Abstracts

Open Gromov-Witten invariants on toric manifolds

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(joint work with K.W. Chan, N.C. Leung, H.H. Tseng)

Let X be a compact toric manifold of complex dimension n and $q \in H^2(X,\mathbb{C})$ be a complexified Kähler class of X. When $-K_X$ is numerically effective, we extract the open Gromov-Witten invariants of X from its mirror map. This gives an open analogue of closed-string mirror symmetry discovered by Candelas-de la Ossa-Green-Parkes [1]. Namely, under mirror symmetry, the computation of open Gromov-Witten invariants is transformed to a PDE problem of solving Picard-Fuchs equations.

The mirror of (X, q) is defined to be a certain holomorphic function W_q on $(\mathbb{C}^\times)^n$ called the superpotential[1]. Closed-string mirror symmetry states that the deformation of W_q encodes Gromov-Witten invariants of X. More precisely, it states that there is an isomorphism

$$QH^*(X,q) \cong \mathrm{Jac}(W_q)$$

as Frobenius algebras, where $QH^*(X,q)$ denotes the small quantum cohomology ring of (X,q) and

$$\mathrm{Jac}(W_q) := \frac{\mathbb{C}[z_1^{\pm 1},\ldots,z_n^{\pm 1}]}{\left\langle z_1 \frac{\partial W_q}{\partial z_1},\ldots,z_n \frac{\partial W_q}{\partial z_n}\right\rangle}$$

is the Jacobian ring of W_q.

Based on physical arguments, Hori-Vafa [6] gave a recipe to write down a Laurent polynomial W_q° from the fan configuration of X. It turns out that W_q° only records the 'leading order terms' and is not equal to W_q in general. The difference $W_q - W_q^\circ$ is called instanton corrections.

Traditionally, the instanton corrections are written down from the PDE approach. Namely, one writes down a Picard-Fuchs system using the fan configuration of X, and solves it explicitly for the 'mirror map' $\check{q}(q)$. Then define

$$W_q^{\mathrm{PF}} := W_{\check{q}(q)}^\circ.$$

When the anti-canonical line bundle $-K_X$ is numerically effective, W_q^{PF} fits into the mirror symmetry framework mentioned above, namely

$$QH^*(X,q) \cong \mathrm{Jac}(W_q^{\mathrm{PF}})$$

as Frobenius algebras [5, 7].

On the other hand, the instanton corrections are realized by Fukaya-Oh-Ohta-Ono [4] using open Gromov-Witten invariants as follows. Let $\mathbf{T} \subset X$ be a Lagrangian toric fiber and $\pi_2(X,\mathbf{T})$ be the set of homotopy classes of maps

[1] More precisely W is a formal series over the Novikov field. Here we assume it is convergent in some neighborhood of $q = 0$, and thus defines a holomorphic function on $(\mathbb{C}^\times)^n$.

$(\Delta, \partial\Delta) \to (X, \mathbf{T})$, where Δ denotes the closed unit disk. For $\beta \in \pi_2(X, \mathbf{T})$, the moduli space $\mathcal{M}_1(\beta)$ of stable disks representing β and its virtual fundamental class $[\mathcal{M}_1(\beta)] \in H_n(\mathbf{T})$ are defined by use of Kuranishi structures. The one-pointed Gromov-Witten invariant associated to β is defined as

$$n_\beta := \int_{[\mathcal{M}_1(\beta)]} \mathrm{ev}^*[\mathrm{pt}]$$

where $[\mathrm{pt}] \in H^n(\mathbf{T})$ is the point class and $\mathrm{ev} : \mathcal{M}_1(\beta) \to \mathbf{T}$ is the evaluation map. Then

$$W_q^{\mathrm{LF}} := \sum_{\beta \in \pi_2(X,\mathbf{T})} n_\beta Z_\beta$$

gives another definition of the instanton-corrected superpotential, where Z_β is an explicitly written monomial for each β. Notice that the above formal sum involves infinitely many terms in general, and is well-defined over the Novikov field. Fukaya-Oh-Ohta-Ono [3] proved that

$$QH^*(X, q) \cong \mathrm{Jac}(W_q^{\mathrm{LF}})$$

as Frobenius algebras.

While W_q^{PF} and W_q^{LF} originates from totally different approaches, they lead to the same mirror symmetry statements. The following conjecture is made in a joint work with Chan, Leung and Tseng [2]:

Conjecture 1 ([2])**.** *Let X be a toric manifold with $-K_X$ numerically effective, and let W^{PF} and W^{LF} be the superpotentials in the mirror as introduced above. Then*

$$W^{PF} = W^{LF}. \tag{1}$$

Conjecture 1 can be proved under the technical assumption that W^{LF} converges analytically (instead of just being a formal sum):

Theorem 2 ([2])**.** *Let X be a toric manifold with $-K_X$ numerically effective, and let W^{PF} and W^{LF} be the superpotentials in the mirror as explained above. Then*

$$W^{PF} = W^{LF}$$

provided that each coefficient of W^{LF} converges in an open neighborhood around $q = 0$.

The above technical assumption on the convergence of W^{LF} is satisfied when $\dim X = 2$, or when X is of the form $\mathbb{P}(K_S \oplus \mathcal{O}_Y)$ for some toric Fano manifold Y.

The function W^{LF} is a generating function of open Gromov-Witten invariants n_β (and thus can be regarded as an object in the 'A-side'), whereas W^{PF} arises from solving Picard-Fuchs equations (and so is an object in the 'B-side'). Using this equality, the task of computing the open Gromov-Witten invariants is transformed to solving Picard-Fuchs equations which has been known to experts. Thus the above equality gives a mirror symmetry method to compute open Gromov-Witten invariants.

References

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Reporter: Lars Kastner