best possible singular objects are ***Lagrangian immersions***. Seidel was the first one who used Lagrangian immersion to derive homological mirror symmetry [Sei11]. Sheridan extended his method to prove homological mirror symmetry for all Fermat-type Calabi-Yau hypersurfaces [She15]. Their groundbreaking discoveries revealed the importance of Lagrangian immersion. Floer theory has been extensively studied by Fukaya-Oh-Ohta-Ono [FOOO09b]. Akaho-Joyce [AJ10] extended the theory for Lagrangian immersions. Figure 2 shows two important examples of Lagrangian immersions.

The key idea is to find a suitable collection of Lagrangian immersions, and glue up their deformation spaces to construct a global mirror. In moduli theory, compactification of the (regular) moduli by singular objects is the essential step. The choice of singular objects to add in is determined by the stability condition.

The most typical singular SYZ fiber is the immersed two-sphere with a single nodal point. In [HKL], we solved the Maurer-Cartan equation (2.1) for its formal deformations. There are two degree-one immersed generators



(A) (B)

FIGURE 2. The left shows the immersed curve invented by Seidel which triggers a lot of recent developments. The right shows a typical singular SYZ fiber, which is an immersed two-sphere with a single nodal point.

U, V for the immersed two-sphere $\mathcal{S}^2$. We proved that $uU + vV$ for all $u, v \in \Lambda_0$ with $\mathrm{val}(uv) > 0$ are unobstructed bounding cochains. Thus we can think of

$$\{(u, v) \in \Lambda_0^2 : \mathrm{val}(uv) > 0\}$$

as the local mirror for the immersed two-sphere.

To make the computation explicit, we have used a symplectic $\mathbb{S}^1$-reduction. Consider $X = \mathbb{C}^2 - \{ab = 1\}$, with the $\mathbb{S}^1$-action

$$(a, b) \mapsto (e^{i\theta} a, e^{-i\theta} b). \tag{3.1}$$

It has a moment map $|a|^2 - |b|^2$, and the quotient map to the reduced space can be identified as $ab : X \to \mathbb{C} - \{1\}$. Then $\mathcal{S}^2 \subset X$ is obtained by taking the intersection of the preimage of a circle passing through 0 in the reduced space and the level $|a|^2 - |b|^2 = 0$. See Figure 3.

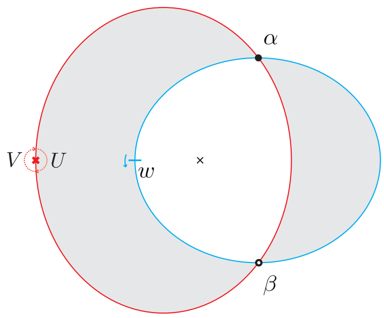


FIGURE 3. The images of the immersed sphere $\mathcal{S}^2$ and a Chekanov torus T^{Ch}.

Next, we glue this deformation space with those of smooth tori (which are smoothings of $\mathcal{S}^2$) via isomorphisms in the Fukaya category. An isomorphism between two objects L_1, L_2 in the Fukaya category is a pair $\alpha \in CF^0(L_1, L_2)$, $\beta \in CF^0(L_2, L_1)$ satisfying

(3.2)

$$m_1(\alpha) = 0,\ m_1(\beta) = 0,\ m_2(\beta, \alpha) = \mathbf{1}_{L_1} + m_1(\gamma_1),\ m_2(\alpha, \beta) = \mathbf{1}_{L_2} + m_1(\gamma_2)$$

for some γ_1, γ_2.

We solved the above equations for an ***isomorphism between $\mathcal{S}^2$ and a Chekanov torus T^{Ch}*** (and also a Clifford torus). The tori are also constructed by taking the intersection of preimages of circles in the reduced space with the level $\{|a|^2 - |b|^2 = 0\}$, see Figure 3. We proved that Equation (3.2) is satisfied if and only if the immersed deformations (u, v) of $\mathcal{S}^2$ and the formal deformations of T^{Ch} by flat connections $\nabla^{(z,w)}$ satisfy the ***gluing formula***

$$u = w,\ uv = 1 + z. \tag{3.3}$$

Such a formula patches the formal deformation spaces of the two Lagrangians $\mathcal{S}^2$ and T^{Ch} together which is crucial to construct a global mirror. The gluing equation $uv = 1 + z$ was also formulated by a different method in the monotone setting via Legendrian topology in [DRET].

It easily follows from A_∞ relations that the isomorphism has the nice property that it preserves disc potentials:

Proposition 3.1. *Let b_i be weak bounding cochains of L_i for $i = 1, 2$, and $W_i(b_i)$ the corresponding disc potentials. Suppose (L_i, b_i) are isomorphic. Then $W_1(b_1) = W_2(b_2)$.*

By the above proposition, if W_1 and the gluing relation between b_1 and b_2 are known, we can deduce W_2.

As an application, we constructed the Floer theoretical mirror of the Grassmannian $\mathrm{Gr}(2, n)$. The disc potential of a smooth torus fiber of the Gelfand-Cetlin system in this case was computed by Nishinou-Nohara-Ueda [NNU10]. Gluing in the immersed spheres (times tori) is important since they correspond to missing critical points not covered by torus charts $(\mathbb{C}^\times)^N$ in general. This provides an enumerative meaning for the Lie-theoretical mirror superpotential of Rietsch [Rie08].

An advantage of such a geometric gluing construction is that, the associated local mirror functors [CHL17] are glued by the isomorphisms in a canonical way [CHLa]. This is important in deriving homological mirror symmetry.

3.2. **Gluing in the Equivariant setting.** All the constructions in the last subsection can be lifted to the equivariant setting. In Section 2, we have explained a local equivariant mirror for a Lagrangian torus under a free T^k-action, which is

$$(\Lambda_0^\times)^{n-k} \times \Lambda_+^k \times \mathrm{Spec}(\Lambda[\lambda_1, \ldots, \lambda_k])$$

together with a superpotential W^T. The Λ_+^k factor comes from boundary deformations appeared in Theorem 2.1. The $(\Lambda_0^\times)^{n-k}$ factor comes from deformations by flat connections in the remaining factor. Below we explain how we can construct equivariant mirrors via Lagrangian Floer theory.

(1) First we take a suitable collection $\{L_i : i \in I\}$ of ***G-invariant Lagrangian immersions*** in X. Typically, L_i are obtained as the inverse images of (possibly degenerate) Lagrangian tori in $X//G$.
(2) For each L_i, we construct a formal family of curved A_∞ algebras

$$\left(C^*(L_i) \otimes H^*(BG), \left\{m_k^{G,(L_i, b_i \otimes \mathbf{1}_{BG})}\right\}_{k=0}^{\infty}\right)$$

for (Novikov convergent) weak Maurer-Cartan formal deformations

$$b_i \in MC^{\mathrm{weak}}(L_i) = \mathrm{Spec}(\mathcal{A}_i).$$

In general $\mathcal{A}_i$ can be a (noncommutative) quiver algebra with relations. The obstruction term

$$m_0^{G,(L_i,b_i\otimes \mathbf{1}_{BG})} = \left(W^{(i)}(b_i) + \sum_\lambda h_\lambda^{(i)}(b_i)\cdot\lambda\right)\mathbf{1}_{L_i}$$

gives the ***equivariant disc potential*** of L_i, where λ runs over the equivariant parameters in $H^2(BG)$. Then the ***local equivariant mirror*** of each L_i is given by

$$\left(\mathrm{Spec}(\mathcal{A}_i\otimes\Lambda[H^2(BG)]),\, W^{(i)}(b_i) + \sum_\lambda h_\lambda^{(i)}(b_i)\cdot\lambda\right) \tag{3.4}$$

which is a family of Landau-Ginzburg models over $H^2(BG)$ (where the fibration map is given by $h_\lambda^{(i)}$).

(3) We fix a tree of isomorphisms (3.2) among the objects (L_i, b_i), which are valid under gluing equations $b_i = \Phi(b_j)$. These gluing equations patch the above local charts (3.4) together to form a ***global equivariant LG mirror*** $\check{X}^{\mathrm{equiv}}$ over $H^2(BG)$.

(4) The subvariety defined by the system of equations

$$h_\lambda^{(i)}(b_i) = 0 \;\forall\lambda$$

is the mirror of the quotient $X//G$.

In (3), we have used the fact that each isomorphism has an equivariant lifting as follows.

Proposition 3.2. *Let G act on (X,ω,J), D an anti-canonical divisor preserved by T, and $L_1, L_2 \subset X - D$ graded Lagrangians preserved by G. Let b_1 and b_2 be degree-one weak bounding cochains of L_1 and L_2 respectively. If $\alpha \in CF_G^0((L_1,b_1),(L_2,b_2))$ is an isomorphism, then $\alpha\otimes\mathbf{1}_{BG} \in CF_G^0((L_1, b_1\otimes\mathbf{1}_{BG}),(L_2, b_2\otimes\mathbf{1}_{BG}))$ is also an isomorphism.*

Example 3.3. *Consider $\mathbb{C}^2 - \{ab = 1\}$, in which we have the immersed sphere $\mathcal{S}^2$ and the Chekanov torus T^{Ch} as explained in the last subsection. The corresponding deformation spaces $\{(u,v)\in\Lambda_0^2 : \mathrm{val}(uv) > 0\}$ and $\{(z,w)\in\Lambda : \mathrm{val}(z) = \mathrm{val}(w) = 0\}$ are glued together by Equation (3.3). This gives the mirror space*

$$\{(u,v)\in\Lambda_0^2 : \mathrm{val}(uv-1) = 0\}.$$

If we take the complex part (namely the intersection of the above space with $\mathbb{C}^2\subset\Lambda_0^2$), then we get

$$\mathbb{C}^2 - \{uv = 1\}$$

which equals the space we start with. ($\mathbb{C}^2$ is self-mirror in this sense.)

Now consider the $\mathbb{S}^1$-action (3.1) which preserves both $\mathcal{S}^2$ and T^{Ch}. By Proposition 3.2, the isomorphism relation (3.3) is still valid. More precisely, we fix a non-trivial spin structure of T^{Ch} in the direction of the $\mathbb{S}^1$-action, and take a boundary deformation $b = xX$ by a dual cycle of the $\mathbb{S}^1$-action. Then the variable z is replaced by $-e^x$, and hence we have the gluing equation $e^x = 1 - uv$. Thus the equivariant mirror is simply

$$\{(u,v)\in\Lambda_0^2 : \mathrm{val}(uv) > 0\}\times\mathrm{Spec}(\Lambda[\lambda])$$

with the equivariant disc potential [KLZ, Section 4]

$$W^{\mathbb{S}^1} = x\lambda = -\lambda \sum_{j=1}^{\infty} \frac{(uv)^j}{j}.$$

4. Equivariant open Gromov-Witten invariants of Lagrangian immersions

4.1. **Open Gromov-Witten invariants.** Open GW invariants play an important role in mirror symmetry. They form the source of quantum corrections. Moreover their generating functions give a canonical way to construct the mirror map [CKYZ99, AZ06, GS11, CLT13, CCLT16, CLLT17, GPS, CHKL17]. Furthermore the generating functions produce important modular forms in many interesting situations [LZ15, KL, FRZZ19].

Open GW invariants can be divided into two types. The first type concerns about stable discs without any output boundary marked point. They contribute to the so-called spacetime superpotential, whose differential gives the obstruction term of the Lagrangian Floer theory. The definition of this type of open Gromov-Witten invariants typically relies on additional symmetries, such as anti-symplectic involutions [PSW08] or $\mathbb{S}^1$-actions [KL01]. They are particularly adapted to the case of threefolds. They behave more similarly to closed Gromov-Witten invariants, and may be computed by localization [GZ02, FLT, FLZ] and open WDVV equations [HS].

The second type comes from stable discs with one output boundary marked point. They are systematically recorded by a generating function called the disc potential, which is the curved term m_0 of Lagrangian Floer theory. L is called to be weakly unobstructed [FOOO09b] if the curved term m_0 equals to $W_L \mathbf{1}_L$ for some W_L. We have invented techniques to compute W_L in various situations, including local Calabi-Yau manifolds [CLT13, CCLT16, KL], semi-Fano toric manifolds [CL14, CLLT17], elliptic and hyperbolic orbifolds [CHKL17], and Grassmannians $Gr(2, n)$ [HKL].

In this section, we focus on (non-compact) toric Calabi-Yau manifolds and explain how to apply the theory in the last two sections to compute one-pointed equivariant open Gromov-Witten invariants, which should be closely related to unmarked (equivariant) open GW invariants via the divisor axiom.

4.2. **Open Gromov-Witten invariants of toric Calabi-Yau manifolds.** Toric Calabi-Yau manifolds are local building blocks of a compact Calabi-Yau manifold. Recall that a toric manifold is encoded combinatorially by a fan. It is a toric Calabi-Yau manifold if all the primitive generators of the fan are contained in a hyperplane defined by $(\nu, \cdot) = 1$ for some dual lattice point ν. In this case there exists a global holomorphic nowhere-zero top form.

Open Gromov-Witten invariants for the ***Lagrangian branes defined by Aganagic-Vafa*** in toric Calabi-Yau threefolds were predicted by physicists [AVb, AVa, AKV02, BKMnP09]. The works [KL01, LLLZ09, GZ02] used $\mathbb{S}^1$-equivariant localization to formulate and compute these invariants. There have been a lot of recent developments [FL13, FLT, FLZ] in formulating and proving the physicists' predictions using localization technique. There are also vast conjectural generalizations of these invariants in relation with knot theory, see for instance [AENV14, TZ].

Let's illustrate by the typical example of a toric Calabi-Yau manifold $K_{\mathbb{P}^{n-1}}$, the total space of the canonical line bundle of the projective space. There is a T^{n-1}-action on $K_{\mathbb{P}^{n-1}}$ preserving the holomorphic volume form, whose symplectic quotient is identified with the complex plane. The Aganagic-Vafa Lagrangian brane L^{AV} for $n = 3$ can be realized as a ray, See Figure 4.

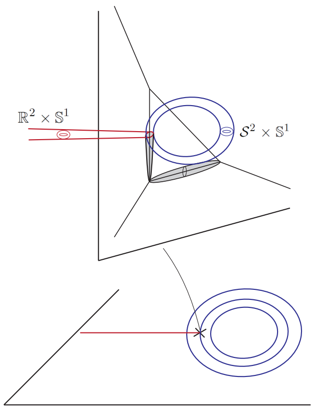


FIGURE 4. The Aganagic-Vafa brane and the Lagrangian fibration on $K_{\mathbb{P}^2}$.

In dimension $n = 3$, Aganagic-Klemm-Vafa predicted that the genus-zero open Gromov-Witten potential of L^{AV} equals to the integral

$$\int \log(-z_1(z_2, q))dz_2 \tag{4.1}$$

where $z_1(z_2, q)$ is obtained by solving the mirror curve equation

$$z_1 + z_2 + \frac{q}{z_1 z_2} + \exp(\phi(q)/3) = 0.$$

In the expression, $\phi(q)$ is the inverse mirror map on the Kähler parameter q, which has an explicit expression by solving the Picard-Fuchs differential equation. The statement was proved by [FL13] via localization.

On the other hand, in [CLL12, CLT13, CCLT16], we used the ***T-duality approach*** of [SYZ96] and ***wall-crossing*** [KS01, KS06, GS06, GS10, GS11, Aur07] to understand mirror symmetry for toric Calabi-Yau manifolds in general dimensions. We constructed the mirror dual of the Lagrangian fibration [Gro01, Gol01]. Moreover, we proved that the generating function of open Gromov-Witten invariants for a Lagrangian toric fiber equals to the inverse mirror map. It states as follows for $K_{\mathbb{P}^{n-1}}$.

Theorem 4.1 ([CLL12, LLW11, CCLT16, Lau15]). *The SYZ mirror of* $K_{\mathbb{P}^{n-1}}$ *equals to*

$$uv = z_1 + \ldots + z_{n-1} + \frac{q}{z_1 \ldots z_{n-1}} + (1 + \delta(q)) \tag{4.2}$$

where $1 + \delta(q)$ *is the generating function of one-pointed Gromov-Witten invariants of a moment-map fiber. Moreover,* $(1+\delta(q))$ *equals to the inverse mirror map* $\exp(\phi(q)/n)$*, which also equals to the Gross-Siebert normalized slab function.*

For instance, let's take the dimension to be $n = 4$. The mirror map for $K_{\mathbb{P}^3}$ is given by $q = Qe^{f(Q)}$, where q is the Kähler parameter for the primitive curve class in $K_{\mathbb{P}^3}$, Q is the corresponding mirror complex parameter, and $f(Q)$ is the hypergeometric series

$$f(Q) = \sum_{k=1}^{\infty} \frac{(4k)!}{k(k!)^4} Q^k.$$

For more about Hodge structures and period integrals for toric Calabi-Yau manifolds, see for instance [Hos06, KM10].

Taking the inverse of the mirror map, we get

$$Q(q) = q - 24q^2 - 396q^3 - 39104q^4 - 4356750q^5 - O(q^6).$$

Then the generating function of open Gromov-Witten invariants of a Lagrangian toric fiber is given by

$$1 + \delta(q) = \exp(f(Q(q))/4) = 1 + 6q + 189q^2 + 14366q^3 + 1518750q^4 + O(q^5).$$

This counts stable discs with one marked point bounded by a moment-map torus fiber.

The relation between the Aganagic-Vafa branes and the SYZ approach was not clear. Our equivariant version of the SYZ construction can explain the relation.

In Section 2, we have formulated and computed the equivariant open Gromov-Witten potential for a smooth SYZ torus fiber. Applying to a toric Calabi-Yau manifold (with an anti-canonical divisor taken away), the equivariant disc potential simply reads

$$W^T = x \cdot \lambda$$

where $T = \mathbb{S}^1$, λ is the corresponding equivariant parameter, and x parametrizes the boundary deformations by a degree-one cycle dual to an orbit loop of the $\mathbb{S}^1$-action.

Moreover, in Section 3, we have explained our gluing method and its equivariant lift. For a toric Calabi-Yau manifold, by studying the pearl trajectories and making use of suitable equivariant Kuranishi perturbations developed by [FOOO10, FOOO11, FOOO16], we can deduce the gluing formula between the torus and an immersed fiber $\mathcal{S}^2 \times T^{n-2}$ as follows. This is a joint work with Hansol Hong, Yoosik Kim and Xiao Zheng.

Theorem 4.2 (Gluing formula [HKLZ]). *Let* $\mathcal{S}^2 \times T^{n-2}$ *be an immersed SYZ fiber in* $K_{\mathbb{P}^{n-1}}$ *depicted in Figure 4. The isomorphism equation* (3.2) *for* $\mathcal{S}^2 \times T^{n-2}$ *and the Chekanov torus* T^{Ch} *is satisfied if and only if the immersed deformations* (u, v) *and the flat connections* $\nabla^{(z_2^{imm},\ldots,z_{n-1}^{imm})}$ *on* $\mathcal{S}^2 \times T^{n-2}$*, the boundary deformations* xX *and the flat connections* $\nabla^{(z_2,\ldots,z_{n-1},w)}$ *on* T^n *satisfy the following gluing formula:*

$$u = w,\ z_i^{imm} = z_i \text{ for } i = 2, \ldots, n-1,\ uv = -e^x + z_2 + \ldots + z_{n-1} + \frac{q}{z_1 \ldots z_{n-1}} + (1+\delta(q)).$$

By Proposition 3.2, the above isomorphism can be lifted to the equivariant setting. Since an isomorphism preserves the equivariant disc potential, the potential for $\mathcal{S}^2 \times T^{n-2}$ can be derived from that for the Chekanov torus T^{Ch} and the gluing formula.

Theorem 4.3 (Equivariant disc potential [HKLZ]). *For $X = K_{\mathbb{P}^{n-1}}$, the $\mathbb{S}^1$-equivariant disc potential of $\mathcal{S}^2 \times T^{n-2}$ equals to*

$$m_0^{\mathbb{S}^1, b\otimes \mathbf{1}_{B\mathbb{S}^1}} = \lambda \cdot \log g(uv, z_2, \ldots, z_{n-1}) \tag{4.3}$$

where λ is the $\mathbb{S}^1$-equivariant parameter, and $-z_1 = g(uv, z_2, \ldots, z_{n-1})$ is a solution to the mirror equation (4.2)*.*

We have focused on the immersed $\mathcal{S}^2 \times T^{n-2}$. We may compare with the Aganagic-Vafa brane L^{AV}. First, note that $\mathcal{S}^2 \times T^{n-2}$ bounds holomorphic polygons which have one or more corners lie in the immersed sectors. These are not present in L^{AV} which is smooth. Thus ***the equivariant disc potential we obtain for $\mathcal{S}^2 \times T^{n-2}$ has more terms than the potential for L^{AV}***: the coefficients of $(uv)^k$ in Equation (4.3) are counting polygons with $2k$ corners.

Setting $u = v = 0$ takes away the contribution from these polygons. The stable discs bounded by L^{AV} are exactly the same as those bounded by $\mathcal{S}^2 \times T^{n-2}$. The only difference is that, $\mathcal{S}^2 \times T^{n-2}$ has two 'branches' around the immersed locus, while L^{AV} is smooth. This is the main reason that $\mathcal{S}^2 \times T^{n-2}$ is unobstructed: the contribution of the stable discs to the obstruction term of $\mathcal{S}^2 \times T^{n-2}$ appear in pairs and cancel within each other. The equivariant obstruction term of L^{AV} should coincide with that for $\mathcal{S}^2 \times T^{n-2}$ given in Equation (4.3).

For instance when the dimension $n = 4$, some of the coefficients a_{jklp} of the equivariant disc potential (4.3), written in the form $\sum -a_{jklp} z_1^j z_2^k q^l (uv)^p$, of $K_{\mathbb{P}^3}$ are given by the following table (for $p = 0$).

ord(q) = 0				
	ord(z_2)			
ord(z_1)	0	1	2	3
0	0	1	1/2	1/3
1	1	1	1	1
2	1/2	1	3/2	2
3	1/3	1	2	10/3

ord(q) = 1				
	ord(z_2)			
ord(z_1)	−1	0	1	2
−1	1	2	3	4
0	2	6	12	20
1	3	12	30	60
2	4	20	60	140

ord(q) = 2					
	ord(z_2)				
ord(z_1)	−2	−1	0	1	2
−2	3/2	6	15	30	105/2
−1	6	36	108	246	480
0	15	108	387	1020	2250
1	30	246	1020	3060	7560
2	105/2	480	2250	7560	20685

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