TORIC, GLOBAL, AND GENERALIZED SYZ

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Abstract. In this talk I introduce some of my recent projects in SYZ mirror symmetry. They include SYZ constructions for compact toric manifolds and local Calabi-Yau manifolds, local conifold transitions, and a generalized version of SYZ which should naturally produce homological mirror symmetry for a class of compact Calabi-Yau manifolds.

Strominger-Yau-Zaslow (SYZ) [46] proposed that mirror symmetry can be understood in terms of duality of special Lagrangian torus fibrations. Namely, the mirror manifold can be constructed by taking fiberwise torus dual of the original manifold, and Lagrangian branes can be transformed to coherent sheaves on the mirror by a real version of Fourier-Mukai transform. The SYZ program reveals the geometric origin of mirror symmetry.

The construction can be summarized into three steps:

(1) *Construct a Lagrangian torus fibration.* Topological and Lagrangian torus fibrations on the Fermat quintic were constructed by Gross [24] and Castano-Bernard-Matessi [3] respectively. Interesting local examples were studied by Gross [23].
(2) *Take T-duality away from singular fibers to obtain the semi-flat mirror.* Each non-singular Lagrangian torus fiber has a dual, and the union of all such dual torus fibers form a (typically non-compact) complex manifold known as the semi-flat mirror. This construction was studied by [35, 36, 29].
(3) *Carry out quantum corrections on the semi-flat mirror to obtain the true mirror.* Fiberwise torus duality away from singular fibers only gives the first-order approximation of the mirror. one needs to capture the additional information of holomorphic discs emanated from singular fibers in order to reconstruct the genuine mirror. These holomorphic discs interact with each other and scatter in a complicated way, whose tropical version was studied by Kontsevich-Soibelman [30] and Gross-Siebert [25] (and [26] gave an excellent survey on their work).

In general discriminant loci of a Lagrangian fibration are rather complicated, and quantum corrections can only be computed order-by-order using the Gross-Siebert program. However, how to carry out quantum corrections on Fourier-Mukai transform between Lagrangian branes and coherent sheaves over the mirror is in general an open problem. Apparently *the way from SYZ to homological mirror symmetry is blocked by quantum corrections occurring up to infinite order.*

There are situations where quantum corrections are relatively simple and can be explicitly handled by the theory of open Gromov-Witten invariants, namely toric Calabi-Yau manifolds [8], compact toric orbifolds [6] and their conifold transitions [32]. These will be the topics in the first and second parts of my talk. In these cases we show that the (inverse) mirror map has an enumerative meaning in terms of open Gromov-Witten invariants [33, 34, 7, 11, 9, 10, 5], which are usually difficult to compute due to the lack of machinery. Our result computes *all* the open Gromov-Witten invariants of Lagrangian torus fibers.

In order to understand the geometric origin of homological and closed-string mirror symmetry *without invoking infinitely many quantum corrections*, we introduced a generalized version of SYZ mirror symmetry [13]. This will be the third part of my talk. The original SYZ approach is based on T-duality in which tori play a central role. The generalized version is based on (immersed) Lagrangian Floer theory rather than T-duality: one considers general Lagrangians (with mild singularities) rather than restricting to tori. An A_∞-functor naturally comes out, which should lead to homological mirror symmetry when the chosen Lagrangian split-generates (such as orbifold projective lines $\mathbb{P}^1_{a,b,c}$ and Fermat hypersurfaces). This explains the *geometric origin of homological mirror symmetry.*

1. Toric SYZ

1.1. **Compact toric manifolds.**

SYZ for compact toric manifolds has been well-studied by the works of [14, 12, 19, 20]. The construction in this case is rather simple and serves as an excellent starting point:

(1) The god-given moment map over a toric manifold X serves as a Lagrangian torus fibration for carrying out SYZ.
(2) All singular fibers are located at the boundary of the moment-map polytope. By taking fiberwise dual one obtains (a domain of) $(\mathbb{C}^\times)^n$ where $n = \dim X$.
(3) Quantum corrections are captured by the weighted sum of *open Gromov-Witten invariants* with one boundary marked point. This gives a function $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ known as the superpotential. [1]

Defining open Gromov-Witten invariants takes tremendous effort, and we will not spend our time there. Roughly speaking, open Gromov-Witten invariants in our context means the *counting of holomorphic discs bounded by a Lagrangian torus fiber passing through a generic chosen point on the boundary*, and the readers are referred to [19, 20, 17, 18] for details.

The two-sphere $\mathbb{P}^1$ with an area form ω is the best example to illustrate these concepts. See Figure 1. One obtains $(\mathbb{C}^\times, W)$ with

$$W = z + \frac{q}{z}$$

where $q = \mathrm{e}^{-\int_{\mathbb{P}^1}\omega}$. The left and right hemispheres are the only discs with non-trivial invariants. They both have invariants 1; summing them weighted by their areas gives the above superpotential with two terms.

While the superpotential for $\mathbb{P}^1$ may look rather boring, note that in general open Gromov-Witten invariants are difficult to compute and the superpotential can have a complicated form. We will go back to this point soon.

[1] More rigorously, one should use Novikov ring instead of complex numbers, especially when $-K_X$ is not nef, because W may not converge over $\mathbb{C}$. But for simplicity we pretend everything is over the complex field $\mathbb{C}$.

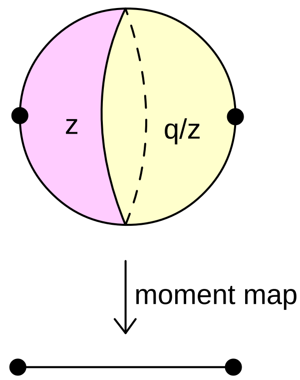


FIGURE 1. The two-sphere and its moment map.

1.2. **Toric Calabi-Yau manifolds.** Toric Calabi-Yaus serve as local examples of Calabi-Yau manifolds. Here Calabi-Yau condition simply means that the canonical divisor K_X is linearly equivalent to 0. Examples include $\mathbb{C}^n$ and the total space of the canonical line bundle $K_{\mathbb{P}^{n-1}}$ of $\mathbb{P}^{n-1}$. (Note that the Fermat-type hypersurfaces in $\mathbb{P}^{n-1}$ can be obtained as the critical locus of a superpotential over $K_{\mathbb{P}^{n-1}}$.)

SYZ for toric Calabi-Yaus is different from SYZ for compact toric manifolds: we want to construct a Calabi-Yau mirror instead of a Landau-Ginzburg mirror, because the toric Landau-Ginzburg mirror $((\mathbb{C}^\times)^n, W)$ does not capture enough information. All holomorphic curves in a toric Calabi-Yau manifold are contained in the toric divisors by maximal principle. These are the crucial elements that the mirror Calabi-Yau variety should capture, yet they are hidden in the boundary strata of the moment map.

The idea is to deform the Lagrangian fibration initially given by the moment map such that (part of) its discriminant locus moves to the interior. Such deformations were explicitly constructed by Goldstein [22] and Gross [23] independently. Figure 3a gives an illustration for the case of A_n-resolution, and the readers are referred to [8] for the details. The main point is, once discriminant locus occurs in the interior, *quantum corrections not only gives rise to an additional superpotential, but also changes the structure of the mirror space (even topologically).*

In my joint work with Chan and Leung [8], we proposed a general procedure, which is different from the Gross-Siebert program, to carry out SYZ construction with quantum corrections. It used Fourier transform of *open Gromov-Witten invariants* while the Gross-Siebert program used tropical geometry. Applying the construction to toric Calabi-Yau manifolds gives:

Theorem 1.1 (Theorem 4.37 of [8]). *Let (X,ω) be a toric Calabi-Yau n-fold. The SYZ mirror $\mathcal{F}_{\mathrm{SYZ}}(X,\omega)$ is $(\check{X},\check{\Omega})$, where*

$$\check{X} = \left\{ (u,v,z) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_{l=1}^{m} (1+\delta_l(q)) Z_l(q,z) \right\}$$

where q is the flat coordinate system on the Kähler moduli of X around the large volume limit, $Z_l(q,z)$ are certain explicit monomials in (q,z) for $l,\ldots,m$ and $\delta_l(q)$ for $l=1,\ldots,m$

are certain explicitly defined generating functions of open Gromov-Witten invariants of X; $\check{\Omega}$ is a holomorphic volume form on $\check{X}$ obtained by taking Fourier transform of ω.

The SYZ mirror $\mathcal{F}_{\mathrm{SYZ}}(X, \omega)$ is defined by a Laurent polynomial, which has the same form as the Hori-Iqbal-Vafa mirror [27]. *One crucial extra feature of the SYZ mirror compared with the Hori-Iqbal-Vafa mirror is that, it is intrinsically expressed in terms of flat coordinates.*

The key feature of this work is *wall-crossing* of open Gromov-Witten invariants. This type of wall-crossing phenomenons was first observed by Auroux [2] in a non-toric Lagrangian fibration on $\mathbb{P}^2$. Open Gromov-Witten invariants bounded by a Lagrangian fiber may undergo sudden change when the fiber moves around. On the other hand, a point moving around the mirror Calabi-Yau never undergoes any sudden change. We will encounter a similar phenomenon in conifold transition presented later in this talk.

Our work [8] explicitly computed wall-crossing for all toric Calabi-Yau manifolds in a uniform way. See Figure 3b. First consider discs emanated from below. A wall separates the base into two chambers. The disc potential below the wall is simply a monomial ζ, while the disc potential above the wall is $\zeta g(z)$. Thus one obtains the function

$$u = \begin{cases} \zeta g(z) & \text{above the wall.} \\ \zeta & \text{below the wall.} \end{cases}$$

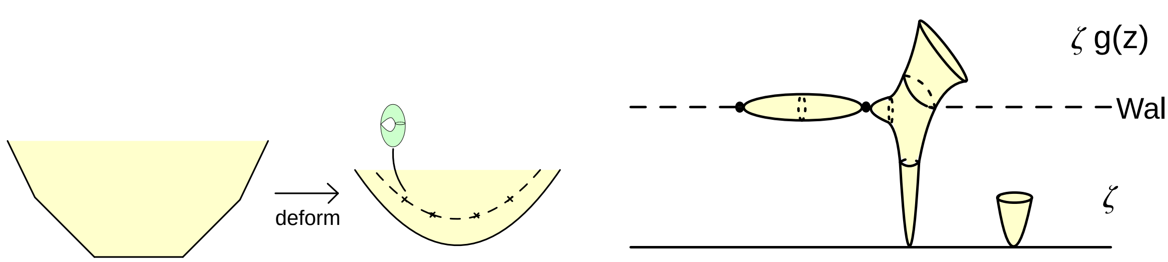
for certain Laurent polynomial g. The same consideration for discs emanated from the above gives the function

$$v = \begin{cases} \zeta^{-1} & \text{above the wall.} \\ \zeta^{-1} g(z) & \text{below the wall.} \end{cases}$$

This explains why the mirror has the form

$$uv = g(z).$$

FIGURE 2. The A_n resolution and wall-crossing.



(A) Deformation of Lagrangian fibration. The left shows the moment map image. After deformation singular fibers occur in the interior as shown on the right. The dotted line is the wall.

(B) Wall-crossing. There is only one disc below the wall, giving rise to the disc potential ζ. Additional discs come up when crossing the wall, giving rise to the disc potential $\zeta g(z)$.

1.3. **Open mirror theorem.** The mirror map is a central object in mirror symmetry. It provides a canonical local isomorphism between the Kähler moduli and the mirror complex moduli near the large complex structure limit. The success of mirror symmetry on counting rational curves in the quintic threefold essentially relies on identifying the mirror map. Mirror map arises from the classical study of deformation of complex structures and Hodge structures and can be computed explicitly by solving Picard-Fuchs equations.

Integrality of the coefficients of mirror maps has been studied [47, 41, 31]. Conjecturally these integers should have enumerative meanings in terms of disc counting. In the tropical setting of toric degenerations a conjecture of this type was made in the foundational work of Gross-Siebert [25].

Let's denote the (inverse) mirror map by

$$\mathfrak{F}_{\text{mirror}} : \mathcal{M}^A \to \check{\mathcal{M}}^B$$

from the (complexified) Kähler moduli $\mathcal{M}^A$ to the moduli space $\check{\mathcal{M}}^B$ of complex structures on the mirror. Now the SYZ construction also gives a map

$$\mathcal{F}_{\text{SYZ}} : \mathcal{M}^A \to \check{\mathcal{M}}^B.$$

This map is defined canonically in terms of open Gromov-Witten invariants.

Using the SYZ map $\mathcal{F}_{\text{SYZ}}$, the *open mirror symmetry conjecture* can be formulated as the following neat equality:

$$\mathfrak{F}_{\text{mirror}} = \mathfrak{F}_{\text{SYZ}}. \tag{1.1}$$

Since the right hand side of Equation (1.1) are expressed in terms of open Gromov-Witten invariants, it gives *an enumerative meaning of the mirror map* $\mathfrak{F}_{\text{mirror}}$.

Furthermore, mirror map $\mathfrak{F}_{\text{mirror}}$ is a classical object and typically can be computed by solving Picard-Fuchs equations. Thus Equation 1.1 gives a *computation of open Gromov-Witten invariants.* This is an open analogue of closed-string mirror symmetry which computes Gromov-Witten invariants by geometry of the mirror side.

Equation 1.1 was recently completely proved in the semi-Fano toric case in my work jointly with Chan, Leung and Tseng, and also for toric Calabi-Yaus in my joint work with Chan, Cho and Tseng:

Definition 1.2. *A Kähler manifold X is said to be semi-Fano if $-K_X$ is numerically effective. It is said to be Calabi-Yau if $-K_X$ is linearly equivalent to* 0*.*

Theorem 1.3 (Theorem 1.2 of [10] and Theorem 1.5 of [5])**.** *For toric Calabi-Yau manifolds and compact toric semi-Fano manifolds, we have*

$$\mathfrak{F}_{\text{mirror}} = \mathfrak{F}_{\text{SYZ}}.$$

Indeed the theorem can be stated in greater generality including orbifolds, but here we restrict ourselves to manifolds for simplicity. Theorem 1.3 was first proved for A_n resolutions [34] and compact toric semi-Fano surfaces [7]. Then in [11] we proved it for local Calabi-Yau manifolds of the form K_Y, where Y is a toric Fano manifold. Assuming convergence of the superpotential, [9] gave a proof for general compact toric semi-Fano manifolds. The convergence assumption was later removed [10]. It was completely proven for toric Calabi-Yau manifolds in [5].

To demonstrate the use of Theorem 1.3, consider $X = K_{\mathbb{P}^2}$. $\mathcal{F}_{\mathrm{mirror}}$ is basically given by inverse of the power series

$$\check{q}\exp\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\frac{(3k)!}{(k!)^3}\check{q}^k\right).$$

Using Theorem 1.3, the open Gromov-Witten invariants can be extracted from the coefficients of the inverse of the above power series. The first few terms are

$$n(l) = -2, n(2l) = 5, n(3l) = -32, n(4l) = 286, n(5l) = -3038, n(6l) = 35870, \ldots$$

where l is the line class of $\mathbb{P}^2$.

We expect Equation (1.1) holds, as long as one can make a suitable sense of the SYZ construction. In the third part of the talk I will introduce a generalized version of SYZ using immersed Lagrangian Floer theory instead of T-duality.

1.4. **Open-closed relations.** The basic idea to prove Theorem 1.3 is to use open-closed relations. The main difficulty for handling open Gromov-Witten invariants is the lack of computational techniques due to the presence of boundary strata of the moduli. On the other hand, computational techniques for closed invariants such as localization and mirror principle [21, 37, 38, 39, 40] are well-developed. Thus we try to identify the open invariants involved in $\mathcal{F}_{\mathrm{SYZ}}$ with certain closed invariants, and then use the mirror principle to compute them. (Another approach using universal unfolding which is more analytic in nature was used in [9]. I will skip it due to lack of time.)

The open-closed relation we start with was the following (see Figure 4 for an illustration in the case $X = \mathbb{F}_2$ is the Hirzebruch surface):

Theorem 1.4 (Open-closed relation I [4, 34])**.** *Let X be either a toric Calabi-Yau manifold, or a compact semi-Fano toric surface. Let $\beta \in \pi_2(X, T)$ be a stable disc class bounded by a Lagrangian torus fiber T. Then there exists an explicit toric manifold $\tilde{X}$ of the same dimension as X and a curve class $\bar{\beta} \in H_2(\tilde{X})$ such that*

$$n_1^X(\beta) = \langle[\mathrm{pt}]\rangle_{0,1,\bar{\beta}}^{\tilde{X}}$$

where $n_1^X(\beta)$ denotes the open Gromov-Witten invariant associated to β.

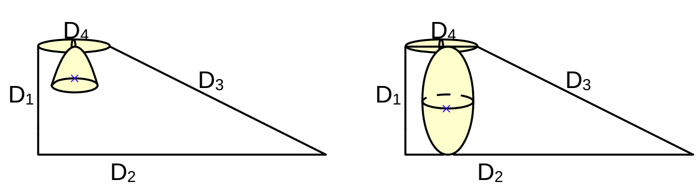


FIGURE 4. Compactifying a disc (with sphere bubblings) in $\mathbb{F}_2$ to a rational curve.

The above theorem transforms open invariants of toric Calabi-Yau manifolds and compact semi-Fano toric surface to closed invariants that we need to compute and thereby deduces

Theorem 1.3 in these cases (together with the use of the mirror theorem [21, 39]). However it poses topological restrictions on the manifold in order to 'close up the discs'. To remove such a restriction, we go up one more dimension to the Seidel space (see Figure 5 for an illustration where $X = \mathbb{P}^1$):

Theorem 1.5 (Open-closed relation II [10])**.** *Let X be a compact semi-Fano toric manifold, D_l be a toric prime divisor and $\beta \in \pi_2(X)$ be a stable disc class bounded by a Lagrangian torus fiber T. Then there exists an explicit Seidel space E associated to X and $\tilde{\beta} \in H_2(E)$ such that*

$$n_{1,1}^X(\beta; D_l) = \langle D_l, [pt]_E \rangle_{0,2,\tilde{\beta}}^E \tag{1.2}$$

where $n_{1,1}^X(\beta; D_l)$ denotes the open Gromov-Witten invariant relative to D_l associated to β.

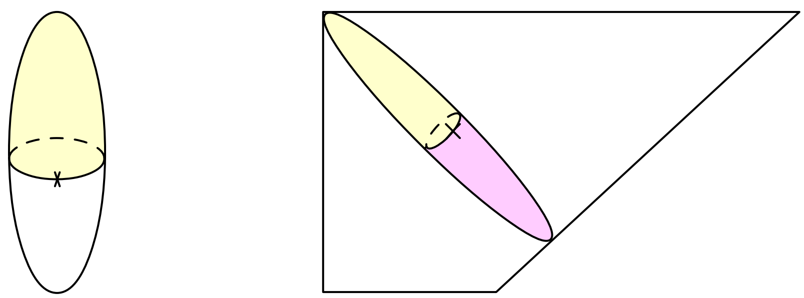


Figure 5. Pushing a disc to the associated Seidel space and compactify it to a sphere. The left hand side shows a disc in X. The right hand side is the associated Seidel space and rational curve.

Note that the Seidel space might not be semi-Fano, even when X itself is semi-Fano. Thus the computation of closed Gromov-Witten invariants of E is more complicated than one might imagine, which involves the use of Seidel representations. The readers are referred to [10] for details.

2. Global SYZ

We have mainly focused around a large volume limit. On the other hand, there are other interesting limit points in the global Kähler moduli. Topologies of Kähler manifolds parametrized by the Kähler moduli undergo non-trivial changes passing from one limit point to the other. An important class is given by conifold transitions: Figure 6 shows a typical local picture of a conifold transition.

It is expected that the picture for the B-side is much simpler and more classical. namely, complex manifolds parametrized by the complex moduli around different limit points are simply related by analytic continuations. There are monodromies around the limit points, but the topology never undergoes sudden change.

Remark 2.1. *Indeed we have already discussed a phenomenon having this sort of flavor over the open moduli, namely, wall-crossing of open Gromov-Witten invariants of Lagrangian torus fibers. There is a chamber structure on the base of the Lagrangian fibration due to*

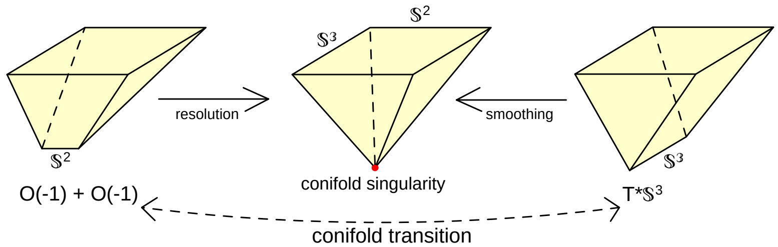


FIGURE 6. Local picture of a conifold transition. The picture on the left shows the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ over $\mathbb{P}^1 = \mathbb{S}^2$.

non-trivial change of open Gromov-Witten invariants across the wall, but the SYZ mirror complex manifold is smooth and does not carry any chamber structure.

There are many existing literatures on wall-crossing (over closed moduli) in the A-side, including wall-crossing of Donaldson-Thomas invariants, crepant resolution conjecture and Landau-Ginzburg correspondence. Here I would like to talk about my work [32] on *SYZ under conifold transitions of toric Calabi-Yau manifolds*. While there are topological changes across a transition, we will see that their SYZ mirrors (as complex manifolds) are related by analytic continuations.

2.1. **Conifold transitions of toric Calabi-Yau.** Conifold transitions of toric Calabi-Yaus can be described by beautiful combinatorics. Let P be a lattice polytope. Placing P to level one and taking cone gives the fan of a toric Gorenstein singularity X. Triangulations of P give (partial) resolutions Y of X. On the other side, due to Altmann [1] Minkowski decompositions of P give (partial) smoothings X_t of X. Then X_t is a conifold transition of Y. See Figure 7 for an example of such a construction.

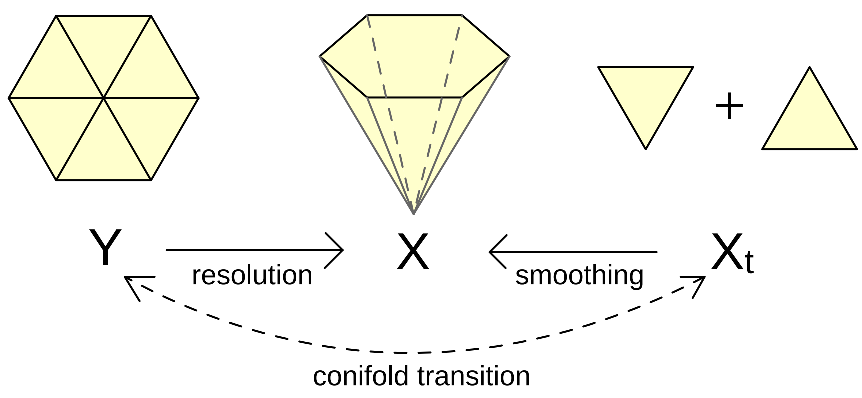


FIGURE 7. An example of conifold transition of local Calabi-Yau manifolds. Triangulation of the hexagon shown on the left corresponds to the total space of canonical line bundle of the del Pezzo surface dP_6. (One of) its conifold transition is shown by the figure on the right.

2.2. **Lagrangian fibrations.** Recall that the first step of SYZ is constructing Lagrangian fibrations. On smoothing of a toric Gorenstein singularity Lagrangian fibrations were constructed by Gross [23]. The construction is similar to that for toric Calabi-Yau manifolds using symplectic reductions.

The base of the Lagrangian fibration is the upper half space. The discriminant loci are topologically duals of the polygons involved in the Minkowski decomposition, and each of them is contained in a hyperplane. See Figure 8. The dotted lines show the discriminant loci. One sees that Lagrangian fibration over the smoothing X_t is obtained by suitably pulling apart the discriminant locus of Lagrangian fibration over the singular variety X.

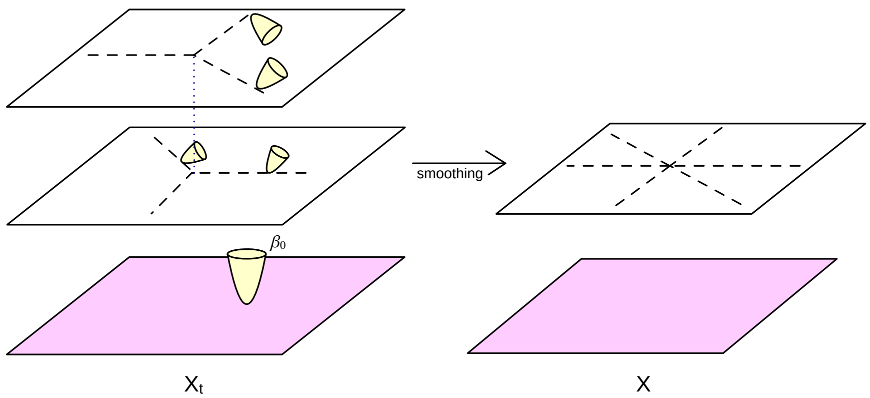


FIGURE 8. Wall-crossing of open Gromov-Witten invariants in a conifold transition of K_{dP_6}.

2.3. **The SYZ mirrors of smoothings of toric Gorenstein singularities.** As in the case of toric Calabi-Yaus, the key phenomenon for quantum corrections is wall-crossing of open Gromov-Witten invariants. As we have discussed, for a toric Calabi-Yau manifold there is a hyperplane (called the wall) in the base, and across the wall open Gromov-Witten invariants jump due to the interaction with holomorphic discs bounded by fibers over the wall.

For its conifold transition, I proved that it has several parallel walls, which are the hyperplanes containing the discriminant loci [32]. See Figure 8 for an illustration. In the lowest chamber there is only one holomorphic disc with non-trivial invariant; There are holomorphic discs (of Maslov index zero) bounded by fibers over the wall; when crossing each of the walls, these holomorphic discs interact with discs below the wall and then produce more holomorphic discs above the wall. By analyzing the open moduli and computing the open Gromov-Witten invariants in each chamber, one obtains the following expression for the SYZ mirror:

Theorem 2.2 (Theorem 1.1 of [32]). *For a (total) smoothing of a toric Gorenstein singularity coming from a Minkowski decomposition of the corresponding polytope, its SYZ mirror is*

$$uv = \prod_{i=0}^{p} \left(1 + \sum_{l=1,\ldots,k_i} z^{u_l^i} \right) \tag{2.1}$$

where u_l^i are vertices of the simplices appearing in the Minkowski decomposition.

Remark 2.3. *Note that* the smoothings $\mathcal{X}_t$ are no longer toric*. Open Gromov-Witten invariants for non-toric cases are usually difficult to compute and only known for certain isolated cases (such as the real Lagrangian in the quintic* [42]*). For this class of non-toric manifolds* all the relevant open Gromov-Witten invariants are computed explicitly*.*

The right hand side of Equation (2.1) gives a factorization of a polynomial whose Newton polytope is P, which is the polytope that we start with to construct the conifold transition. Thus SYZ mirror symmetry realizes the magical duality between decomposition of polytopes and factorization of polynomials. The following diagram summarizes the dualities that we have and their relations:

$$\begin{tikzcd}
\text{Polytope decompositions} \arrow[rr, leftrightarrow, "\text{Newton}"] \arrow[dr, leftrightarrow, "\text{Altmann}"'] & & \text{Polynomial factorizations} \arrow[dl, leftrightarrow, "\text{SYZ}"] \\
& \text{Smoothings of toric Gorenstein singularity} &
\end{tikzcd}$$

2.4. **SYZ of conifold transitions of local Calabi-Yaus.** Theorem 2.2 gives an explicit expressions for SYZ mirrors of conifold transitions of toric Calabi-Yau manifolds. On the other hand, the SYZ mirrors of toric Calabi-Yau manifolds can be explicitly written down in terms of the mirror map due to the validity of Equation (1.1). By simple algebraic manipulations the following theorem on the behavior of SYZ mirrors under conifold transition was obtained:

Theorem 2.4 (Theorem 1.2 of [32])**.** *Let Y be a toric Calabi-Yau manifold and $\mathcal{X}_t$ be a conifold transition of Y. Then their SYZ mirrors $\check{\mathcal{X}}$ and $\check{Y}_q$ are connected by an analytic continuation: there exists an invertible change of coordinates $q(\check{q})$ and a specialization of parameters $\check{q} = \underline{\check{q}}$ such that*

$$\check{\mathcal{X}} = \check{Y}_{q(\check{q})}|_{\check{q}=\underline{\check{q}}}.$$

While the toric Calabi-Yau manifold Y and its conifold transition $\mathcal{X}_t$ is different even in the topological level, their SYZ mirrors (and also their generating functions of open Gromov-Witten invariants) belong to the same family of complex manifolds and they are related by analytic continuations.

Remark 2.5. *Similar story holds for SYZ mirror symmetry under crepant resolutions, see* [6, 5]*.*

3. Generalized SYZ

We have mentioned in the very beginning of this lecture that apparently there is a gap between SYZ and homological mirror symmetry: the order-by-order quantum corrections are so complicated and they hinder our understanding on homological mirror symmetry from the SYZ perspective.

It would be great if one has a mirror construction such that homological mirror symmetry naturally arises. What follows is an introduction to my work [13] jointly with Cho and Hong in this direction.

3.1. **The construction.** The main idea is the following: instead of restricting to Lagrangian torus fibrations in the original SYZ approach, we use a Lagrangian immersion $\mathbf{L}$ and its deformation theory to construct a Landau-Ginzburg model W.

The three steps of doing SYZ mentioned in the beginning of this lecture are modified to the following:

(1) *Construct a suitable Lagrangian immersion $\bar{L}$.* This should be oriented and (relatively) spin. For simplicity it is assumed to have transverse self-intersections. For the purpose of mirror symmetry we may want to require $\bar{L}$ split-generates the Fukaya category, although we do not need such an assumption in the construction.

(2) *We take a (weakly-unobstructed) deformation space V of $\bar{L}$ as the semi-flat mirror.* Recall that for a torus T, the dual torus T^* is given by
$$T^* = \{\nabla : \nabla \text{ is a flat } U(1) \text{ connection on } T\} = H^1(T,\mathbb{R})/H^1(T,\mathbb{Z})$$
which is the imaginary part of the space of complexified Lagrangian deformations of T:
$$H^1(T,\mathbb{R}) \oplus \mathbf{i}\,(H^1(T,\mathbb{R})/H^1(T,\mathbb{Z})).$$
Thus the deformation space of a Lagrangian immersion plays the role of the dual of a Lagrangian torus. Note that deformations of an immersed Lagrangian not just include the usual Lagrangian deformations but also smoothings at immersed points.

(3) *The quantum corrections are given by countings of J-holomorphic polygons bounded by $\bar{L}$.* The semi-flat mirror comes from deformations and only has information about a neighborhood of $\bar{L}$. Again one uses holomorphic discs to capture information of the ambient space X. They form a generating function W. Then (V,W) forms a Landau-Ginzburg model, and we call this a generalized SYZ mirror.

One advantage of such a construction is that it avoids complicated scattering and gluing, and so the Landau-Ginzburg model (V,W) comes out in a direct and natural way. This also matches the general philosophy that Landau-Ginzburg model is easier to work with than Calabi-Yau model (and it is an important topic to study the correspondence between the two).

The best example to illustrate the construction is the two-dimensional pair-of-pants $X = \mathbb{P}^1 - \{p_1,p_2,p_3\}$. Seidel [43] introduced a specific Lagrangian immersion $L \subset X$ shown in Figure 9 to prove homological mirror symmetry for genus-two curves (which indeed works for all genus shown by Efimov [16]). Later Sheridan [45, 44] generalized the construction to higher dimensions and proved homological mirror symmetry for Fermat-type hypersurfaces.

L has three immersed points, and they give three independent directions of (weakly) unobstructed deformations labelled by x,y,z. The only (holomorphic) polygon passing through a generic point of L with x,y,z's as vertices and having Maslov-index two is the triangle shown in Figure 9, which corresponds to the monomial xyz. Thus the generalized SYZ mirror of the pair of pants is $W : \mathbb{C}^3 \to \mathbb{C}$ given by
$$W = xyz.$$

We can also use the Seidel Lagrangian L to construct the generalized SYZ mirror $(\mathbb{C}^3, W)$ of the orbifold projective line $\mathbb{P}^1_{a,b,c}$ $(a,b,c \geq 1)$, which is an orbifold compactification of the pair

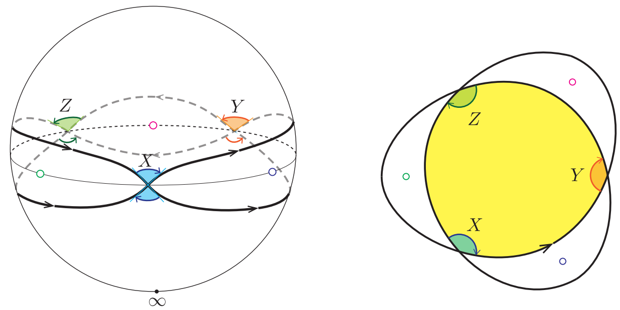


FIGURE 9. The Seidel Lagrangian. Two pictures show the same Lagrangian immersion from different viewpoints. The three dots on the equator are punctured when X is a pair-of-pants, or they are orbifold points when X is an orbifold projective line. The shaded triangle on the right contributes to the term xyz of the mirror superpotential.

of pants. One important thing is, the three independent directions of deformations labelled by x, y, z are weakly unobstructed, due to cancellations between holomorphic polygons and its reflection about the equator. Then our construction gives the generalized SYZ mirror of $\mathbb{P}^1_{a,b,c}$, whose leading terms are

$$x^a + y^b + z^c + \sigma(q)xyz.$$

Notice that $\mathbb{P}^1_{a,b,c}$ can be written as a G-quotient of a Riemann surface Σ. Then the generalized SYZ mirror of the Riemann surface Σ is also given by the same superpotential W, but over the quotient $\mathbb{C}^3/\hat{G}$ where $\hat{G}$ is the group of characters of G (which is just isomorphic to G itself because G is Abelian). When $1/a + 1/b + 1/c \geq 1$, which corresponds to the case that Σ has genus less than or equal to one, the superpotential W has finitely many terms; when $1/a + 1/b + 1/c < 1$, which corresponds to the case that Σ has genus greater than one, it has infinitely many terms. This gives the generalized SYZ mirror of a Riemann surface Σ.

One can also run the above procedures for (quotients of) the Fermat-type hypersurfaces

$$\tilde{X} = \left\{[z_0 : \ldots : z_{n+1}] \in \mathbb{P}^{n+1} : z_0^{n+2} + \ldots + z_{n+1}^{n+2} = 0\right\}$$

using the Lagrangian immersion constructed by Sheridan [45]. There are n degree-one independent deformation directions labelled by $x_1, \ldots, x_n$. Assuming that they are weakly unobstructed, we obtain a generalized SYZ mirror $(\mathbb{C}^{n+2}/\mathbb{Z}^{n+2}, W_n)$ of Fermat-type hypersurfaces where W_n has leading terms

$$\sum_{i=1}^{n} x_i^n + \sigma(q)x_1 \ldots x_n.$$

3.2. **Open mirror symmetry.** We have discussed an enumerative meaning of mirror map in the toric case in part I. Now let's consider the compact Calabi-Yau case. Presumably this is much more difficult. Since we have developed generalized SYZ, which also gives a map from the Kähler moduli of X to the complex moduli of the Landau-Ginzburg model W, Equation (1.1) makes sense. Thus we obtain a formulation of open mirror symmetry in this way.

We can actually prove it for elliptic curves:

Theorem 3.1 (Theorem 1.4 of [13])**.** *The mirror map equals to the generalized SYZ map for an elliptic curve E (or more precisely its quotient $E/\mathbb{Z}_3 = \mathbb{P}^1_{3,3,3}$ where E has a complex multiplication by cube root of unity.)*

Let me explain more on the geometric meaning of the above theorem. The Seidel Lagrangian (Figure 9) in $E/\mathbb{Z}_3$ lifted to E is a union of three circles, see Figure 10. The generalized SYZ mirror is of the form

$$W = (x^3 + y^3 + z^3) - \frac{\psi(q)}{\phi(q)} xyz$$

where $\phi(q)$ and $\psi(q)$ are generating series counting triangles with vertices at x, x, x and at x, y, z respectively, see Figure 10. These generating series can be computed explicitly (note that their coefficients have signs which require careful treatments).

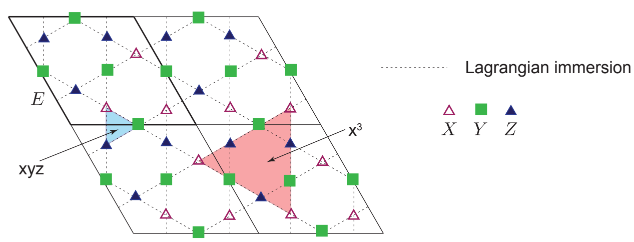


FIGURE 10. Polygon countings in the elliptic curve E. The parallelograms are fundamental domains of the elliptic curve E. The dotted lines show the three circles which are lifts of the Seidel Lagrangian in $E/\mathbb{Z}_3$.

On the other side, let $\pi_A(\check{q})$ and $\pi_B(\check{q})$ be the periods of E which satisfy the Picard-Fuchs equation

$$u''(\check{q}) + \frac{3\check{q}^2}{\check{q}^3 + 27} u'(\check{q}) + \frac{\check{q}}{\check{q}^3 + 27} u(\check{q}) = 0.$$

The inverse series of

$$q(\check{q}) = \pi_B(\check{q})/\pi_A(\check{q})$$

is what we refer as the mirror map, and it can be explicitly written as $\check{q}(q) = -3a(q)$, where

$$a(q) = 1 + \frac{1}{3}\left(\frac{\eta(q)}{\eta(q^9)}\right)^3 = 1 + \frac{1}{3} q^{-1} \left(\frac{\prod_{k=1}^{\infty}(1-q^k)}{\prod_{k=1}^{\infty}(1-q^{9k})}\right)^3.$$

We can verify that $\check{q}(q)$ equals to $\frac{\psi(q)}{\phi(q)}$. Thus the mirror map has an enumerative meaning of counting triangles. Note that the η-function also has rich number-theoretical meanings.

In [13] we conjecture that Equality (1.1) between mirror map and generalized SYZ map continues to hold for Fermat hypersurfaces in all dimensions, and this would give an enumerative meaning of mirror maps of Fermat hypersurfaces.

3.3. **Localized mirror functor.** Homological mirror symmetry proposed by Kontsevich [28] states that Lagrangian submanifolds correspond to matrix factorizations in the Landau-Ginzburg mirror. Currently the main approach to prove homological mirror symmetry is to compare generators and their relations (hom spaces) on both sides and show that they are (quasi-)isomorphic. This does not explain why homological mirror symmetry works.

Our generalized version of SYZ construction naturally gives an A_∞-functor $\mathcal{LM}^{\mathbf{L}}$ from the Fukaya category of X to the category of matrix factorization of W and hence explains the geometric origin of homological mirror symmetry:

Theorem 3.2 (Theorem 1.1 of [13]). *We have an A_∞-functor*

$$\mathcal{LM}^{\mathbf{L}} : \mathcal{F}uk_{\,\lambda}(X) \to \mathcal{MF}(W - \lambda).$$

Here, $\mathcal{F}uk_{\,\lambda}(X)$ is the Fukaya category of X (as an A_∞-category) whose objects are weakly unobstructed Lagrangians with potential value λ, and $\mathcal{MF}(W-\lambda)$ is the dg category of matrix factorizations of $W - \lambda$.

The key is the A_∞ relation for Lagrangian Floer theory of the pair of Lagrangians $\mathbf{L}$ and L':

$$(m_1^{\mathbf{L},L'})^2 = W(b) - \lambda \tag{3.1}$$

where λ is the disc potential of L', and $m_1^{\mathbf{L},L'}$ is an A_∞ operator defined by counting holomorphic strips bounded by $\mathbf{L}$ and L'. This equality follows from the compactification of moduli space of strips of Maslov index two, which may degenerate into either broken strip (contributing to m_1^2), or a constant strip with a disc bubble. Then a matrix factorization δ of W can be defined using $m_1^{\mathbf{L},L'}$, which satisfies

$$\delta^2 = W - \lambda. \tag{3.2}$$

We prove that the association of the matrix factorization δ to L' can be extended to an A_∞-functor from the Fukaya A_∞-category of unobstructed Lagrangians to the dg category of matrix factorizations of W. The proof employs the A_∞ equations of the Fukaya category, and is similar to the statement that the Hom functor in Yoneda embedding is an A_∞-functor.

We prove our functor is an equivalence for the orbifold projective line $X = \mathbb{P}^1_{a,b,c}$. Namely,

Theorem 3.3 (Theorem 1.3 of [13]). *Let $X = \mathbb{P}^1_{a,b,c}$ and W be its generalized SYZ mirror. The A_∞-functor $\mathcal{LM}^{\mathbf{L}}$ in Theorem 3.2 derives an equivalence of triangulated categories*

$$D^\pi(\mathrm{Fuk}(\mathbb{P}^1_{a,b,c})) \cong D^\pi(\mathcal{MF}(W)).$$

Indeed $\mathcal{LM}^{\mathbf{L}}(\mathbf{L})$ can be written (by some non-trivial change of coordinates) in the following simple form:

(3.3)
$$\left(\bigwedge\nolimits^{*}_{new}\langle X,Y,Z\rangle, xX \wedge^{new} (\cdot) + yY \wedge^{new} (\cdot) + zZ \wedge^{new} (\cdot) + w_x\, \iota_X^{new} + w_y\, \iota_Y^{new} + w_z\, \iota_Z^{new}\right),$$

This type of matrix factorizations was proved to split-generate the derived category of matrix factorizations by Dyckerhoff [15].

References

1. K. Altmann, *The versal deformation of an isolated toric Gorenstein singularity*, Invent. Math. **128** (1997), no. 3, 443–479.
2. D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor*, J. Gökova Geom. Topol. GGT **1** (2007), 51–91.
3. R. Castano-Bernard and D. Matessi, *Conifold transitions via affine geometry and mirror symmetry*, preprint, arXiv:1301.2930.
4. K. Chan, *A formula equating open and closed Gromov-Witten invariants and its applications to mirror symmetry*, Pacific J. Math. **254** (2011), no. 2, 275–293.
5. K. Chan, C.-H. Cho, S.-C. Lau, and H.-H. Tseng, *Gross fibrations, SYZ mirror symmetry, and open Gromov-Witten invariants for toric Calabi-Yau orbifolds*, preprint, arXiv:1306.0437.
6. ———, *Lagrangian Floer superpotentials and crepant resolutions for toric orbifolds*, preprint, arXiv:1208.5282.
7. K. Chan and S.-C. Lau, *Open Gromov-Witten invariants and superpotentials for semi-Fano toric surfaces*, IMRN (2013), doi:10.1093/imrn/rnt050.
8. K. Chan, S.-C. Lau, and N.C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds*, J. Differential Geom. **90** (2012), no. 2, 177–250.
9. K. Chan, S.-C. Lau, N.C. Leung, and H.-H. Tseng, *Open Gromov-Witten invariants and mirror maps for semi-Fano toric manifolds*, preprint, arXiv:1112.0388.
10. ———, *Open Gromov-Witten invariants and Seidel representations for toric manifolds*, preprint (2012), arXiv: 1209.6119.
11. K. Chan, S.-C. Lau, and H.-H. Tseng, *Enumerative meaning of mirror maps for toric Calabi-Yau manifolds*, Adv. Math. (2013), doi:10.1016/j.aim.2013.05.018.
12. K. Chan and N.C. Leung, *Mirror symmetry for toric Fano manifolds via SYZ transformations*, Adv. Math. **223** (2010), no. 3, 797–839.
13. C.-H. Cho, H. Hong, and S.-C. Lau, *Localized mirror functor for Lagrangian immersions, and homological mirror symmetry for $\mathbb{P}^1_{a,b,c}$*, preprint, arXiv:1308.4651.
14. C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds*, Asian J. Math. **10** (2006), no. 4, 773–814.
15. Tobias Dyckerhoff, *Compact generators in categories of matrix factorizations*, Duke Math. J. **159** (2011), no. 2, 223–274.
16. A. I. Efimov, *Homological mirror symmetry for curves of higher genus*, Adv. Math. **230** (2012), no. 2, 493–530.
17. K. Fukaya, Y.-G. Oh, H. Ohta, and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction. Part I*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.
18. ———, *Lagrangian intersection Floer theory: anomaly and obstruction. Part II*, AMS/IP Studies in Advanced Mathematics, vol. 46, American Mathematical Society, Providence, RI, 2009.
19. ———, *Lagrangian Floer theory on compact toric manifolds. I*, Duke Math. J. **151** (2010), no. 1, 23–174.
20. ———, *Lagrangian Floer theory on compact toric manifolds II: bulk deformations*, Selecta Math. (N.S.) **17** (2011), no. 3, 609–711.
21. A. Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics (Kyoto, 1996), Progr. Math., vol. 160, Birkhäuser Boston, Boston, MA, 1998, pp. 141–175.
22. E. Goldstein, *Calibrated fibrations on noncompact manifolds via group actions*, Duke Math. J. **110** (2001), no. 2, 309–343.
23. M. Gross, *Examples of special Lagrangian fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 81–109.
24. ———, *Topological mirror symmetry*, Invent. Math. **144** (2001), no. 1, 75–137.
25. M. Gross and B. Siebert, *From real affine geometry to complex geometry*, Ann. of Math. (2) **174** (2011), no. 3, 1301–1428.

26. ______, *An invitation to toric degenerations*, Surveys in differential geometry. Volume XVI. Geometry of special holonomy and related topics, Surv. Differ. Geom., vol. 16, Int. Press, Somerville, MA, 2011, pp. 43–78.
27. K. Hori, A. Iqbal, and C. Vafa, *D-branes and mirror symmetry*, preprint 2000, arXiv:hep-th/0005247 .
28. M. Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.
29. M. Kontsevich and Y. Soibelman, *Homological mirror symmetry and torus fibrations*, Symplectic geometry and mirror symmetry (Seoul, 2000), World Sci. Publ., River Edge, NJ, 2001, pp. 203–263.
30. ______, *Affine structures and non-Archimedean analytic spaces*, The unity of mathematics, Progr. Math., vol. 244, Birkhäuser Boston, Boston, MA, 2006, pp. 321–385.
31. C. Krattenthaler and T. Rivoal, *On the integrality of the Taylor coefficients of mirror maps*, Duke Math. J. **151** (2010), no. 2, 175–218.
32. S.-C. Lau, *Open Gromov-Witten invariants and SYZ under local conifold transitions*, preprint (2013), arXiv:1305.5279.
33. S.-C. Lau, N.C. Leung, and B. Wu, *A relation for Gromov-Witten invariants of local Calabi-Yau threefolds*, Math. Res. Lett. **18** (2011), no. 5, 943–956.
34. ______, *Mirror maps equal SYZ maps for toric Calabi-Yau surfaces*, Bull. London Math. Soc. **44** (2012), 255–270.
35. N.C. Leung, *Mirror symmetry without corrections*, Comm. Anal. Geom. **13** (2005), no. 2, 287–331.
36. N.C. Leung, S.-T. Yau, and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform*, Adv. Theor. Math. Phys. **4** (2000), no. 6, 1319–1341.
37. B. Lian, K. Liu, and S.-T. Yau, *Mirror principle. I*, Asian J. Math. **1** (1997), no. 4, 729–763.
38. ______, *Mirror principle. II*, Asian J. Math. **3** (1999), no. 1, 109–146, Sir Michael Atiyah: a great mathematician of the twentieth century.
39. ______, *Mirror principle. III*, Asian J. Math. **3** (1999), no. 4, 771–800.
40. ______, *Mirror principle. IV*, Surveys in differential geometry, Surv. Differ. Geom., VII, Int. Press, Somerville, MA, 2000, pp. 475–496.
41. B. Lian and S.-T. Yau, *Integrality of certain exponential series*, **2** (1998), 215–227.
42. R. Pandharipande, J. Solomon, and J. Walcher, *Disk enumeration on the quintic 3-fold*, J. Amer. Math. Soc. **21** (2008), no. 4, 1169–1209.
43. P. Seidel, *Homological mirror symmetry for the genus two curve*, J. Algebraic Geom. **20** (2011), no. 4, 727–769.
44. N. Sheridan, *Homological Mirror Symmetry for Calabi-Yau hypersurfaces in projective space*, preprint (2011), arXiv:1111.0632.
45. ______, *On the homological mirror symmetry conjecture for pairs of pants*, J. Differential Geom. **89** (2011), no. 2, 271–367.
46. A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror symmetry is T-duality*, Nuclear Phys. B **479** (1996), no. 1-2, 243–259.
47. V. V. Zudilin, *On the integrality of power expansions related to hypergeometric series*, Mat. Zametki **71** (2002), no. 5, 662–676.

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