QUANTUM CORRECTIONS AND WALL-CROSSING VIA LAGRANGIAN INTERSECTIONS

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ABSTRACT. This article introduces the past and ongoing works on quantum corrections in SYZ from the author's perspective. It emphasizes on a method of gluing local pieces of mirrors using isomorphisms between immersed Lagrangians, which is an ongoing joint work with Cho and Hong. It gives a canonical construction of mirrors and generalizes the SYZ setting.

1. INTRODUCTION: SYZ, QUANTUM CORRECTIONS AND WALL-CROSSING

Let (X,ω) be a symplectic manifold. Roughly speaking, a mirror of (X,ω) is a variety $\check{X}$ whose complex geometry reflects the symplectic geometry of (X,ω). In terms of homological mirror symmetry (HMS) formulated by Kontsevich [Kon95], we should have

$$D^{\pi}\mathrm{Fuk}(X) \cong D^b(\check{X}).$$

Mirror symmetry transforms symplectic geometry, which stems from Hamiltonian dynamics and its quantizations, to complex geometry, which is more well-understood and computable.

Strominger-Yau-Zaslow [SYZ96] proposed to understand mirror symmetry by using Lagrangian fibrations. Suppose the mirror $\check{X}$ is a smooth complex manifold. Tautologically $\check{X}$ is the moduli space of $p \in \check{X}$. The endomorphism space is $\mathrm{Ext}^*(\mathscr{O}_p,\mathscr{O}_p) \cong \bigwedge^* \mathbb{C}^n \cong H^*(T,\mathbb{C})$, the cohomology of a torus T. It suggests that $\mathscr{O}_p$ is likely to be mirror to a Lagrangian torus in X (whose Floer cohomology equals to the usual cohomology). A nice assumption is that deformations of a certain Lagrangian torus produce a Lagrangian fibration, and the moduli space of such tori is simply the base of the fibration.[1]

Now assume that X has a Lagrangian torus fibration. The dual torus fibration should be a mirror of X. Moreover it asserted that there should be a real version of *Fourier-Mukai transform* [LYZ00] which realizes homological mirror symmetry. The SYZ program leads to a lot of important developments in mirror symmetry, including Gross-Siebert program [GS11] and family Floer theory [Fuk02, Tu14, Tu15, Aboa, Abob].

Quantum correction coming from singular fibers of the Lagrangian fibration is the most interesting part of the SYZ program. Torus duality away from singular fibers give only the first-order approximation of the mirror variety. We need to

[1]Note that when the mirror is a Landau-Ginzburg model (a singularity defined by a superpotential), this derivation breaks down and we should consider deformations of more general vanishing cycles or even immersed Lagrangians.

correct it by holomorphic discs emanated from singular fibers, which roughly speaking probe the geometry of singular fibers. They lead to the famous wall-crossing phenomenon, whose algebraic aspect has been studied extensively by Kontsevich-Soibelman [KS, KS06] and Gross-Siebert [GS06, GS10, GS11] in the SYZ context.

In this article, I will introduce a symplectogeometric method of capturing the quantum corrections in an ongoing joint work of the author with Cho and Hong. There are several existing methods to capture quantum corrections (for instance the works of Fukaya [Fuk02], Auroux [Aur07, Aur09], Chan-Lau-Leung [CLL12], Pascaleff [Pas14], Abouzaid-Auroux-Katzarkov [AAK16], Lin [Lin17], and Yu [Yu16]). The method introduced here can be explicitly carried out in many cases, directly related to HMS, and can be applied to a more general setting involving immersed Lagrangians (rather than SYZ tori). In the example of $\mathbb{C}^2$, this method was introduced by Seidel [Sei, Prop. 11.8] to deduce the wall-crossing function.

We shall stick with the following two examples throughout the article. In later sections we will use various perspectives to explain how the mirrors come up from the first principle.

Example 1.1 ($K_{\mathbb{P}^1}$)**.** *Let X be the total space of the canonical line bundle over $\mathbb{P}^1$. It is a toric Calabi-Yau 2-fold whose fan is generated by the primitive vectors $(-1,1),(0,1),(1,1)$. Toric Calabi-Yau manifolds and their mirrors serve as important local models for Lagrangian fibrations on compact Calabi-Yau manifolds. In* [CnBM09]*, Castaño-Bernard and R. and Matessi glued these local models and constructed a Lagrangian fibration on the Fermat Calabi-Yau threefold (see also the works of Ruan* [Rua01, Rua02, Rua03] *and Gross* [Gro01b]*).*

Let μ_1 be the moment map of the $\mathbb{S}^1$-action corresponding to the vector $(1,0)$. Let w be the toric holomorphic function corresponding to the covector $(0,1)$. X has the Lagrangian fibration $(\mu_1, |w-1|)$ which was found by Gross [Gro01a] *and Goldstein* [Gol01] *in a more general context. See Figure 1.*

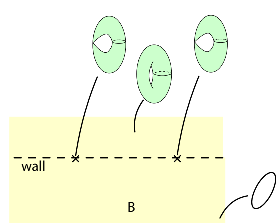


FIGURE 1. A Lagrangian fibration on $K_{\mathbb{P}^1}$.

The 'wall' in this example is given by $\mathbb{R}\times\{1\}$ in the base of the fibration. It contains two singular points of the fibration. In terminology of Gross-Siebert [GS11]*, one singular point should give the slab function $1+z$ and the other should give $1+qz^{-1}$ (where q is the Kähler parameter associated with the zero section $\mathbb{P}^1 \subset K_{\mathbb{P}^1}$. Thus the overall slab function of this wall is $(1+z)(1+qz^{-1})$, and the* *mirror should be*

$$uv = (1+z)(1+qz^{-1})$$

which agrees with Hori-Vafa mirror [HV, HIV] *(via the mirror map).*

Geometrically, the wall is the collection of Lagrangian fibers which bound holomorphic discs of Maslov index zero, namely the discs contained in the divisor $w = 0$. *Unfortunately counting of Maslov-zero holomorphic discs is hard to define, since disc bubbling occurs and produces codimension-one boundaries of their moduli spaces. In this article we shall explain methods to extract the information.*

Example 1.2 (Four-punctured sphere)**.** *Let* $X = \mathbb{P}^1 - \{a,b,c,d\}$ *where* a,b,c,d *are four chosen points in* $\mathbb{P}^1$. [AAE+13] *proved HMS for punctured spheres (and later* [Lee15] *proved HMS for punctured Riemann surfaces). In particular*

$$D^\pi \mathrm{Fuk}(X) \cong D^b \mathrm{MF}(W)$$

where W *is a holomorphic function defined on the resolved conifold; it is toric whose fan is generated by the primitive vectors* $(0,0,1),(1,0,1),(0,1,1),(1,1,1)$. W *is the toric holomorphic function corresponding to the covector* $(0,0,1)$.

We would like to understand why W *defined above is the mirror of* X *more systematically. More precisely, we would like to develop a general geometric recipe of constructing mirrors such that homological mirror symmetry naturally follows. This example is particularly interesting to us since SYZ cannot be applied directly. (Note that* X *has a Lagrangian fibration whose base is a trivalent graph which is not a manifold.) A mirror construction by using SYZ on conic fibrations was found by* [AAK16].

In this article we shall see that the method in Section 3 which uses isomorphisms of Lagrangians can be applied to a pair-of-pants decomposition and gives a canonical way to construct mirrors. For this example it produces the above W *defined on the resolved conifold.*

Another way is to use the noncommutative mirror construction developed in [CHLb]. *The recipe produces* W *defined on the noncommutative resolution of the resolved conifold. It is explained in Section 4. The results produced by the two methods agree to each other in the sense that they have equivalent derived categories.*

2. Capturing quantum corrections by Maslov-two discs

In Example 1.1, we have seen that Maslov-zero holomorphic disc is the key ingredient in SYZ mirror symmetry. Maslov-zero disc is a central object in the theory of topological vertex [AKMnV05, LLLZ09, FL13] for toric Calabi-Yau manifolds which has a very rich structure.

Unfortunately the moduli spaces of Maslov-zero discs have codimension-one boundaries and it is very difficult to define the associated countings (and usually auxiliary data is needed). In the case of K3 surfaces, Lin [Lin17] defined and computed their open Gromov-Witten invariants by hyper-Kähler techniques.

In the following we review a method used in [Aur07, Aur09, CLL12] to detect the contribution of Maslov-zero discs by using holomorphic discs of Maslov index two. In the next section I shall introduce a more canonical method which is also applicable to Lagrangian immersions.

We can consider holomorphic discs of Maslov index two bounded by torus fibers. This produces a superpotential W defined above and below the wall. (W is not quite well-defined right on the wall since it depends on the choice of Kuranishi perturbations due to the existence of Maslov-zero discs.) From the expressions of W above and below the wall, we can read off the wall-crossing effect and obtain the slab function.[2]

The main idea is, the Maslov-two discs interact with the Maslov-zero discs and hence detect the effect of walls. Namely, it could happen that

$$\beta' = \beta + \beta_0$$

where β', β, β_0 are holomorphic disc classes of Maslov index $2, 2, 0$ respectively, bounded by a fiber at the wall. The superpotential jumps across the wall and it detects the slab function.

Let's take Example 1.1 to illustrate. There is only one holomorphic disc class bounded by a fiber below the wall. It is emanated from the boundary divisor $w = 0$. It propagates and gives a holomorphic disc class bounded by a fiber above the wall. During the propagation, it can also interact with holomorphic discs of Maslov index zero over the wall. It turns out that three more holomorphic disc classes are produced, corresponding to the terms z, qz^{-1} and q in the slab function $1 + q + z + qz^{-1}$. (Note that holomorphic disc classes corresponding to other terms z^k for $k \neq 0, 1, -1$ could also occur. It is a non-trivial computation that all other contributions are zero [FOOO12, CL10].)

In order to compute all the disc contributions precisely, we used virtual technique and compactified stable discs to stable spheres [Cha11, LLW11, LLW12, CL10]. [FOOO12] gave another method to compute them using degeneration. We fully carry out the computation for all toric Calabi-Yau orbifolds [CCLT16] and compact semi-Fano toric manifolds [CLLT], and showed that they exactly coincide with the mirror map and Seidel representation.

Note that this method requires the presence of Maslov-two discs. In particular it works when the Lagrangian fibration has boundary divisors where the Maslov-two discs are emanated from. In the case of surfaces, the work of Yu [Yu16] counted holomorphic cylinders (and hence does not need boundary divisors) in the setting of non-Archimedian geometry.

Another drawback of this method is that, it is not directly connected to HMS. Namely, we match the m_0-terms of torus fibers by hand, and one has to prove HMS separately. The method of Fukaya [Fuk02, Tu14, Aboa, Abob] is more canonical. Namely, he used a diffeomorphism which is isotopic to identity to identify one Lagrangian torus fiber above and one below the wall. It gives a family of

[2] More precisely we should take the generating function of discs emanated from each boundary divisor D, namely those disc classes β with intersection number 1 for D and 0 for all other divisors.

almost complex structures and a homotopy between the two associated A_∞ algebras. This gives an isomorphism between their (weak) Maurer-Cartan spaces. However the isomorphism is abstract and it is hard to make it explicit. In the next section, we introduce a simpler method to obtain and compute the isomorphism (which does not even require that the two Lagrangians are diffeomorphic.)

3. Gluing by isomorphisms between Lagrangians

In an ongoing work with Cho and Hong, we are developing a method to capture quantum corrections by establishing isomorphisms between formally deformed Lagrangian branes. An isomorphism is simply a morphism (given by an intersection point) from a Lagrangian brane to another one which has an inverse morphism (in the strict sense) in the Fukaya category.

An advantage is that the method works uniformly for both Example 1.1 and 1.2. It works for all toric Calabi-Yau manifolds and punctured Riemann surfaces (and indeed many more other interesting examples). Moreover, it canonically induces a mirror functor realizing HMS which will be studied in another paper.

The method of deducing the wall-crossing function by considering morphisms between Lagrangian tori above and below the wall has been used by Seidel [Sei, Prop. 11.8] in the example of $\mathbb{C}^2$. We will apply this method to all toric Calabi-Yau manifolds and their mirrors.

3.1. Isomorphism between Lagrangian torus fibers in $K_{\mathbb{P}^1}$.

Recall that we have the Lagrangian fibration $(\mu_1, |w-1|)$ on $K_{\mathbb{P}^1}$. Fix $b_1 \in (c_1, c_2)$, where c_1 and c_2 are the singular values of μ_1. The symplectic reduction $\mu_1^{-1}\{b_1\}/\mathbb{S}^1$ can be identified with $\mathbb{C}$ by the $\mathbb{S}^1$-invariant function w. The Lagrangian fiber at (b_1, b_2) is the preimage in $\mu_1^{-1}\{b_1\}$ of a circle centered at $w = 1$ with radius b_2 in the symplectic reduction.

We shall consider the deformation spaces of two Lagrangian torus fibers, one is above and one is below the wall $b_2 = 1$. The two Lagrangians are the preimages of the two circles with radius $b_2 > 1$ and $b_2 < 1$ respectively. *We will glue their deformation spaces (which are treated as local pieces of the mirror) by establishing an isomorphism between them as objects in the Fukaya category.*

An advantage of this method over considering wall-crossing of Maslov-two holomorphic discs in Section 2 is that, we do not need the boundary divisor $\{w = 1\}$. In particular it is better adapted to general Calabi-Yau settings (where there is no boundary divisor at all). *From now on we consider $K_{\mathbb{P}^1} - \{w = 1\}$ with the Lagrangian fibration $(\mu_1, |w-1|)$.*

Now the trick is, we move the two Lagrangians by an isotopy such that they intersect with each other. See Figure 2. This method was invented by Seidel [Sei, Prop. 11.8] to understand the wall-crossing for $\mathbb{C}^2$. We call the two Lagrangian tori to be L_1 and L_2 respectively. Since the isotopy never hits the wall (and hence no Maslov-zero holomorphic disc is involved in the process), it does not change the gluing. We shall use their clean intersections to produce the isomorphism we need for gluing.

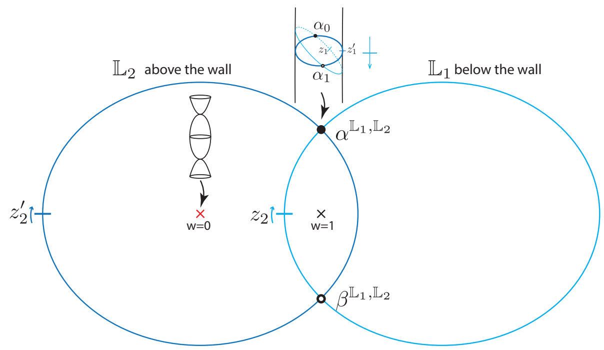


FIGURE 2. Images of L_1 and L_2 in the symplectic reduction.

The deformation space of L_i is given by the set of flat $\mathbb{C}^\times$-connections on L_i. To fix the gauge, we take the two hyper-tori in L_1 which are the circle fiber over z_2 and the section marked by z_1 in Figure 2. This is similar to the setting of [CHL14] for compact toric manifolds. The parallel transport along a path passing through the hypertorus marked by z_i is the multiplication by z_i. We do the same thing for L_2 and denote the corresponding holonomies by z_i' for $i = 1,2$.

In short, we have two local pieces of mirrors $(\mathbb{C}^\times)^2$ whose coordinates are z_i and z_i' for $i = 1,2$ respectively. Moreover we have the A_∞ algebras

$$\left(H^*(L_1), \{m_k^{z_1,z_2} : k \geq 0\}\right) \text{ and } \left(H^*(L_2), \left\{m_k^{z_1',z_2'} : k \geq 0\right\}\right)$$

which are given by counting stable discs with one output marked point bounded by L_i weighed by the holonomies along their boundaries. In the following, we shall find isomorphisms between the objects $(L_1, \vec{z})$ and $(L_2, \vec{z}')$.

L_1 and L_2 intersect cleanly along two disjoint circles marked by α^{L_1,L_2} and β^{L_1,L_2} in Figure 2. The Morse-Bott Floer complex $\mathrm{CF}((L_1,\vec{z}),(L_2,\vec{z}'))$ is generated by cochains of these circles. To reduce to usual transverse situation, one may further perturb one of the Lagrangian L_i to produce the transversal intersection points α_0, α_1 and β_1, β_2. The subscripts $0,1,2$ denote the degrees when they are treated as morphisms from L_1 to L_2. Take $\alpha_0^{L_1,L_2}$ as a degree-zero chain in $\mathrm{CF}((L_1,\vec{z}),(L_2,\vec{z}'))$.

Theorem 3.1. *$\alpha_0^{L_1,L_2}$ is an isomorphism*

$$(L_1,\vec{z}) \xrightarrow{\cong} (L_2,\vec{z}')$$

in the Fukaya category if and only if $z_1' = z_1$ and $z_2' = -z_2(1+z_1)(1+qz_1^{-1})$.

Proof. Let's compute the Floer differential d of the complex $\mathrm{CF}((L_1,\vec{z}),(L_2,\vec{z}'))$ applied to $\alpha_0^{L_1,L_2}$, which is counting stable strips from an input intersection point to an output intersection point. The result is a degree-one element in $\mathrm{CF}((L_1,\vec{z}),(L_2,\vec{z}'))$, which is a linear combination of $\alpha_1^{L_1,L_2}$ and $\beta_1^{L_1,L_2}$.

Over the point marked by α^{L_1,L_2} in Figure 2, there are two holomorphic strips with the same area, say ϵ. They give the term $\mathbf{T}^{\epsilon}\left(z_1 - z_1'\right)\alpha_1^{L_1,L_2}$ in $d(\alpha_0^{L_1,L_2})$. (Here $\mathbf{T}$ is the formal parameter in the Novikov ring. Indeed we do not have convergence issue here and so the reader can replace it by e^{-1}.) There is no other strip from $\alpha_0^{L_1,L_2}$ to $\alpha_1^{L_1,L_2}$.

Next consider the holomorphic strips from $\alpha_0^{L_1,L_2}$ to $\beta_1^{L_1,L_2}$. By topological reason, these strips must project to either the left or the right region bounded by the two circles in the w-plane (see Figure 2). (Note that it cannot project to the middle region since the divisor $w = 1$ has been removed in the current setting.)

The region on the right does not contain any singular point of the holomorphic fibration. There is exactly one rigid holomorphic strip (which is a constant section) from $\alpha_0^{L_1,L_2}$ to $\beta_1^{L_1,L_2}$ over this region. (It does not pass through any holonomy hypertorus.) It gives the term $-\mathbf{T}^{\Delta}\beta_1^{L_1,L_2}$ where Δ is its symplectic area.

The region on the left contains the singular point $w = 0$. By Riemann mapping theorem (which is used to smooth out the corners of the strips), *stable strips over this region are one-to-one corresponding to holomorphic discs bounded by a toric fiber of the toric Calabi-Yau manifold $K_{\mathbb{P}^1}$ passing through a generic marked point α_0*. We already know that four stable discs are contributing from the disc pontential of $K_{\mathbb{P}^1}$. They give

$$-\mathbf{T}^{\Delta} z_2 (z_2')^{-1}(1+z_1)(1+qz_1^{-1})\beta_1^{L_1,L_2}.$$

In summary, we have

$$d(\alpha_0^{L_1,L_2}) = \mathbf{T}^{\epsilon}(z_1 - z_1')\alpha_1^{L_1,L_2} - \mathbf{T}^{\Delta}\left(1 + z_2(z_2')^{-1}(1+z_1)(1+qz_1^{-1})\right)\beta_1^{L_1,L_2}$$

Thus $d(\alpha_0^{L_1,L_2}) = 0$ if and only if

$$\begin{cases} z_1' = z_1 \\ z_2' = -z_2(1+z_1)(1+qz_1^{-1}). \end{cases}$$

We have $m_2(\alpha_0^{L_1,L_2}, \beta_0^{L_2,L_1}) = \mathbf{T}^{\Delta}\,\mathbf{1}$ (contributed from the strip over the region on the right passing through a generic marked point), and hence the degree zero element $\beta_0^{L_2,L_1} \in \mathrm{HF}^0((L_2, \vec{z}'), (L_1, \vec{z}))$ gives the inverse of $\alpha_0^{L_1,L_2}$. Thus $\alpha_0^{L_1,L_2}$ defines an isomorphism. $\square$

By the above theorem, we glue the two copies of $(\mathbb{C}^{\times})^2$ by

$$z_1' = z_1 \text{ and } z_2' = -z_2(1+z_1)(1+qz_1^{-1}).$$

It gives

$$\left\{(u, v, z_1) \in \mathbb{C}\times\mathbb{C}\times\mathbb{C}^{\times} : uv + (1+z_1)(1+qz_1^{-1}) = 0\right\} - \{u = v = 0\}$$

where the copy $\operatorname{Spec}\mathbb{C}\left[z_1^{\pm1}, z_2^{\pm1}\right]$ is embedded by $u = z_2^{-1}$, and the copy $\operatorname{Spec}\mathbb{C}\left[(z_1')^{\pm1}, (z_2')^{\pm1}\right]$ is embedded by $z_1 = z_1', v = z_2'$. [3]

[3] It differs from the Hori-Vafa mirror $uv = (1+z_1)(1+qz_1^{-1})$ by a negative sign, which can be easily absorbed away by taking $u_1 \mapsto -u_1$.

Remark 3.2. *Note that we have lost two points* $\{u = v = 0, z_1 = -1 \text{ or } -q\}$ *in this construction. It corresponds to the two immersed Lagrangian*

$$L_{0,j} = \{\mu_1 = c_j, |w - 1| = 1\}$$

for $j = 1,2$ *(recall that* c_j *are the singular values of* μ_1*). To recover these points, we need to glue the deformation spaces* $\mathbb{C} \times \mathbb{C}$ *of* $L_{0,j}$ *to the deformation spaces of* L_1 *and* L_2*. This is an ongoing joint work with Cho and Hong.*

3.2. **Mirror construction for a punctured sphere.** This method also works for gluing in the mirror construction for a pair-of-pants decomposition. Let's illustrate by the 4-punctured sphere in Example 1.2.

Consider the pair-of-pants decomposition as shown in Figure 3. We have two pair-of-pants in the decomposition. In each pair-of-pants, we take an immersed Lagrangian as shown in the figure. This is the Lagrangian used by Seidel [Sei11] for proving HMS for the pair-of-pants and genus-two Riemann surfaces. We make a family of immersed Lagrangians in X in which the Seidel Lagrangian in each pair-of-pants is a member. To do this, we smooth out one of the immersed points in one of the Seidel Lagrangian (labeled by X_1 in Figure 3). The immersed Lagrangian becomes a union of two circles after the smoothing, and we move it to a neighboring pair-of-pants, degenerating it to a Seidel Lagrangian by forming another immersed point (labeled by X_2). As a result a pair-of-pants decomposition induces a family of immersed Lagrangians over a graph.

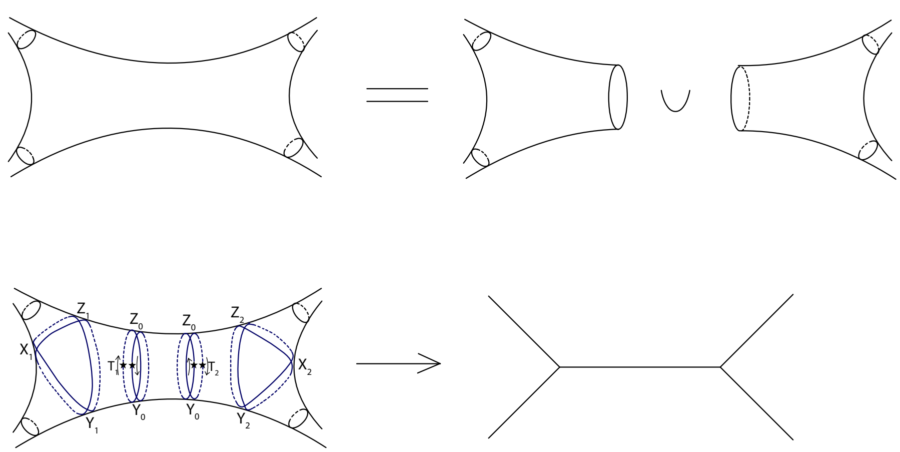


FIGURE 3. A pair-of-pants decomposition induces a family of immersed Lagrangians over a polyhedral complex.

The formal deformation space is spanned by degree-one Floer generators of a Seidel Lagrangian in a pair-of-pants is $\mathbb{C}^3$. Namely we have three immersed points, and each point gives one deg = 1 and one deg = 0 generators. (Here the degree is $\mathbb{Z}_2$-valued since we use $\mathbb{Z}_2$ grading for the Floer theory.) Let's call these generators to be X, Y, Z and $\bar{X}, \bar{Y}, \bar{Z}$ respectively. Then the formal deformation space consists of the elements $b = xX + yY + zZ$ where $x, y, z \in \mathbb{C}^3$.

If the Seidel Lagrangian L is exact (such that the front and back triangles bounded by the Seidel Lagrangian have the same symplectic area), then each formal deformation $b = xX + yY + zZ$ is weakly unobstructed [CHLa] in the sense of [FOOO09], namely $m_0^b = W(x, y, z) \cdot \mathbf{1}_L$. Roughly speaking it means the superpotential W is well-defined.

Let's denote their deformations by

$$\boldsymbol{b}_i = x_i X_i + y_i Y_i + z_i Z_i$$

for $i = 1,2$. Then we have two copies of $\mathbb{C}^3$ coming from the Seidel Lagrangians L_i in the two pair-of-pants.

We also have an immersed Lagrangian L_0, which is a smoothing of L_1 at the immersed generator X_1 and can also be described as a smoothing of L_2 at the immersed generator X_2. It is a union of two circles intersecting each other at two points. The two intersection points give two deg = 1 generators Y_0, Z_0 and two deg = 0 generators $\bar{Y}_0, \bar{Z}_0$. Moreover we have flat $\mathbb{C}^\times$ connections taken as deformations of the circles. Let's fix the gauge by taking the two points marked by T_1 in Figure 3 where the parallel transport are given by multiplication by $t_1 \in \mathbb{C}^\times$ when passing through the points (in the indicated direction). T_1 corresponds to the immersed point X_1 of L_1. Alternatively we can take the gauge to be the two points marked by T_2, corresponding to the immersed point X_2 of L_2. It is easy to check that these deformations are weakly unobstructed if we take L_0 to be exact. Thus for L_0 we have the deformations

$$(\boldsymbol{b}_0 = y_0 Y_0 + z_0 Z_0, \nabla^{t_1})$$

parametrized by $\mathbb{C}^\times \times \mathbb{C}^2$, and the deformations

$$(\boldsymbol{b}_0' = y_0' Y_0 + z_0' Z_0, \nabla^{t_2})$$

parametrized by $\mathbb{C}^\times \times \mathbb{C}^2$.

We need to glue the deformation spaces of L_1, L_0, L_2 together to form a global geometry. In this simple example, we can use the method in Section 2 to determine the gluing. Namely we compute the local superpotentials defined by counting Maslov-two discs bounded by each Lagrangian and extract the gluing data from the jump of the superpotential. This is done in Section 3.2.1 below.

However, the gluing data that we obtain in this way is not quite canonical. Namely, there may be more than one ways of gluing to match the local superpotentials. In Section 3.2.2, we shall use isomorphisms between immersed Lagrangians to extract the gluing data. It incorporates the whole Fukaya algebras rather than just the m_0-parts of the algebras.

3.2.1. *Gluing by Maslov-two discs.* First consider the effect of smoothing of L_1. The superpotential of L_1 is $x_1 y_1 z_1$ corresponding to the two triangles bounded by L_1. For L_0, the superpotential of L_0 is $t_1 y_0 z_0$ corresponding to the two bigons bounded by L_0 (and they pass through the holonomy points marked by T_1 once). From this we infer that the gluing should be given by

$$t_1 = x_1, y_0 = y_1, z_0 = z_1.$$

Similarly the gluing between L_2 and L_0 is $t_2 = x_2, y_0' = y_2, z_0' = z_2$. Here we assume that the triangles and bigons have the same areas and have ignored the area contribution for simplicity.

Then we consider the effect of gauge change for L_0. The holonomy points T_1 and T_2 describe two different gauges for flat $\mathbb{C}^\times$ connections. Let's take a homotopy from the gauge given by T_1 to that given by T_2 as follows.

(1) There are two holonomy points marked by T_1. Move the one in the front part of the surface to the back through the immersed point Z_0 along one of the circles in L_0.
(2) Similarly move the other holonomy point in the back to the front through the immersed point Y_0.
(3) Take the inverse of the holonomy in order to match with that given by T_2.

When a holonomy point moves across an immersed point of the Lagrangian, there is a non-trivial change of coordinates in the Floer theory. In the first step the superpotential changes from $t_1 y_0 z_0$ to $y_0 z_0'$. Thus we should have the coordinate change $z_0' = t_1 z_0$. In the second step the superpotential changes from $y_0 z_0$ to $t_1^{-1} y_0' z_0$. Thus we should have the coordinate change $y_0' = t_1 y_0$. In the final step we have $t_2 = t_1^{-1}$.

Combing the above change of coordinates, the gluing of the two deformation spaces $\mathbb{C}^3$ is

$$t_2 = t_1^{-1}, y_2 = t_1 y_1, z_2 = t_1 z_1.$$

on $\mathbb{C}^\times \times \mathbb{C}^2$. The resulting manifold is the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, see Figure 4. The mirror is the Landau-Ginzburg model $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1), W)$ that we have described in Example 1.2.

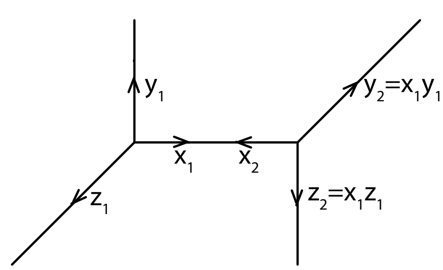


FIGURE 4. The resulting mirror space by gluing the deformation spaces of the immersed Lagrangians is $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. The right hand side is its toric diagram.

Note that the gluing depends on the choice of homotopy for gauge change. Different choices *result in different mirror models.* For instance, there is another choice which results in the Landau-Ginzburg model $(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0), W')$. Nevertheless the critical loci of the superpotentials are isomorphic and so they have the same derived category of singularities.

3.2.2. *Gluing by isomorhpism of objects.* The above method finds the coordinate changes by hand in order to match the local superpotentials. However it does not derive the coordinate changes from the first principle. When we consider more complicated examples (such as compact symplectic manifolds), it is hard to find coordinate changes by hand. Below we give a canonical method to determine the coordinate change.

We shall only focus on smoothing of the immersed point X_1 in L_1. Gauge change is indeed conceptually more standard and hence omitted.

To have explicit isomorphisms, we deform the Seidel Lagrangian L_1 such that it intersects with L_0. See Figure 5. The figure shows the pair-of pants containing L_0 and L_1. (One of the three punctures is the infinity point of the figure.) We also label the areas of the regions bounded by L_0 and L_1 by A_i for $i = 1,\dots,7$.

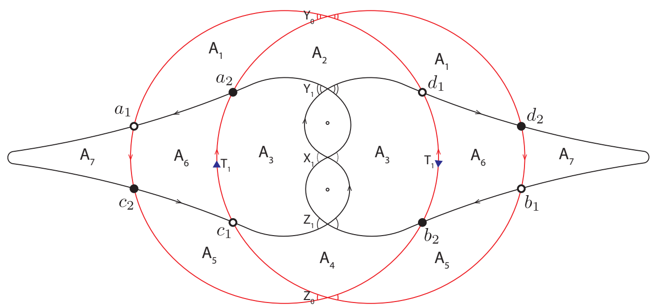


FIGURE 5. The Seidel Lagrangian L_1 and the double-circle L_0. They intersect at eight points marked by a_i, b_i, c_i, d_i for $i = 1,2$ respectively.

The Seidel Lagrangian L_1 and the double-circle L_0 intersect at eight points marked by a_i, b_i, c_i, d_i for $i = 1,2$ respectively. Take $a_1 + b_1$ as an element in

$$\mathrm{CF}^0((L_0, \boldsymbol{b}_0, \nabla^{t_1}), (L_1, \boldsymbol{b}_1)).$$

(Recall that $\boldsymbol{b}_1 = x_1 X_1 + y_1 Y_1 + z_1 Z_1$ and $\boldsymbol{b}_0 = y_0 Y_0 + z_0 Z_0$.)

Proposition 3.3. *For the 4-punctured sphere, $a_1 + b_1 \in \mathrm{CF}^0((L_0, \boldsymbol{b}_0, \nabla^{t_1}), (L_1, \boldsymbol{b}_1))$ gives an isomorphism from $(L_0, \boldsymbol{b}_0, \nabla^{t_1})$ to $(L_1, \boldsymbol{b}_1)$ if and only if $t_1 = \mathbf{T}^{A_1+A_2+A_3+A_4+A_5-A_7} x_1$, $y_0 = \mathbf{T}^{A_7-A_2-2A_1} y_1$, $z_0 = \mathbf{T}^{A_7-A_4-2A_5} z_1$ (where $x_1 \neq 0$).*

Proof. Consider $d(a_1+b_1)$, which is a linear combination of the deg $= 1$ elements a_2, b_2, c_2, d_2. There is no strip from b_1 to c_2, and so the coefficient of c_2 is merely contributed from strips from a_1 to c_2, which equals to $\mathbf{T}^{A_1+A_2+A_3+A_4+A_5} t_1^{-1} x_1 - \mathbf{T}^{A_7}$. See the top two strips in Figure 6. In order to have $d(a_1 + b_1) = 0$, we need $t_1 = \mathbf{T}^{A_1+A_2+A_3+A_4+A_5-A_7} x_1$.

Now consider the coefficient of a_2. The contribution from a_1 is $-\mathbf{T}^{A_1} y_0$, and the contribution from b_1 is $\mathbf{T}^{A_3+A_4+A_5} y_1 x_1 t_1^{-1} = \mathbf{T}^{A_3+A_4+A_5-(A_1+A_2+A_3+A_4+A_5-A_7)} y_1$. See the middle two strips in Figure 6. In order to have $d(a_1 + b_1) = 0$, we need $y_0 = \mathbf{T}^{A_7-A_2-2A_1} y_1$. Similarly the output to b_2 equals to zero implies that $z_0 = \mathbf{T}^{A_7-A_4-2A_5} z_1$. See the bottom two strips in Figure 6.

We have $m_2(a_1+b_1, c_2+d_2) = \mathbf{T}^{A_1+A_2+A_3+A_4+A_5} t_1^{-1} x_1 \mathbf{1}_{L_0} = \mathbf{T}^{A_7}\mathbf{1}_{L_0}$ (where $c_2 + d_2$ is regarded as an element in $\mathrm{CF}^0((L_1, \boldsymbol{b}_1),(L_0,\boldsymbol{b}_0))$). (Fixing a generic marked point, there is either one strip from a_1 to c_2 or one from b_1 to d_2 passing through it. There is no strip from a_1 to d_2 nor from b_1 to c_2.) Hence c_2+d_2 serves as the inverse of a_1+b_1 (up to the multiple $\mathbf{T}^{A_7}$). $\square$

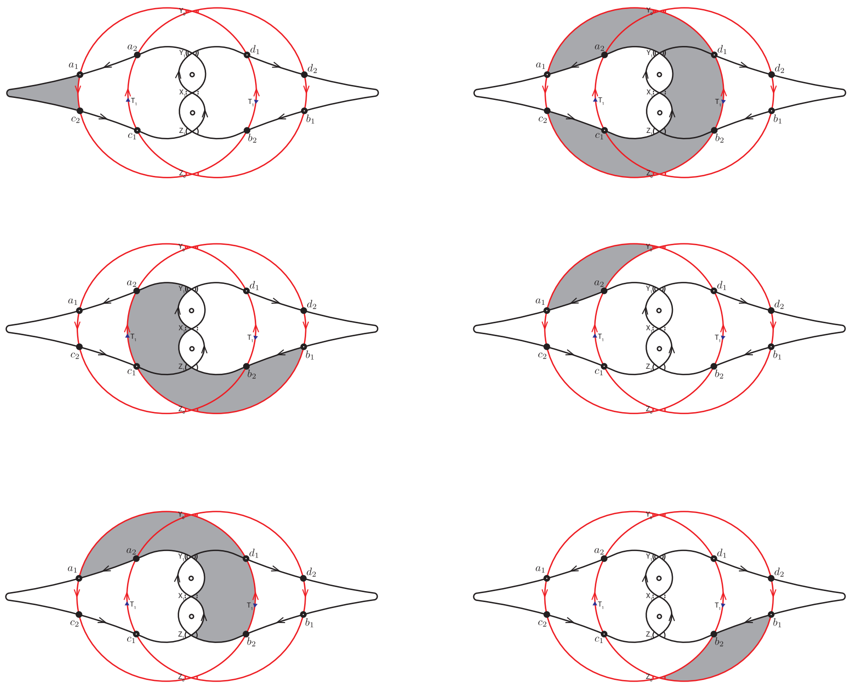


FIGURE 6. The strips contributing to the differential of a_1+b_1.

By the above proposition, we obtain the gluing

$$t_1 = \mathbf{T}^{A_1+A_2+A_3+A_4+A_5-A_7} x_1, \; y_0 = \mathbf{T}^{A_7-A_2-2A_1} y_1, \; z_0 = \mathbf{T}^{A_7-A_4-2A_5} z_1$$

between the deformation space $\mathbb{C}^3$ of L_1 and $\mathbb{C}^\times \times \mathbb{C}^2$ of L_0. Note that it automatically matches the superpotential, namely, we have

$$\mathbf{T}^{A_1+A_6+A_5} t_1 y_0 z_0 = \mathbf{T}^{A_3+A_6+A_7} x_1 y_1 z_1.$$

In general the coordinate changes receive much more complicated quantum corrections coming from holomorphic discs in non-exact situations. For instance, consider the compactification $\bar{X} = \mathbb{S}^2$ by filling in the four punctures, which still contains the family of immersed Lagrangians depicted in Figure 3. Even for this simple example, the gluing is already rather non-trivial and it is better to use our systematic method to deduce the gluing.

Since the method is the same as above, we skip the detail and state the result below.

Proposition 3.4. *For the compact sphere, the above method gives the gluing*

$$\begin{aligned} t_1 =& \mathbf{T}^{A_1+A_2+A_3+A_4+A_5-A_7} x_1, \\ y_0 =& \mathbf{T}^{A_7-A_2-2A_1} y_1 + \mathbf{T}^{A_3+A_4+A_5+A_8-A_1} t_1^{-1}, \\ z_0 =& \mathbf{T}^{A_7-A_4-2A_5} z_1 - \mathbf{T}^{A_1+A_2+A_3+A_9-A_5} t_1^{-1} \end{aligned}$$

between the deformation spaces $\mathbb{C}^3$ *and* $\mathbb{C}^\times \times \mathbb{C}^2$ *of* L_1 *and* L_0 *respectively.*

The additional terms t_1 and t_1^{-1} are contributed by the extra strip from a_1 to b_2 (see Figure 7) and that from b_1 to a_2 respectively. They pass through the punctures (in the region A_8 and A_9) and hence were not counted in the four-punctured sphere before.

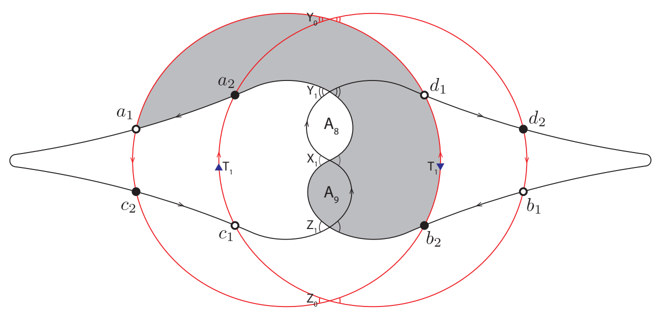


FIGURE 7. An extra strip contributing to the term t_1 in Proposition 3.4.

One can directly verify that the above change of coordinates matches the local superpotentials, namely

$$\begin{aligned} &\mathbf{T}^{A_3+A_6+A_7} x_1 y_1 z_1 - \mathbf{T}^{A_3+A_6+A_7+A_9} y_1 + \mathbf{T}^{2A_1+A_2+A_3+A_4+2A_5+A_6+A_7+A_{10}} x_1 + \mathbf{T}^{A_3+A_6+A_7+A_8} z_1 \\ =& \mathbf{T}^{A_1+A_5+A_6} t_1 y_0 z_0 + \mathbf{T}^{A_1+A_6+A_5+2A_7+A_{10}} t_1 + \mathbf{T}^{A_1+A_2+2A_3+A_4+A_5+A_6+A_8+A_9} t_1^{-1} \end{aligned}$$

where the first expression is the superpotential for L_1 and the second one is that for L_0. A_{10} is the area of the outer region in Figure 7 in the sphere. The terms t_1 and t_1^{-1} in the second expression correspond to the left and right hemispheres bounded by one of the circles in L_0. It is rather amazing that the area terms all match automatically in our method.

Since the gluing constructed in this way is canonical, we should have a natural mirror functor for HMS. This will be studied in a separate paper.

4. LANDAU-GINZBURG MIRROR FROM A FINITE COLLECTION OF LAGRANGIANS

In [CHLa, CHLb], we developed a program of constructing noncommutative mirrors using a finite collection of Lagrangians. The construction naturally comes with a mirror functor realizing HMS. It was applied to derive mirror symmetry for elliptic and hyperbolic orbifolds, punctured Riemann surfaces and local Calabi-Yau threefolds associated to Hitchin systems. It also has applications to deformation quantizations [Kon03]. In this section, we will briefly explain how to use this method to capture quantum corrections.

4.1. **General framework.** The procedure is the following. Let $\mathbf{L} = \{L_1, \ldots, L_N\}$ be a collection of spin oriented connected compact Lagrangian immersions in X which intersect each other transversely. We use the deformation and obstruction theory of $\mathbf{L}$ to construct a non-commutative Landau-Ginzburg model $(\mathscr{A}, W)$, together with a non-trivial functor from $\mathrm{Fuk}(X)$ to the category of matrix factorizations (W-twisted complexes) of $(\mathscr{A}, W)$. The functor is automatically injective on the morphism space $\mathrm{HF}(\mathbf{L}, \mathbf{L})$. In particular if $\mathbf{L}$ and its image under the functor are both generators, and their endomorphism spaces have the same dimension, then the functor derives homological mirror symmetry.

The formal deformation space of $\mathbf{L}$ are given by degree one endomorphisms. (If we just have $\mathbb{Z}_2$ grading, 'degree one' means 'odd degree'.) They are described by a directed graph Q (so-called a quiver). The path algebra ΛQ is regarded as the noncommutative space of formal deformations of $\mathbf{L}$. Each edge e of Q corresponds to an odd-degree Floer generator X_e and a formal dual variable $x_e \in \Lambda Q$. We take the formal deformation $\boldsymbol{b} = \sum_e x_e X_e$, where x_e is taken as an element in the path algebra ΛQ.

Obstruction of $\boldsymbol{b}$ is defined by counting of holomorphic polygons bounded by $\mathbf{L}$. Roughly speaking, it is captured by the superpotential W, which is defined as a weighted count of holomorphic polygons passing through a marked point:

$$W = \sum_{\beta} n_\beta q^\beta x^{\partial\beta} \tag{4.1}$$

where n_β is the counting of holomorphic polygons in class β passing through a marked point, q^β records the symplectic area of β, and $x^{\partial\beta}$ records the self-intersection points hit by the corners of the polygon. Note that $x^{\partial\beta}$ is an element in the path algebra ΛQ; in particular we also record the order of the corners of each polygon.

However such a counting is not well-defined in general, since it depends on the position of the marked point. We need to consider *weakly-unobstructed* deformations, and W is only well-defined for such deformations. *Weakly unobstructedness* was introduced by Fukaya-Oh-Ohta-Ono [FOOO09].

We extend the notion of weakly unobstructedness by Fukaya-Oh-Ohta-Ono [FOOO09] to the current noncommutative setting. The corresponding Maurer-Cartan equation is

$$m_0^{\boldsymbol{b}} := \sum_{k=0}^{\infty} m_k(\boldsymbol{b}, \ldots, \boldsymbol{b}) = \sum_{i=1}^{k} W_i(\boldsymbol{b}) \mathbf{1}_{L_i}$$

where m_k is the A_∞ operations defined by counting holomorphic polygons bounded by $\mathbf{L}$, and $\mathbf{1}_{L_i}$ is the Floer-theoretical unit corresponding to the fundamental class of L_i. $m_0^{\boldsymbol{b}}$ is an analog of the exponential map, which integrates infinitesimal deformations to actual small deformations. Essentially the equation means that the counting W is well-defined and in particular does not depend on the position of the marked point.

The solution space is given by the quiver algebra with relations $\mathscr{A} = \Lambda Q / R$ where R is the two-sided ideal generated by weakly unobstructed relations. The relations are coefficients of m_0^b in all generators other than $\mathbf{1}_{L_i}$. As a result, we obtain a noncommutative Landau-Ginzburg model

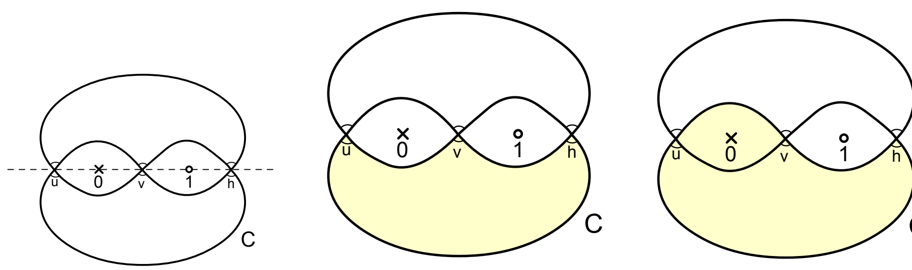
$$\left(\mathscr{A}, W = \sum_i W_i\right).$$

Theorem 4.1 ([CHLb])**.** *There exists an A_∞-functor $\mathscr{F}^{\mathbf{L}} : \mathrm{Fuk}(X) \to \mathrm{MF}(\mathscr{A}, W)$, which is injective on $H^\bullet(\mathrm{Hom}(\mathbf{L}, U))$ for any object U in* $\mathrm{Fuk}(X)$.

An important feature is that the Landau-Ginzburg superpotential W constructed in this way is automatically a central element in $\mathscr{A}$. In particular we can make sense of $\mathscr{A}/\langle W\rangle$ as a hypersurface singularity defined by 'the zero set' of W.

4.2. **Landau-Ginzburg mirror of a toric Calabi-Yau manifold.** Take $K_{\mathbb{P}^1} - \{w = 1\}$ as an example. In Section 2 and 3.1 we have given two methods to construct its mirror. Here we sketch the third method which produces a Landau-Ginzburg mirror using an immersed Lagrangian. It is an ongoing work with Cho and Hong.

The immersed Lagrangian L we use is the inverse image in $\mu_1^{-1}\{b_1\}$ of the immersed curve C in the w-plane shown in Figure 8a, where b_1 is taken in the interval (c_1, c_2) where c_1, c_2 are the two singular values of μ_1. L has clean self-intersections which are circles over the three points marked by u, v, h in C.



(A) The curve C. (B) The uvh-polygon. (C) The polygons in $hf(z)$.

FIGURE 8. The curve C and polygons bounded by C.

We equip L with a non-trivial spin structure by fixing a generic point in the curve C (which is not any of its immersed points). Denote the three generators which has base degree 1 and fiber degree 0 by U, V, H corresponding to the three immersed points. Take the formal deformations $\boldsymbol{b} = uU + vV + hH$ for $u, v, h \in \mathbb{C}$. We also have a flat $\mathbb{C}^\times$ connection in the fiber circle direction over C; let's denote its holonomy by $z \in \mathbb{C}^\times$.

Using cancellation in pairs of holomorphic polygons due to symmetry along the dotted line shown in Figure 8a, we can show that

Proposition 4.2. $(L, \boldsymbol{b}, \nabla^z)$ *is weakly unobstructed.*

Thus the superpotential associated to $(L, \boldsymbol{b}, \nabla^z)$ is well-defined. We can compute it explicitly.

Theorem 4.3. *The superpotential of* $(L, \boldsymbol{b}, \nabla^z)$ *is*

$$W = -uvh + h(1+z)(1+qz^{-1})$$

defined on $((u,v,h),z) \in \mathbb{C}^3 \times \mathbb{C}^\times$. *Its critical locus is*

$$\check{X} = \left\{((u,v),z) \in \mathbb{C}^2 \times \mathbb{C}^\times : uv = (1+z)(1+qz^{-1})\right\}.$$

Proof. Since the smooth fibers are conics which topologically do not bound any non-constant discs, the image of a Maslov-two disc must be either one of the regions shown in Figure 8. For the region with corners u, v, h, there is no singular conic fiber and hence there is only one holomorphic polygon over it passing through a generic marked point (corresponding to the constant section). This gives the term $-uvh$. For the region with one corner h, by Riemann mapping theorem the polygons over it are one-to-one corresponding to those bounded by a toric fiber of $K_{\mathbb{P}^1}$. They contribute $h(1+z)(1+qz^{-1})$ to W. □

4.3. **Non-commutative mirror of the 4-punctured sphere.** We can also apply the method in this section to the 4-punctured sphere in Example 1.2. By [CHLb], the result is a noncommutative resolution of the conifold (with a superpotential), which agrees with the resolved conifold constructed in Section 3.2 using the method of pair-of-pants decomposition, in the sense that their derived categories are equivalent.

Consider a collection of two circles $\mathbf{L} = \{L_1, L_2\}$ shown in Figure 9. This collection arises from the coamoeba of $\{1+x+y+axy=0\} \subset (\mathbb{C}^\times)^2$ for $a \neq 1$. This is a special case of the branes studied by a lot of physics literature (see for instance [FHKV08, OY09, NY10, UY11]) on dimer models and brane tilings. Moreover, HMS using dimer models for punctured Riemann surface was proved by Bocklandt [Boc16].

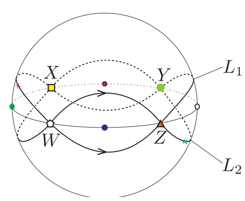


FIGURE 9. A collection of Lagrangian circles on the sphere.

By the construction described in Section 4.1, we obtain the following.

Theorem 4.4 ([CHLb])**.** *The formal weakly unobstructed deformation space of* $\mathbf{L}$ *is given by* $\mathscr{A}$ *with the superpotential* $W \in \mathscr{A}$ *where* $(\mathscr{A}, W)$ *is defined as follows.*

(1) *Q is the directed graph with two vertices v_1, v_2, two arrows $\{y, w\}$ from v_1 to v_2 and two arrows $\{x, z\}$ from v_2 to v_1.*
(2) $\mathscr{A} = \frac{\Lambda Q}{(\partial \Phi_0)}$ *is the noncommutative resolution of the conifold, where*

$$\Phi := xyzw - wzyx.$$

(3) *The superpotential $W = xyzw + wzyx$ lies in the center of $\mathscr{A}$.*

Moreover we have the A_∞-functor

$$\mathscr{F}^{\mathbf{L}}: \mathrm{WF}(\mathbb{P}^1 - \{a,b,c,d\}) \to \mathrm{MF}(\mathscr{A}, W)$$

deriving HMS. (Here WF *denotes the wrapped Fukaya category.)*

In this way we can also construct mirrors of the orbifolded compactifications $\mathbb{P}^1(p,q,r,s)$ where $p,q,r,s \in \mathbb{N}$ denote the isotropy orders. We have the same quiver Q, but the spacetime superpotential Φ and the worldsheet superpotential W become much more complicated. The case $p = q = r = s = 2$ is of particular interest: it is the elliptic orbifold $E/\mathbb{Z}_2$ and the expression of W involves important modular forms. The readers are referred to [CHLb, LZ15, BRZ15] for more detail.

Theorem 4.5 ([CHLb])**.** *For $\mathbb{P}^1(2,2,2,2) = E/\mathbb{Z}_2$, there is a T^2-family of Lagrangian branes $(\mathbf{L}_t, \nabla^\lambda)$ where $(t-1) \in \mathbb{R}/2\mathbb{Z}$ and $\lambda \in U(1)$ such that*

(1) the associated noncommutative algebras are $\mathscr{A}_{(\lambda,t)} = \frac{\Lambda Q}{(\partial \Phi_{(\lambda,t)})}$ where

$$\Phi = a(\lambda,t)\, xyzw + b(\lambda,t)\, wzyx + \frac{1}{2} c(\lambda,t)\, ((wx)^2 + (yz)^2) + \frac{1}{2} d(\lambda,t)\, ((xy)^2 + (zw)^2).$$

(2) The coefficients $(a(\lambda,t) : b(\lambda,t) : c(\lambda,t) : d(\lambda,t))$ defines an embedding $T^2 \to \mathbb{P}^3$ onto the complete intersection of two quadrics (which is an elliptic curve) defined by

$$x_1 x_3 = x_2 x_4$$
$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + \sigma x_1 x_3 = 0$$

where $\sigma = \frac{\psi}{\phi}$ (which is the inverse mirror map) and

$$\phi(q_d) = \sum_{k,l\geq 0}^{\infty} (4k+1) q_d^{(4k+1)(4l+1)} + \sum_{k,l\geq 0}^{\infty} (4k+3) q_d^{(4k+3)(4l+3)},$$
$$\psi(q_d) = \sum_{k,l\geq 0}^{\infty} (k+l+1) q_d^{(4k+1)(4l+3)}.$$

(3) The family of noncommutative algebra $\mathscr{A}_{\lambda,t}/(W_{\lambda,t})$ near $t = 0, \lambda = 1$ gives a quantization of the complete intersection given by the above two quadratic equations in $\mathbb{C}^4$ in the sense of [EG10].

5. AN APPLICATION: MIRROR OF ATIYAH FLOP AND STABILITY CONDITIONS

As an application, we can use SYZ and its generalization to study mirror symmetry over a global moduli. There are three important geometric scenarios: flop, crepant resolution, and geometric transition. In this section we focus on Atiyah flop in the work of [FHLY]. The readers are referred to [CCLT14] for crepant resolution and [Lau14, KL] for geometric transition in this approach.

The Atiyah flop contracts a $(-1,-1)$ curve and resolves the resulting conifold singularity by a small blow-up, getting a $(-1,-1)$ curve in another direction, see Figure 10. We would like to answer the question: what is the mirror operation of the Atiyah flop?

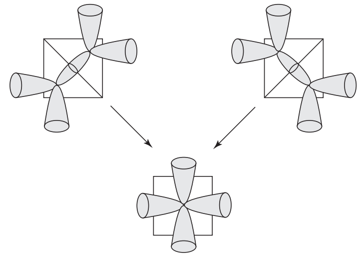


FIGURE 10. The Atiyah flop.

Consider the SYZ mirror of a conifold singularity, which is well-known by the works of [Gro01a, CLL12, CnBM14, AAK16, CPU, KL]. A conifold singularity is given by $u_1 v_1 = u_2 v_2$ in $\mathbb{C}^4$. There are two different choices of anti-canonical divisors which turn out to be mirror to each other, namely $D_1 = \{u_2 v_2 = 1\}$ and $D_2 = \{(u_2 - 1)(v_2 - 1) = 0\}$. Now consider the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, with the divisor D_2 deleted. Its SYZ mirror is given by the deformed conifold

$$X = \{(u_1, v_1, u_2, v_2, z) \in \mathbb{C}^4 \times \mathbb{C}^\times : u_1 v_1 = z + q, u_2 v_2 = z + 1\}$$

The deformed conifold contains a Lagrangian sphere whose image in the z-coordinate projection is the interval $[-q, -1] \subset \mathbb{C}$. See Figure 11. (Here we just consider $q > 1$ to be a real parameter.) The Lagrangian sphere is mirror to the holomorphic sphere in the resolved conifold.

Now take the Atiyah flop. The Kähler moduli of the resolved conifold is the punctured real line $\mathbb{R} - \{0\}$, consisting of two Kähler cones $\mathbb{R}_+$ and $\mathbb{R}_-$ of the resolved conifold and its flop respectively. A serves as the standard coordinate and flop takes $A \in \mathbb{R}_+$ to $-A \in \mathbb{R}_-$. Thus the Atiyah flop amounts to switching A to $-A$, or equivalently q to q^{-1}. As a result, the SYZ mirror changes from $X = \{u_1 v_1 = z + q, u_2 v_2 = z + 1\}$ to $\{u_1 v_1 = z + q^{-1}, u_2 v_2 = z + 1\}$.

However the above two manifolds are symplectomorphic to each other, and hence they are just equivalent from the viewpoint of symplectic geometry. Unlike Atiyah flop in complex geometry, the mirror operation does not produce a new symplectic manifold. It is not very surprising since symplectic geometry is much softer than complex geometry.

We need to endow a symplectic threefold with additional geometric structures in order to make it more rigid, so that the effect of the mirror flop can be seen. In [FHLY] we considered two kinds of geometric structures, namely Lagrangian fibrations, and Bridgeland stability conditions on the derived Fukaya category. Let's focus on stability conditions here.

Suppose we have a Bridgeland stability condition $(Z, \mathcal{S})$ on the derived Fukaya category, where Z is a homomorphism of the K group to $\mathbb{C}$ known as the central charge, and $\mathcal{S}$ is a collection of objects in the derived Fukaya category which are said to be stable. (It satisfies some axioms [Bri07], the most important one being the Harder-Narasimhan property.) In an ideal geometric situation, $\mathcal{S}$ is the collection of graded special Lagrangians with respect to a certain holomorphic volume form Ω, and Z is given by the period $\int_\cdot \Omega$.

Then the mirror flop (along a certain Lagrangian sphere S) should be understood as a change of stability conditions $(Z,\mathscr{S}) \mapsto (Z^\dagger,\mathscr{S}^\dagger)$. Namely given a Lagrangian $L \in \mathscr{S}$, the mirror flop is a certain surgery $L^\dagger$ of L (along the sphere S). Thus we obtain another collection of Lagrangians $\mathscr{S}^\dagger$, and we impose that $Z^\dagger(L^\dagger) = Z(L)$. The mirror flop should be realized as an involution on the moduli space of stability conditions.

Unfortunately it is hard to construct stability conditions in general. Using the methods of mirror construction introduced in previous sections, we can construct stability conditions via mirror symmetry. In [FHLY] we focused on the deformed conifold and the result is the following. We will study deformed generalized conifolds and cotangent bundles of lens spaces in ongoing papers.

Theorem 5.1 ([FHLY]). *The mirror construction in* [CHLb] *applied to the deformed conifold X produces the noncommutative resolved conifold $\mathscr{A} = \frac{\Lambda Q}{(\partial \Phi_0)}$ given in Theorem 4.4. In particular we have the equivalence of triangulated categories*

$$D\mathrm{Fuk}(X) \cong D\mathrm{mod}(\mathscr{A}).$$

The collection of Lagrangians we use to construct the mirror in the above theorem is depicted in Figure 11. It consists of two Lagrangian spheres, whose images in the base of the double conic fibration $z : X = \{u_1 v_1 = z+q, u_2 v_2 = z+1\} \to \mathbb{C}$ are curves shown in the figure.

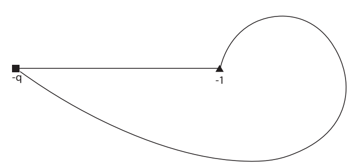


FIGURE 11. The two Lagrangians in the deformed conifold to construct the mirror.

Note that the mirror of the deformed conifold X is the noncommutative resolved conifold $\mathscr{A}$ rather than the Landau-Ginzburg model $(\mathscr{A}, W)$ (which is mirror to the 4-punctured sphere in Section 4.3).

Stability conditions on $D\mathrm{mod}(\mathscr{A})$ have been extensively studied by Toda [Tod08] and Nagao-Nakajima [NN11]. Combining with the above mirror statement, we proved the following.

Theorem 5.2 ([FHLY]). *Let X be the deformed conifold $\{u_1 v_1 = z + q, u_2 v_2 = z + 1, z \neq 0\}$ (where $q \neq 1$). Equip X with the holomorphic volume form $\Omega = dz \wedge du_1 \wedge du_2$. There exists a collection $\mathscr{S}$ of graded special Lagrangians which defines a stability condition $(Z,\mathscr{S})$ on $D\mathrm{Fuk}(X)$.*

The flop $(Z^\dagger, \mathscr{S}^\dagger)$ *is another stability condition respecting the holomorphic volume form* $\rho^*\Omega_{X^\dagger}$*, where* ρ *is a symplectomorphism from* X *to*

$$X^\dagger = \{u_1 v_1 = z+1, u_2 v_2 = z + 1/q : z \neq 0\}$$

and $\Omega_{X^\dagger} = dz \wedge du_1 \wedge du_2$ *on* $X^\dagger$.

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