Modularity of Open Gromov-Witten Potentials of Elliptic Orbifolds

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Abstract

We study the modularity of the genus zero open Gromov-Witten potentials and its generating matrix factorizations for elliptic orbifolds. These objects constructed by Lagrangian Floer theory are a priori well-defined only around the large volume limit. It follows from modularity that they can be analytically continued over the global Kähler moduli space.

1 Introduction

The mirror of an elliptic $\mathbb{P}^1$ orbifold $\mathbb{P}^1_{a,b,c}$ is a Landau-Ginzburg mirror: it is determined by a polynomial

$$W^{\mathrm{mir}} = x^a + y^b + z^c + \sigma xyz\,, \tag{1.1}$$

where σ is a complex parameter. Mirror symmetry asserts that symplectic geometry of $\mathbb{P}^1_{a,b,c}$ is reflected from the complex geometry of W^{mir}, and vice versa. While the orbifold $\mathbb{P}^1_{a,b,c}$ is only of dimension one, its Gromov-Witten theory is very interesting and receives a lot of attention in the context of mirror symmetry and integrable systems, see for instance [MT08, Tak10, Ros10, MR11, ST11, KS11, MS12, ET13, LLS13, SZ14].

The paper [CHL13] proposed a systematic construction of Landau-Ginzburg mirror and a homological mirror functor using Lagrangian Floer theory. For an elliptic $\mathbb{P}^1$ orbifold $\mathbb{P}^1_{a,b,c}$, where $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, the construction produces a polynomial $W_q(x,y,z)$ whose coefficients are convergent series in the Kähler parameter q of $\mathbb{P}^1_{a,b,c}$. The polynomial W_q can be rearranged to the form of W^{mir} by an explicit change of coordinates in (x,y,z). It is called to be the open Gromov-Witten potential because it is obtained by counting holomorphic polygons bounded by a fixed Lagrangian, which is the Lagrangian immersion constructed by Seidel [Sei11].

The open Gromov-Witten potential $W_q(x,y,z)$ is a priori defined only around the point $q=0$, the so-called large volume limit of the Kähler moduli space. In this paper, we show that indeed it can be extended to certain global moduli space:

Theorem 1.1. *Let $W_q(x,y,z)$ be the open Gromov-Witten potential of an elliptic $\mathbb{P}^1$ orbifold $\mathbb{P}^1_{a,b,c}$ where $(a,b,c)=(3,3,3)$ or $(2,4,4)$. The coefficients of $W_q(x,y,z)$, which are functions in q, are modular forms of certain weight k for the modular group $\Gamma=\Gamma(3)$ or $\Gamma(4)$ respectively. Hence the potential extends to be a section of the line bundle $\mathcal{K}^{\frac{k}{2}}$ over the product $\mathbb{C}^3\times(\Gamma\backslash\mathcal{H}^*)$, where $\mathcal{K}$ is the pull back of the canonical line bundle of the modular curve $\Gamma\backslash\mathcal{H}^*$.*

2010 *Mathematics subject classification.* 14N35, 14N10, 11FXX

The proof is arithmetic in nature. We explicitly express the open Gromov-Witten potential in terms of the Dedekind η-function and Eisenstein series, and use known expressions for modular forms with respect to the groups $\Gamma = \Gamma(3)$ and $\Gamma(4)$. We expect the same statement holds for the case $(a, b, c) = (2, 3, 6)$, see Section 3.3 for more details.

Remark 1.2. The theorem also holds for the elliptic orbifold $\mathbb{P}^1_{2,2,2,2}$, namely the coefficients of the open Gromov-Witten potential of $\mathbb{P}^1_{2,2,2,2}$ are modular forms for the modular group $\Gamma(2)$. See Section 3.4. In this case W is defined on the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ rather than $\mathbb{C}^3$, and its critical locus is the zero section $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ rather than an isolated point. Thus we separate this case from the above theorem.

For an elliptic $\mathbb{P}^1$ orbifold $\mathbb{P}^1_{a,b,c}$, the mirror functor produces a particular matrix factorization M of W_q, which is an odd endomorphism δ on $\bigwedge^* \mathbb{C}^3$ satisfying $\delta^2 = W_q \cdot \mathrm{Id}$. This matrix factorization has the important property that it split generates the derived category of matrix factorizations, and it is mirror to the Seidel Lagrangian. Using similar arithmetic techniques, we can express M in terms of modular forms.

Theorem 1.3. *Let M be the matrix factorization of the open Gromov-Witten potential $W_q(x, y, z)$ which is mirror to the Seidel Lagrangian in $\mathbb{P}^1_{a,b,c}$, where $(a, b, c) = (3, 3, 3)$ or $(2, 4, 4)$. The matrix entries of M are polynomials in x, y, z whose coefficients are modular forms of weight k for the modular group $\Gamma = \Gamma(3)$ or $\Gamma(4)$ respectively.*

Why modularity is expected can be explained as follows. The Seidel Lagrangian in the elliptic orbifold $\mathbb{P}^1_{a,b,c} = E/\mathbb{Z}_r$, where $r = 3, 4, 6$ for the $(a, b, c) = (3, 3, 3), (2, 4, 4), (2, 3, 6)$ case respectively, can be lifted to r copies of Lagrangians in the elliptic curve E. Thus the moduli space around the large volume limit under consideration on the symplectic side is the moduli space of Kähler structure of *E together with a particular choice of r Lagrangians*. The mirror is the family of elliptic curves decorated with structures of r-torsion points, whose moduli space turns out to be the modular curve $\Gamma \backslash \mathcal{H}^*$. Mirror symmetry asserts that the A-side moduli is globally isomorphic to the B-side moduli. Thus $\Gamma \backslash \mathcal{H}^*$ should also be the global Kähler moduli. Our results confirm that the open Gromov-Witten potential, which is originally just defined around the large volume limit, naturally extend to this global Kähler moduli space.

Remark 1.4. Modularity of closed Gromov-Witten potentials for elliptic curves and elliptic orbifolds is derived in a series of works including [Dij95, KZ95, EO01, OP06, MR11, ST11, Li12, SZ14]. For discussions on modularity of some higher dimensional Calabi-Yau varieties, interested readers are referred to [BCOV93, BCOV94, AGNT95, KV95, MM99, KKRS05, KM08, ABK08, AS12, KMW12, ASYZ14, PT14] and references therein for details.

Structure of the paper

In Section 2, we review some basic materials on modular forms and elliptic curve families defined over some modular curves. In Section 3, we recall the construction of the Seidel Lagrangian and prove the modularity for the potentials W. In Section 4, we prove the modularity for the matrix factorizations M. We discuss why modularity is expected from the perspective of mirror symmetry and give one further example in Section 5.

Acknowledgment

We are grateful to Shing-Tung Yau for constant support and encouragement. We thank Kathrin Bringmann and Larry Rolen for email correspondences and helpful discussions on mock modular forms. The second author thanks Murad Alim, Emanuel Scheidegger, and Shing-Tung Yau for fruitful collaborations on related projects. He also thanks Teng Fei, Todor Milanov, Yongbin Ruan, Yefeng Shen for very useful discussions on various aspects of elliptic orbifolds and modular forms. We thank the referees for helpful comments and pointing out the importance of the study of the elliptic orbifold $\mathbb{P}^1_{2,2,2,2}$.

Part of the work is done while the second author was a graduate student at the mathematics department at Harvard. We would like to thank the department for providing an excellent research atmosphere. J. Z. is supported by the Perimeter Institute for Theoretical Physics. Research at Perimeter Institute is supported by the Government of Canada through Industry Canada and by the Province of Ontario through the Ministry of Economic Development and Innovation.

2 Preliminaries on modular forms

In this section we give a quick review on some background material about modular forms and modular curves. They are essential to our study because global Kähler moduli space of elliptic orbifolds will be identified as modular curves by using mirror symmetry. The open Gromov-Witten potentials and matrix factorizations will be written in terms of modular forms, which are global sections of the corresponding line bundles over modular curves. The material presented here is largely taken from a joint work [ASYZ14] of the second author.

Throughout this paper, we fix $q = \exp 2\pi i\tau$, with τ is the coordinate on the upper-half plane $\mathcal{H}$. The quantity $-2\pi i\tau$ can be regarded as parametrizing the (complexified) symplectic area of an elliptic orbifold (and so q defines a local coordinate near the large volume limit $q = 0$ on the complexified Kähler moduli space of the elliptic orbifold).

2.1 Modular groups and modular forms

The generators and relations for the group $\mathrm{SL}(2,\mathbb{Z})$ are given by the following:

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S^2 = -I, \quad (ST)^3 = -I. \tag{2.1}$$

We will consider in this paper the following congruence subgroups called Hecke subgroups of $\Gamma(1) = \mathrm{PSL}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})/\{\pm I\}$

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \middle| \, c \equiv 0 \mod N \right\} < \Gamma(1). \tag{2.2}$$

Some other groups that we are interested in are the principal congruence subgroups

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \middle| \, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod N \right\} < \Gamma(1). \tag{2.3}$$

One has $\Gamma(N) < \Gamma_0(N) < \Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$.

A modular form of weight k for the congruence subgroup Γ of $\mathrm{PSL}(2, \mathbb{Z})$ is a function $f : \mathcal{H} \to \mathbb{C}$ satisfying the following conditions:

- $f(\gamma\tau) = j_\gamma(\tau)^k f(\tau), \quad \forall \gamma \in \Gamma$, where j is called the j-automorphy factor and is defined by

$$j : \Gamma \times \mathcal{H} \to \mathbb{C}, \quad \left(\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto j_\gamma(\tau) := (c\tau + d)\,.$$

- f is holomorphic on $\mathcal{H}$.

- f is "holomorphic at the cusps" in the sense that the function

$$\tau \mapsto j_\gamma(\tau)^{-k} f(\gamma\tau) \tag{2.4}$$

  is holomorphic at $\tau = i\infty$ for any $\gamma \in \Gamma(1)$.

The second and third conditions in the above can be equivalently described as saying that f is holomorphic on the modular curve $X_\Gamma = \Gamma \backslash \mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, i.e., $\mathcal{H} \cup \mathbb{Q} \cup \{i\infty\}$. The first condition means that f can be formulated as a holomorphic section of a line bundle over X_Γ.

We will need to be able to take roots of modular forms. For this purpose we introduce the notion of multiplier system. A multiplier system of weight k for Γ is a function $v : \Gamma \to \mathbb{C}$ such that $|v(\gamma)| = 1$ and $v(\gamma_1\gamma_2) = w(\gamma_1, \gamma_2)v(\gamma_1)v(\gamma_2)$ for some $w(\gamma_1, \gamma_2)$. We then define modular forms of weight k with the multiplier system v by replacing the j-automorphy factor in (2.4) by the new automorphy factor $v(\gamma)j_\gamma(\tau)$, see for example [Ran77] for details. The simplest case is when v depends only on the entry d of γ. In the following we will be mostly dealing with the case where v is given by a Dirichlet character χ. The space of modular forms with the multiplier system χ for Γ forms a graded differential ring and is denoted by $M_*(\Gamma, \chi)$. Similarly we have the ring of even weight modular forms denoted by $M_{\mathrm{even}}(\Gamma, \chi)$. When χ is trivial, we shall often omit it and simply write $M_*(\Gamma)$.

Example 2.1. Taking the group Γ to be the full modular group $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$. Then $M_*(\Gamma(1)) = \mathbb{C}[E_4, E_6]$, where E_4, E_6 are the familiar Eisenstein series defined by

$$E_4(\tau) = 1 + 240 \sum_{d=1}^{\infty} \sigma_3(d) q^d, \quad q = e^{2\pi i \tau}, \quad \sigma_3(d) = \sum_{k:\, k|d} k^3\,,$$
$$E_6(\tau) = 1 - 504 \sum_{d=1}^{\infty} \sigma_3(d) q^d, \quad q = e^{2\pi i \tau}, \quad \sigma_5(d) = \sum_{k:\, k|d} k^5\,.$$

The Eisenstein series $E_2(\tau) = 1 - 24 \sum_{d=1}^{\infty} \sigma_1(d) q^d$ is not a modular form, but a so-called quasi-modular form [KZ95] for $\Gamma(1)$, since it transforms according to

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) + \frac{12}{2\pi i} c(c\tau + d), \quad \forall\, \tau \in \mathcal{H}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)\,.$$

2.2 Ring of modular forms

Now we consider modular forms (with possibly non-trivial multiplier systems) for the Hecke subgroups $\Gamma_0(N)$ with $N = 2, 3, 4$ and the subgroup $\Gamma_0(1^*)$ which is the unique index two normal subgroup of $\Gamma(1) = \mathrm{PSL}(2, \mathbb{Z})$. All of them are of genus zero in the sense that the corresponding modular curves[1] $X_0(N) := \Gamma_0(N)\backslash\mathcal{H}^*$ are genus zero Riemann surfaces. Each of the corresponding modular curves X_Γ has three singular points: two (equivalence classes) of cusps[2] $[i\infty], [0]$, and the third one is a cusp or an elliptic point, depending on the modular group. It is a quadratic elliptic point $[\tau] = [i]$ for $N = 2$, cubic elliptic point $[\tau] = [\exp 2\pi i/3]$ for $N = 3$ and $N = 1^*$, and a cusp $[\tau] = [1/2]$ for $N = 4$. For a review of these facts, see for instance [Ran77].

We can choose a particular Hauptmodul (i.e., a generator for the rational function field of the genus zero modular curve) $\alpha(\tau)$ for the corresponding modular group such that the two cusps are given by $\alpha = 0, 1$ respectively, and the third one is $\alpha = \infty$. It is given by $\alpha(\tau) = C^r(\tau)/A^r(\tau)$, where $r = 6, 4, 3, 2$ for the cases $N = 1^*, 2, 3, 4$ respectively. The functions[3] $A(\tau), C(\tau) = \alpha(\tau)^{\frac{1}{r}} A(\tau), B(\tau) = (1 - \alpha(\tau))^{\frac{1}{r}} A(\tau)$ are given in Table 1 below. See [BB91, BBG95] and also [Mai09, Mai11] for a review on the modular forms A, B, C.

Table 1: *η-expansions of A, B, C for $\Gamma_0(N), N = 1^*, 2, 3, 4$*

N	A	B	C
1^*	$E_4(\tau)^{\frac{1}{4}}$	$\left(\frac{E_4(\tau)^{\frac{3}{2}} + E_6(\tau)}{2}\right)^{\frac{1}{6}}$	$\left(\frac{E_4(\tau)^{\frac{3}{2}} - E_6(\tau)}{2}\right)^{\frac{1}{6}}$
2	$\frac{(2^6\eta(2\tau)^{24} + \eta(\tau)^{24})^{\frac{1}{4}}}{\eta(\tau)^2\eta(2\tau)^2}$	$\frac{\eta(\tau)^4}{\eta(2\tau)^2}$	$2^{\frac{3}{2}}\frac{\eta(2\tau)^4}{\eta(\tau)^2}$
3	$\frac{(3^3\eta(3\tau)^{12} + \eta(\tau)^{12})^{\frac{1}{3}}}{\eta(\tau)\eta(3\tau)}$	$\frac{\eta(\tau)^3}{\eta(3\tau)}$	$3\frac{\eta(3\tau)^3}{\eta(\tau)}$
4	$\frac{(2^4\eta(4\tau)^8 + \eta(\tau)^8)^{\frac{1}{2}}}{\eta(2\tau)^2} = \frac{\eta(2\tau)^{10}}{\eta(\tau)^4\eta(4\tau)^4}$	$\frac{\eta(\tau)^4}{\eta(2\tau)^2}$	$2^2\frac{\eta(4\tau)^4}{\eta(2\tau)^2}$

Throughout this paper we shall write A_N, B_N, C_N for the $\Gamma = \Gamma_0(N)$ case for these quantities when potential confusion might arise.

The explicit expressions for these quantities in terms of θ-functions and q-series can be found in a lot of literature. By using the θ-expansions therein for these generators, one can easily see that

$$A_2^2 = A_4^2 + C_4^2, \quad C_2^2 = 2A_4C_4. \tag{2.5}$$

The following results are classical:

$$\begin{aligned} M_{\mathrm{even}}(\Gamma_0(2)) &= \mathbb{C}[A_2^2, B_2^4], \\ M_*(\Gamma_0(3), \chi_{-3}) &= \mathbb{C}[A_3, B_3^3], \\ M_*(\Gamma_0(4), \chi_{-4}) &= \mathbb{C}[A_4, B_4^2]. \end{aligned}$$

[1] The $N = 1^*$ case is anomalous, more details are given in Section 2.3. For further discussion, see [Mai09].

[2] Here we use the notation $[\tau]$ to denote the Γ-equivalence class of $\tau \in \mathcal{H}^*$.

[3] Throughout this work, when we take factional powers of modular forms and modular functions, we always take the principal branch of the logarithm.

Here $\chi_{-3}(d) = \left(\frac{-3}{d}\right)$ is the Legendre symbol and it gives the non-trivial Dirichlet character for the modular forms. Similarly, $\chi_{-4}(d) = \left(\frac{-4}{d}\right)$. From these we can derive the following results:

$$M_*(\Gamma(3)) = \mathbb{C}[A_3, C_3]\,, \tag{2.6}$$

$$M_*(\Gamma(4)) = \mathbb{C}[A_4, C_4, C_2]/\langle C_2^2 - 2A_4C_4\rangle\,. \tag{2.7}$$

For the modular group $\Gamma(2)$, the ring of modular forms is isomorphic to that for $\Gamma_0(4)$ by using the 2-isogeny which gives an isomorphism between the modular groups. See for instance [BKMS01, Seb02, Mai11] and references therein for details of all these results.

2.3 Geometric moduli in terms of modular forms

In this section, we shall discuss some basic facts about the geometry and arithmetic of the elliptic curve families of E_n type[4], $n = 5, 6, 7, 8$. They are closely related to[5] the elliptic orbifolds which are the main focus of this work, as we shall see in the sequel.

The equations for the elliptic curve families are given by

$$\begin{aligned}
n = 5 \quad &: \quad \mathbb{P}^3[1,1,1,1][2,2] : x_1^2 + x_3^2 - z^{-\frac{1}{2r}} x_2 x_4 = 0\,,\\
&\qquad\qquad\qquad\qquad x_2^2 + x_4^2 - z^{-\frac{1}{2r}} x_1 x_3 = 0\,,\\
n = 6 \quad &: \quad \mathbb{P}^2[1,1,1][3] : x_1^3 + x_2^3 + x_3^3 - z^{-\frac{1}{r}} x_1 x_2 x_3 = 0\,,\\
n = 7 \quad &: \quad \mathbb{P}^2[1,1,2][4] : x_1^4 + x_2^4 + x_3^2 - z^{-\frac{1}{r}} x_1 x_2 x_3 = 0\,,\\
n = 8 \quad &: \quad \mathbb{P}^2[1,2,3][6] : x_1^6 + x_2^3 + x_3^2 - z^{-\frac{1}{r}} x_1 x_2 x_3 = 0\,,
\end{aligned} \tag{2.8}$$

where the numbers r are given by $2, 3, 4, 6$ for $n = 5, 6, 7, 8$, respectively.

The j-invariants for these elliptic curve families are summarized here, see [LY96, LMW97, KLRY96, CKYZ99] for more details.

$$E_5 : \begin{cases} x_1^2 + x_3^2 - z^{-\frac{1}{4}} x_2 x_4 &= 0 \\ x_2^2 + x_4^2 - z^{-\frac{1}{4}} x_1 x_3 &= 0 \end{cases} \qquad j(z) = \frac{(1 + 224z + 256z^2)^3}{z(1 - 16z)^4}\,. \tag{2.9}$$

The base of this family of elliptic curves is the modular curve $X_0(4)$. It has three singular points: two cusp classes $[i\infty], [0]$ corresponding to $z = 0, 1/16$ respectively; and the cusp class $[1/2]$ corresponding to $z = \infty$.

$$E_6 :\; x_1^3 + x_2^3 + x_3^3 - z^{-\frac{1}{3}} x_1 x_2 x_3 = 0\,, \quad j(z) = \frac{(1 + 216z)^3}{z(1 - 27z)^3}\,. \tag{2.10}$$

The base of this family of elliptic curves is the modular curve $X_0(3)$. It has three singular points: two cusp classes $[i\infty], [0]$ corresponding to $z = 0, 1/27$ respectively; and the cubic elliptic point $[ST^{-1}(\rho)]$ corresponding to $z = \infty$, where $\rho = \exp(2\pi i/3)$.

[4] The names come from the fact that the total spaces of the elliptic curve families correspond to the E_n del Pezzo surfaces, see for instance [KMW12] for further explanation.

[5] In fact, for the $n = 6, 7, 8$ cases these are, up to reparametrization, the simple elliptic singularities [Sai74] $E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$ and are mirror to the elliptic orbifolds, see [MR11].

$$E_7:\ x_1^4+x_2^4+x_3^2-z^{-\frac{1}{4}}x_1x_2x_3=0\,,\quad j(z)=\frac{(1+192z)^3}{z(1-64z)^2}\,. \tag{2.11}$$

The base of this family of elliptic curves is the modular curve $X_0(2)$. It has three singular points: two cusp classes $[i\infty],[0]$ corresponding to $z=0,1/64$ respectively; and the quadratic elliptic point $[(1+i)/2]=[ST^{-1}(i)]$ corresponding to $z=\infty$.

$$E_8:\ x_1^6+x_2^3+x_3^2-z^{-\frac{1}{6}}x_1x_2x_3=0\,,\quad j(z)=\frac{1}{z(1-432z)}\,. \tag{2.12}$$

The base of this family of elliptic curves is the curve $X_0(1^*)=\Gamma_0(1^*)\backslash\mathcal{H}^*$, where $\Gamma_0(1^*)$ is the unique index 2 normal subgroup of $\Gamma(1)=\mathrm{PSL}(2,\mathbb{Z})$. It has three singular points: two cusp classes $[i\infty],[0]$ corresponding to $z=0,1/432$ respectively; and the cubic elliptic point $[\rho]$ corresponding to $z=\infty$.

The Hauptmodul for the corresponding modular group given in the previous section is related to the parameter z by $\alpha=\kappa_N z$, where κ_N is given $432,64,27,16$ for $n=8,7,6,5$ (i.e., $N=1^*,2,3,4$), respectively. For reference, we now summarize the related quantities in Table 2 below. Here the number r is given by $r=12/\nu$, with ν being the index of the subgroup in the full modular group $\mathrm{PSL}(2,\mathbb{Z})$.

Table 2: *Arithmetic numbers*

n	5	6	7	8
N	4	3	2	1^*
r	2	3	4	6
κ_N	16	27	64	432

Remark 2.2. The Picard-Fuchs operators of the above elliptic curves of E_n type have the form

$$\mathcal{L}_{\text{Picard-Fuchs}}=\theta^2-\alpha(\theta+\frac{1}{r})(\theta+1-\frac{1}{r}),\quad \theta=\alpha\frac{\partial}{\partial\alpha}\,. \tag{2.13}$$

Denote $A(\alpha)={}_2F_1(\frac{1}{r},1-\frac{1}{r},1;\alpha)$ to be the regular period at $\alpha=0$ of the elliptic curve family. Then the modular form $A(\tau)$ given in the previous section is actually given by $A(\alpha(\tau))$. One also has $\tau(\alpha)=\frac{i}{\sqrt{N}}A(1-\alpha)/A(\alpha)$. Therefore, the triple $A(\tau),B(\tau),C(\tau)$ introduced earlier can be reconstructed from the periods, see [BB91, BBG95, Mai09]. This fact was used in [ASYZ14, Zho14] in studying modularity in Gromov-Witten theory and mirror symmetry for some non-compact Calabi-Yau threefolds.

In Section 3 and Section 4 below, we will be mainly working with the A-model of the elliptic orbifolds, that is, studying the dependence of the generating functions of genus zero open Gromov-Witten invariants on the complexified Kähler structure. In Section 5, we will comment on how mirror symmetry maps the symplectic geometry data of elliptic orbifolds to the complex geometry data of the elliptic curve families described in this section. This would then give an explanation of why modularity is expected.

3 Open Gromov-Witten potentials of elliptic orbifolds

In this section, we study modularity of open Gromov-Witten potentials of elliptic orbifolds. First let us have a quick glance on the construction of open Gromov-Witten potentials in [CHL13, CHKL14] using immersed Lagrangian Floer theory.

Given a Kähler orbifold X, we fix a Lagrangian immersion $\mathcal{L}$, which is assumed to be oriented and (relatively) spin, and not passing through the orbifold points of X. Moreover we assumed that it has transverse self-intersections for simplicity. Let $\iota : \tilde{\mathcal{L}} \to X$ denote the normalization of $\mathcal{L}$. We assume that $\tilde{\mathcal{L}}$ is connected.

We use the deformations and obstructions of $\mathcal{L}$ to construct a Landau-Ginzburg model (V, W). It is called to be the generalized SYZ construction: it uses deformations of an immersed Lagrangian to construct the mirror, while SYZ uses a Lagrangian torus fibration for the same purpose. The detailed deformation theory for Lagrangian immersion, which is captured by an A_∞ algebra $\left(H, \{m_k\}_{k=0}^{\infty}\right)$, was developed in [AJ10]. Here we only sketch the needed ingredients.

Each transverse self-intersection point a corresponds to two immersed generators X_a^0, X_a^1 of the Floer complex of $\mathcal{L}$. Intuitively they represent the two ways of smoothings of the self-intersection point. For a formal deformation

$$b = \sum_a (x_a^0 X_a^0 + x_a^1 X_a^1) \in H = \bigoplus_a \mathrm{Span}_{\mathbb{C}}\{X_a^0, X_a^1\}\,, \tag{3.1}$$

where the sum is over all self-intersection points a, we have the deformed m_0-term

$$m_0^b = \sum_{k=0}^{\infty} m_k(b, \ldots, b) = \sum_{k=0}^{\infty} \sum_{\substack{(a_1,\ldots,a_k) \\ (s_1,\ldots,s_k)}} m_k(X_{a_1}^{s_1}, \ldots, X_{a_k}^{s_k}) x_{a_1}^{s_1} \ldots x_{a_k}^{s_k}\,, \tag{3.2}$$

which is a singular chain in the fiber product $\tilde{\mathcal{L}} \times_\iota \tilde{\mathcal{L}}$. Roughly speaking it is a sum of the boundary evaluation images of holomorphic polygons bounded by $\mathcal{L}$ (weighted by their symplectic areas) with corners at the immersed generators.

Then we choose a subspace V of H whose elements $b \in V$ have odd degrees and satisfy the so-called weak Maurer-Cartan equation [FOOO09]

$$m_0^b = W(b) \mathbb{1}_{\tilde{\mathcal{L}}}\,, \tag{3.3}$$

where $\mathbb{1}_{\tilde{\mathcal{L}}}$ denotes the fundamental class of $\tilde{\mathcal{L}}$. Such deformations b are called to be weakly unobstructed. This defines a function W on V, and we call it to be the open Gromov-Witten potential of $\mathcal{L}$ because coefficients of W are obtained by counting pseudoholomorphic polygons bounded by the immersed Lagrangian $\mathcal{L}$.

To construct the open Gromov-Witten potential (or so-called Landau-Ginzburg mirror) of elliptic $\mathbb{P}^1$ orbifolds, we take $\mathcal{L}$ to be the Lagrangian immersion constructed by Seidel [Sei11]. It has three self-intersection points as depicted in Figure 1. We take the formal deformations $b = xX + yY + zZ$, where X, Y, Z are immersed generators of odd degrees as shown in the figure. By [CHL13, Lemma 7.5], these deformations are weakly unobstructed. Thus we obtain an open Gromov-Witten potential $W(x, y, z)$.

Note that the potential $W(x, y, z)$ depends on the Kähler parameter of the elliptic $\mathbb{P}^1$ orbifold, which parametrizes the sizes of the holomorphic polygons. Thus W can be

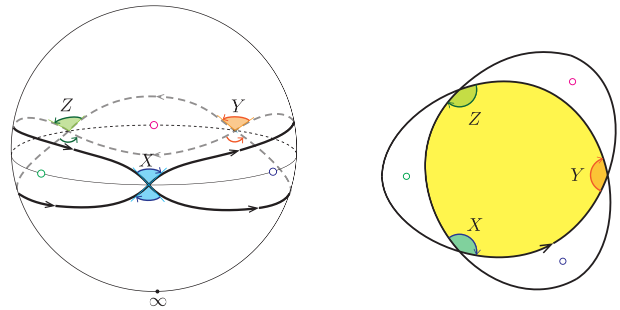

Figure 1: *The Seidel Lagrangian. The two pictures above show the same Lagrangian immersion from different viewpoints. The three dots on the equator represent orbifold points. The shaded triangle on the right contributes to the term $-q_d xyz$ of the open Gromov-Witten potential, where $q_d = \exp(-A)$, A is the symplectic area of the shaded triangle.*

identified as a map from the Kähler moduli of the $\mathbb{P}^1$ orbifold to the complex moduli of holomorphic functions. We call this to be the generalized SYZ map because it arises from the generalized SYZ construction described above.

The explicit expression of W and the generalized SYZ map were computed in [CHL13, Section 6.1] for the elliptic orbifold $\mathbb{P}^1_{3,3,3}$ and in [CHKL14, Section 9 and 10] for the elliptic orbifolds $\mathbb{P}^1_{2,4,4}$ and $\mathbb{P}^1_{2,3,6}$. In the rest of this section we shall study modularity of the coefficients of the open Gromov-Witten potential.

Remark 3.1. There is another elliptic orbifold curve which is not listed above, namely $\mathbb{P}^1_{2,2,2,2}$ which is the $\mathbb{Z}_2$-quotient of some elliptic curve. A similar construction scheme for its open Gromov-Witten potential can be carried out, which involves more than one Lagrangian immersions. The details about the construction of the open Gromov-Witten potential and the mirror functor will be given in a forthcoming work [CHL15]. In this paper, we will state the result of the open Gromov-Witten potential and discuss its modularity.

3.1 $(3,3,3)$ case

Theorem 3.2. *[CHKL14] The open Gromov-Witten potential for $\mathbb{P}^1_{3,3,3}$ is*

$$W = \phi(q_{\mathrm{d}})(x^3 + y^3 + z^3) - \psi(q_{\mathrm{d}})xyz\,, \tag{3.4}$$

where

$$\phi(q_{\mathrm{d}}) = \sum_{k=0}^{\infty}(-1)^{3k+1}(2k+1)q_{\mathrm{d}}^{3(12k^2+12k+3)}\,, \tag{3.5}$$

and

$$\psi(q_{\mathrm{d}}) = -q_{\mathrm{d}} + \sum_{k=1}^{\infty}\left((-1)^{3k+1}(6k+1)q_{\mathrm{d}}^{(6k+1)^2} + (-1)^{3k}(6k-1)q_{\mathrm{d}}^{(6k-1)^2}\right)\,. \tag{3.6}$$

Here $q_{\mathrm{d}} = \exp(-\mathrm{area}(\Delta))$, with Δ the minimal triangle bounded by the Seidel Lagrangian.

Consider the elliptic curve E_ρ with $j(E_\rho) = 0$, it can be realized, say, by $x_1^3 + x_2^3 + x_3^3 = 0$ in $\mathbb{P}^2$. Its quotient[6] by the $\mathbb{Z}_3$ automorphism is $\mathbb{P}^1_{3,3,3}$. The Kähler parameter q of the elliptic curve is related with q_{d} by $q = q_{\mathrm{d}}^{24}$. Here the subscript 'd' stands for 'disk'. Throughout this paper, by abuse of notation, we will use for example the notation $\phi(q)$ to denote the quantity $\phi(q_{\mathrm{d}}(q))$.

An easy computation shows the following.

Theorem 3.3. *Both ϕ and ψ (when expressed in q) are modular forms of formal weight $3/2$ with the same multiplier system for the modular group $\Gamma(3)$.*

Proof. Simple algebra shows that

$$\phi(q_{\mathrm{d}}) = \frac{1}{2} \sum_{k=-\infty}^{\infty} (-1)^{k+1} (2k+1) q_{\mathrm{d}}^{9(2k+1)^2} = \sum_{r \in \mathbb{Z}+\frac{1}{2}} (-1)^{r+\frac{1}{2}} r q_{\mathrm{d}}^{36r^2} .$$

Recall that for the Jacobi theta function (here $v = \exp 2\pi i z$)

$$\theta_1(v, q) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} (-1)^r v^r q^{\frac{1}{2}r^2} ,$$

we have

$$\partial_v|_{v=1} \theta_1(v, q) = \sum_{r \in \mathbb{Z}+\frac{1}{2}} (-1)^r r q^{\frac{1}{2}r^2} .$$

Therefore, we obtain

$$\phi(q_{\mathrm{d}}) = \mathbf{i} \partial_v \theta_1(1, q_{\mathrm{d}}^{72}) = \mathbf{i} \partial_v \theta_1(1, q^3) . \tag{3.7}$$

To compute ψ, we use the identity

$$\psi(q_{\mathrm{d}}) - 3\phi(q_{\mathrm{d}}) = \sum_{l=0}^{\infty} (-1)^{l+1} (2l+1) q_{\mathrm{d}}^{(2l+1)^2} = \mathbf{i} \partial_v \theta_1(1, q_{\mathrm{d}}^8) .$$

Or alternatively, $\phi(q_{\mathrm{d}}) - \psi(q_{\mathrm{d}}^9) = -3\phi(q_{\mathrm{d}}^9)$. Hence

$$\psi(q_{\mathrm{d}}) = \mathbf{i} (\partial_v \theta_1(1, q_{\mathrm{d}}^8) + 3\partial_v \theta_1(1, q_{\mathrm{d}}^{72})) . \tag{3.8}$$

where $q_{\mathrm{d}}^8 = q^{1/3}$ and $q_{\mathrm{d}}^{72} = q^3$. The Jacobi theta function θ_1 satisfies

$$2\pi \mathbf{i} (v \partial_v)|_{v=1} \theta_1 = \partial_z|_{z=0} \theta_1 = -2\pi \eta(q)^3 .$$

That is, $\partial_v|_{v=1} \theta_1 = \mathbf{i} \eta(q)^3$. It follows that $\phi(q), \psi(q)$ can be written in terms of the η–functions as follows:

$$\phi(q) = -\eta(q^3)^3 , \tag{3.9}$$

$$\psi(q) = -(\eta(q^{\frac{1}{3}})^3 + 3\eta(q^3)^3) . \tag{3.10}$$

[6]For example, this action could be realized as $[x_1, x_2, x_3] \mapsto [\exp(2\pi i/3)x_1, x_2, x_3]$ and should not be confused with the action of the group of 3-torsion points which moves the origin of the elliptic curve and thus is not an automorphism.

Remark 3.4. Recall that for the Hesse-Dixon model for elliptic curves: $x^3+y^3+z^3-(\gamma+3)xyz=0$, we have

$$\gamma(\tau)+3=3\frac{A_3(\tau)}{C_3(\tau)}=3\left(1+\frac{\eta(q)^{12}}{3^3\eta(q^3)^{12}}\right)^{\frac{1}{3}}\,, \tag{3.11}$$

where γ is a Hauptmodul for the modular group $\Gamma(3)$, see for example [Mai09] for details. The Hauptmodul is also called to be the mirror map since it gives a map between the Kähler moduli, parametrized by τ, and the complex moduli parametrized by γ. One can check that (see [BBG94])

$$\frac{\psi(\tau)}{\phi(\tau)}=\gamma(\tau)+3\,. \tag{3.12}$$

That is, the generalized SYZ map is identical to the mirror map, as has been deduced in Theorem 6.5 of [CHL13] in a different way. This will be explained further in Section 5, where we see that actually the geometry defined by the open Gromov-Witten potential W coincides with the Hesse-Dixon model.

Using the results in Section 2, we know that $\eta(q^3)^3=3^{-\frac{9}{8}}B_3^{\frac{3}{8}}(\tau)C_3^{\frac{9}{8}}(\tau)$ is a modular form for $\Gamma_0(3)$ with possibly non-trivial multiplier system. In particular, it is so for $\Gamma(3)$. Therefore, this is also true for $\eta(q^{\frac{1}{3}})^3$ since $\gamma(\tau)$ is modular with respect to $\Gamma(3)$ according to the above remark. Moreover, ϕ,ψ must have the same multiplier system since γ has a trivial one. Hence the conclusion follows.

□

3.2 $(2,4,4)$ case

Theorem 3.5. *[CHKL14] The open Gromov-Witten potential of $\mathbb{P}^1_{2,4,4}$ is*

$$W=q_{\mathrm{d}}^6x^2-q_{\mathrm{d}}xyz+d_y(q_{\mathrm{d}})y^4+d_z(q_{\mathrm{d}})z^4+d_{yz}(q_{\mathrm{d}})y^2z^2, \tag{3.13}$$

where

$$d_y(q_{\mathrm{d}})=d_z(q_{\mathrm{d}})=\sum_{0\leq r}(2r+1)q_{\mathrm{d}}^{16(2r+1)^2-4}+\sum_{0\leq r<s}(2r+2s+2)q_{\mathrm{d}}^{16(2r+1)(2s+1)-4}\,, \tag{3.14}$$

$$d_{yz}(q_{\mathrm{d}})=\sum_{r\geq1,s\geq1}\big(-(4r+4s-2)q_{\mathrm{d}}^{16(2r-1)2s-4}+(2r+2s)q_{\mathrm{d}}^{64rs-4}\big)\,. \tag{3.15}$$

The parameter $q_{\mathrm{d}}=\exp(-\mathrm{area}(\Delta))$, where Δ is the minimal disc bounded by the Seidel Lagrangian in $\mathbb{P}^1_{2,4,4}$, is related to the Kähler parameter q of the elliptic curve by $q=q_{\mathrm{d}}^{32}$.

We can rewrite d_y and d_{yz} in terms of the Eisenstein series $E_2(q)$ as follows. First we recall that

$$\sum_{m,n\geq1}mq^{mn}=\sum_{n=1}^{\infty}\sigma_1(n)q^n=\frac{1}{24}(1-E_2(q))\,. \tag{3.16}$$

This identity implies that

$$\sum_{m,n\,\text{even}} mq^{mn} = 2\sum_{a,b} aq^{4ab} = \frac{1}{12}(1 - E_2(q^4))\,.$$

$$\sum_{\substack{m\,\text{odd}\\ n\,\text{even}}} mq^{mn} = \sum_{\substack{m\\ n\,\text{even}}} mq^{mn} - \sum_{\substack{m\,\text{even}\\ n\,\text{even}}} mq^{mn} = \frac{1}{24}(1 - E_2(q^2)) - \frac{1}{12}(1 - E_2(q^4))$$
$$= \frac{1}{24}(-1 - E_2(q^2) + 2E_2(q^4))\,.$$

$$\sum_{\substack{m\,\text{even}\\ n\,\text{odd}}} mq^{mn} = \sum_{\substack{m\,\text{even}\\ n}} mq^{mn} - \sum_{\substack{m\,\text{even}\\ n\,\text{even}}} mq^{mn} = \frac{1}{12}(1 - E_2(q^2)) - \frac{1}{12}(1 - E_2(q^4))$$
$$= \frac{1}{12}(E_2(q^4) - E_2(q^2))\,.$$

$$\sum_{\substack{m\,\text{odd}\\ n\,\text{odd}}} mq^{mn} = \sum_{m,n} mq^{mn} - \sum_{\substack{m\,\text{odd}\\ n\,\text{even}}} mq^{mn} - \sum_{\substack{m\,\text{even}\\ n\,\text{odd}}} mq^{mn} - \sum_{\substack{m\,\text{even}\\ n\,\text{even}}} mq^{mn}$$
$$= -\frac{E_2(q)}{24} + \frac{1}{8}E_2(q^2) - \frac{1}{12}E_2(q^4)\,.$$

where the sums are all over positive integers. Therefore,

$$\begin{aligned}
d_y(q_{\mathrm{d}}) &= \frac{1}{2}\sum_{r,s\geq 0}(2r+2s+2)q_{\mathrm{d}}^{16(2r+1)(2s+1)-4} \\
&= \frac{1}{2}q_{\mathrm{d}}^{-4}\sum_{m,n\,\text{odd}}(m+n)q_{\mathrm{d}}^{16mn} \\
&= q_{\mathrm{d}}^{-4}\sum_{m,n\,\text{odd}} mq_{\mathrm{d}}^{16mn} \\
&= q^{-\frac{1}{8}}\left(-\frac{E_2(q^{\frac{1}{2}})}{24} + \frac{1}{8}E_2(q) - \frac{1}{12}E_2(q^2)\right),
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
d_{yz}(q_{\mathrm{d}}) &= \sum_{r\geq 1, s\geq 1}\big(-(4r+4s-2)q_{\mathrm{d}}^{16(2r-1)2s-4} + (2r+2s)q_{\mathrm{d}}^{64rs-4}\big) \\
&= -2q_{\mathrm{d}}^{-4}\sum_{\substack{m\,\text{odd}\\ n\,\text{even}}}(m+n)q^{\frac{mn}{2}} + q_{\mathrm{d}}^{-4}\sum_{m,n\,\text{even}}(m+n)q^{\frac{mn}{2}} \\
&= -2q^{-\frac{1}{8}}\left(-\frac{1}{24} - \frac{E_2(q)}{8} + \frac{E_2(q^2)}{6}\right) + \frac{1}{6}q^{-\frac{1}{8}}(1 - E_2(q^2)) \\
&= q^{-\frac{1}{8}}\left(\frac{1}{4} + \frac{E_2(q)}{4} - \frac{E_2(q^2)}{2}\right).
\end{aligned} \tag{3.18}$$

We now apply the results for modular forms of the group $\Gamma_0(2)$ in Section 2.2. For this case, it is easy to see (for example, by dimension reasons) that

$$A_2^2(q) = 2E_2(q^2) - E_2(q)\,.$$

It is the generator for $M_2(\Gamma_0(2))$. Moreover, by using the η-expressions for the modular forms A_2, B_2, C_2, we get, see e.g., [Mai11],

$$\begin{aligned} A_2^2(q^2) &= \frac{1}{4}(A_2^2(q) + 3B_2^2(q)), \\ C_2^2(q^2) &= \frac{1}{4}(A_2^2(q) - B_2^2(q)), \\ A_2^4(q) &= B_2^4(q) + C_2^4(q). \end{aligned}$$

Thus, we obtain

$$d_y(q) = \frac{q^{-\frac{1}{8}}}{24}\left(A_2^2(q^{\frac{1}{2}}) - A_2^2(q)\right) = \frac{1}{8}q^{-\frac{1}{8}} \cdot C_2^2(q), \tag{3.19}$$

$$d_{yz}(q) = q^{-\frac{1}{8}}\left(\frac{1}{4} - \frac{A_2^2(q)}{4}\right). \tag{3.20}$$

Using the θ-expansions for the modular forms of $N = 2, 4$ cases and the results on $M_*(\Gamma(4))$, we know that both $A_2^2 = A_4^2 + C_4^2$ and $C_2^2 = 2A_4C_4$ are modular forms of $\Gamma(4)$. On can redefine the variables x, y, z suitably to get rid of the constant $1/4$ and the multiplicative factor $q^{-\frac{1}{8}}$. Then the quantities d_y, d_{yz} become true modular forms.

Under the following change of coordinates in (x, y, z)

$$x \mapsto q_d^{-3}(x + \frac{q_d^{-2}}{2}d_y^{-\frac{1}{4}}d_z^{-\frac{1}{4}}yz), \quad y \mapsto d_y^{-\frac{1}{4}}y, \quad z \mapsto d_z^{-\frac{1}{4}}z,$$

the potential W in (3.13) can be rewritten as

$$W = x^2 + y^4 + z^4 + \sigma(q_{\mathrm{d}})y^2z^2, \tag{3.21}$$

where the generalized SYZ map is

$$\sigma(q_{\mathrm{d}}) := \frac{d_{yz}(q_{\mathrm{d}}) - (4q_{\mathrm{d}}^4)^{-1}}{d_y(q_{\mathrm{d}})} = -\frac{2A_2^2(q)}{C_2^2(q)}. \tag{3.22}$$

Explicitly $\sigma(q_{\mathrm{d}})$ is the series

$$\sigma(q_{\mathrm{d}}) = -\frac{1}{4q_{\mathrm{d}}^{16}} - 5q_{\mathrm{d}}^{16} + \frac{31q_{\mathrm{d}}^{48}}{2} - 54q_{\mathrm{d}}^{80} + \frac{641q_{\mathrm{d}}^{112}}{4} - 409q_{\mathrm{d}}^{144} + \frac{1889q_{\mathrm{d}}^{176}}{2} + \dots \tag{3.23}$$

We now show that $\sigma(q_{\mathrm{d}}(q))$, which comes from generating functions of polygon counting, is the inverse mirror map of the elliptic curve obtained by setting $W = 0$ in (3.13) (again see Section 5 for explanation). We can express the inverse mirror map of the elliptic curve explicitly in terms of η-functions as follows. By the result on elliptic curve families of E_7 type in Section 2.3, the inverse mirror map (as the inverse of the map $a \mapsto \exp 2\pi i\tau(a)$) for

$$x^2 + y^4 + z^4 + axyz = 0 \tag{3.24}$$

is

$$a(q) = 2^{\frac{3}{2}} \frac{A_2(q)}{C_2(q)} . \tag{3.25}$$

To change (3.24) to the form of (3.21), we replace x by $x - \frac{a}{2}yz$ in (3.24) and obtain

$$x^2 + y^4 + z^4 - \frac{a^2}{4} y^2 z^2 = 0 , \tag{3.26}$$

and so the inverse mirror map is

$$-\frac{a^2(q)}{4} = -\frac{2A_2^2(q)}{C_2^2(q)} . \tag{3.27}$$

This coincides with $\sigma(q_{\mathrm{d}}(q))$ in (3.22). As a result, we conclude that

Corollary 3.6. *The generalized SYZ map equals to the inverse mirror map for $\mathbb{P}^1_{2,4,4}$.*

Remark 3.7. We can express everything in terms of the Dedekind η-function

$$\eta(q) = q^{\frac{1}{24}} \left(1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}} \right) \right) .$$

More precisely, from the η-expansions in Section 2, we have

$$A_2(q) = \frac{(2^6 \eta(q^2)^{24} + \eta(q)^{24})^{\frac{1}{4}}}{\eta(q)^2 \eta(q^2)^2} ,$$

$$C_2(q) = 2^{\frac{3}{2}} \frac{\eta(q^2)^4}{\eta(q)^2} .$$

Thus

$$\sigma(q_{\mathrm{d}}(q)) = -2 \left(1 + \frac{\eta(q)^{24}}{2^6 \eta(q^2)^{24}} \right)^{\frac{1}{2}} . \tag{3.28}$$

3.3 $(2, 3, 6)$ case

Theorem 3.8. *[CHKL14] The open Gromov-Witten potential W for $\mathbb{P}^1_{2,3,6}$ is*

$$W = q_{\mathrm{d}}^6 x^2 - q_{\mathrm{d}} xyz + c_y(q_{\mathrm{d}}) y^3 + c_z(q_{\mathrm{d}}) z^6 + c_{yz2}(q_{\mathrm{d}}) y^2 z^2 + c_{yz4}(q_{\mathrm{d}}) yz^4 , \tag{3.29}$$

where

$$A(n, a, b, c) := \binom{n+2}{2} - \binom{a+1}{2} - \binom{b+1}{2} - \binom{c+1}{2}, \tag{3.30}$$

$$c_y(q_{\mathrm{d}}) = \sum_{a \geq 0} (-1)^{a+1} (2a+1) q_{\mathrm{d}}^{48A(a-1,0,0,0)+9}; \tag{3.31}$$

$$c_{yz2}(q_{\mathrm{d}}) = \sum_{n \geq a \geq 0} \left((-1)^{n-a} (6n - 2a + 8) q_{\mathrm{d}}^{48A(n,a,0,0)-4} + (2n+4) q_{\mathrm{d}}^{48A(n,a,n-a,0)-4} \right); \tag{3.32}$$

$$c_{yz4}(q_{\mathrm{d}}) = \sum_{a,b \geq 0, n \geq a+b} (-1)^{n-a-b} (6n - 2a - 2b + 7) q_{\mathrm{d}}^{48A(n,a,b,0)-17}; \tag{3.33}$$

$$c_z(q_{\mathrm{d}}) = \sum (-1)^{n-a-b-c} \left(\frac{6n - 2a - 2b - 2c + 6}{\eta(n, a, b, c)} \right) \cdot q_{\mathrm{d}}^{48A(n,a,b,c)-30} . \tag{3.34}$$

The summation in the expression of $c_z(q_{\mathrm{d}})$ is taken over $(n,a,b,c) \in T_1 \coprod T_2 \coprod T_3 \coprod T_6$,

$$\begin{aligned} T_6 =& \{(3a,a,a,a)\ : a \geq 0\}, \\ T_3 =& \{(n,a,a,a)\ : n > 3a \geq 0\}, \\ T_2 =& \{(a+b+c,a,b,c)\ : a,b,c \geq 0 \text{ such that } a < \min(b,c) \text{ or } a = c < b\}, \\ T_1 =& \{(a+b+c+k,a,b,c)\ : k \in \mathbb{Z}_{>0}, a,b,c \text{ are distinct non-negative integers such that} \\ & \quad a < \min(b,c) \text{ or } a = c < b\}, \end{aligned}$$

and $\eta(n,a,b,c) = i$ for $(n,a,b,c) \in T_i$.

By the change of coordinates in (x,y,z),

$$\begin{aligned} x &\mapsto q_d^{-3}(x + \frac{1}{2}q_d^{-2}c_y^{-\frac{1}{3}}syz + \frac{s^3(1-4q_d^4c_{yz2})}{24q_d^6c_y}z^3)\,, \\ y &\mapsto c_y^{-\frac{1}{3}}(y + s^2\frac{1-4q_d^4c_{yz2}}{12q_d^4c_y^{\frac{2}{3}}}z^2)\,, \quad z \mapsto sz\,, \end{aligned}$$

where

$$s = 864^{\frac{1}{6}}q_d^2c_y^{\frac{1}{3}}(-1+12q_d^4c_{yz2}-48q_d^8c_{yz2}^2+72q_d^8c_yc_{yz2}+64q_d^{12}c_{yz2}^3-288q_d^{12}c_yc_{yz2}c_{yz4}+864q_d^{12}c_y^2c_z)^{-\frac{1}{6}}\,,$$

the open Gromov-Witten potential in (3.29) can be written as

$$x^2 + y^3 + z^6 + \sigma(q_{\mathrm{d}})yz^4\,, \tag{3.35}$$

where the generalized SYZ map is

$$\begin{aligned} \sigma(q_{\mathrm{d}}) = &\left(c_{yz4}(q_{\mathrm{d}}) - \frac{c_{yz2}^2(q_d)}{3c_y(q_{\mathrm{d}})} - (48q_d^8c_y(q_{\mathrm{d}}))^{-1} + \frac{c_{yz2}(q_{\mathrm{d}})}{6q_{\mathrm{d}}^4c_y(q_{\mathrm{d}})}\right)c_y^{-\frac{1}{3}}(q_{\mathrm{d}}) \\ &\cdot\left(c_z(q_{\mathrm{d}}) + \frac{2c_{yz2}^3(q_{\mathrm{d}})}{27c_y^2(q_{\mathrm{d}})} - \frac{c_{yz2}(q_{\mathrm{d}})c_{yz4}(q_{\mathrm{d}})}{3c_y(q_{\mathrm{d}})} - (864q_d^{12}c_y^2(q_{\mathrm{d}}))^{-1} + \frac{c_{yz2}(q_{\mathrm{d}})}{72q_{\mathrm{d}}^8c_y^2(q_{\mathrm{d}})}\right. \\ &\left. - \frac{c_{yz2}^2(q_{\mathrm{d}})}{18q_{\mathrm{d}}^4c_y^2(q_{\mathrm{d}})} + \frac{c_{yz4}(q_{\mathrm{d}})}{12q_{\mathrm{d}}^4c_y(q_{\mathrm{d}})}\right)^{-\frac{2}{3}}. \end{aligned} \tag{3.36}$$

By direct computation, $\sigma(q) := \sigma(q_{\mathrm{d}}(q))$ takes the form

$$\begin{aligned} \sigma(q) = -\frac{3}{2^{2/3}} \cdot (&1 + 576q + 235008q^2 + 109880064q^3 + 53449592832q^4 \\ &+ 26574124961664q^5 + \dots) \end{aligned} \tag{3.37}$$

and so $\sigma(q) = -\frac{3}{2^{2/3}}$ at $q = 0$.

We now show that $\sigma(q_{\mathrm{d}}(q))$ is the inverse mirror map for the elliptic curve defined by setting W in (3.35) to be zero, where $q = q_{\mathrm{d}}^{48}$. We also give an explicit expression of the inverse mirror map in terms of modular functions. First, by the results in Section 2.3 the inverse mirror map for

$$x^2 + y^3 + z^6 + axyz \tag{3.38}$$

is

$$a = -(432)^{\frac{1}{6}} \cdot \frac{E_4^{\frac{1}{4}}}{((E_4^{\frac{3}{2}} - E_6)/2)^{\frac{1}{6}}} . \tag{3.39}$$

where E_4 and E_6 are the Eisenstein series. Again as before we are now considering the elliptic curve family given by $W = 0$. Then we apply a change of coordinates in (x, y, z) to change (3.38) to the form in (3.35). This is achieved by first replacing x by $x - \frac{a}{2}yz$ to change the term xyz to y^2z^2, and then replacing y to $y + \frac{a^2}{12}z^2$ to replace the term y^2z^2 to yz^4. As a result, (3.38) is changed to

$$x^2 + y^3 + z^6 - \frac{3a^4}{2^{\frac{2}{3}}(864 - a^6)^{\frac{2}{3}}} yz^4 . \tag{3.40}$$

By substituting a in (3.39) into the above expression, we obtain that the inverse mirror map for the elliptic curve $x^2 + y^3 + z^6 + syz^4$ given by

$$s(q) = \frac{-3E_4^3(q)}{2^{\frac{2}{3}} E_6^2(q)} . \tag{3.41}$$

One can do a computational check that $s(q)$ has the same expression in (3.37) as $\sigma(q)$.

Remark 3.9. Similar to the other cases, we expect the quantities $c_y, c_{yz2}, c_{yz4}, c_z$ to be modular forms up to addition and multiplication by some factors which are not essential, so that the generalized SYZ map in (3.36) coincides with the expression given in (3.41). This is true for c_y. In fact, we have

$$\begin{aligned} c_y(q_{\mathrm{d}}) &= q_{\mathrm{d}}^9 \sum_{a \geq 0} (-1)^{a+1}(2a+1) q_{\mathrm{d}}^{24a(a+1)} \\ &= q_{\mathrm{d}}^3 \sum_{a \geq 0} (-1)^{a+1}(2a+1) q_{\mathrm{d}}^{24(a+\frac{1}{2})^2} \\ &= 2q^{\frac{1}{16}} \sum_{r \geq 0, r \in \mathbb{Z}+\frac{1}{2}} (-1)^{r+\frac{1}{2}} r q^{\frac{1}{2}r^2} \\ &= q^{\frac{1}{16}} \mathbf{i} \partial_v \theta_1|_{v=1} \\ &= -q^{\frac{1}{16}} \eta(q)^3 . \end{aligned} \tag{3.42}$$

Also the second term in c_{yz2} (which counts parallelograms) is

$$\begin{aligned} & q_{\mathrm{d}}^{-4} \sum_{n \geq a \geq 0} (2n+4) q_{\mathrm{d}}^{48(a+1)(n-a+1)} \\ =\ & q_{\mathrm{d}}^{-4} \sum_{a \geq 0, b \geq 0} (2(a+b)+4) q_{\mathrm{d}}^{48(a+1)(b+1)} \\ =\ & 2q_{\mathrm{d}}^{-4} \sum_{a \geq 1, b \geq 1} (a+b) q_{\mathrm{d}}^{48ab} \\ =\ & 2q_{\mathrm{d}}^{-4} \cdot \frac{1}{12}(1 - E_2(q_{\mathrm{d}}^{48})) \\ =\ & \frac{1}{6} q^{-\frac{1}{12}} (1 - E_2(q)) . \end{aligned} \tag{3.43}$$

We conjecture that the rest are quasi-modular forms as introduced by [KZ95] and the overall coefficients are modular forms. See Section 4.3 for further discussions.

3.4 $(2,2,2,2)$ case

The remaining case of elliptic orbifolds is $\mathbb{P}^1_{2,2,2,2}$. It can be constructed as a quotient of an elliptic curve E by $\mathbb{Z}_2$, where $1 \in \mathbb{Z}_2$ acts by $z \mapsto -z \in E$. The generalized SYZ mirror construction in this case is rather different, namely it involves more than one reference Lagrangians. The construction is given in [CHL15], here we quote the result below. It turns out that the mirror is not an isolated singularity, and hence Saito's theory of primitive forms does not apply directly to this case.

Theorem 3.10. *[CHL15] The open Gromov-Witten potential of* $\mathbb{P}^1_{2,2,2,2}$ *is*

$$W = \phi(q_d)((xy)^2 + (xw)^2 + (zy)^2 + (zw)^2) + \psi(q_d)xyzw \tag{3.44}$$

defined on the resolved conifold $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) = (\mathbb{C}^4 - Z)/\mathbb{C}^\times$, *where* (x, y, z, w) *are the standard coordinates of* $\mathbb{C}^4$, $Z = \{x = z = 0\}$, $\mathbb{C}^\times$ *acts by* $\lambda \cdot (x, y, z, w) = (\lambda x, \lambda^{-1}y, \lambda z, \lambda^{-1}w)$, *and*

$$\begin{aligned} \phi(q_d) &= \sum_{k,l\geq 0}^{\infty} (4k+1)q_d^{(4k+1)(4l+1)} + \sum_{k,l\geq 0}^{\infty} (4k+3)q_d^{(4k+3)(4l+3)}, \\ \psi(q_d) &= \sum_{k,l\geq 0}^{\infty} (k+l+1)q_d^{(4k+1)(4l+3)}. \end{aligned}$$

The parameter $q_{\mathrm{d}} = \exp(-\mathrm{area}(\Delta))$, where Δ is a certain holomorphic square in $\mathbb{P}^1_{2,2,2,2}$, is related to the Kähler parameter q of the elliptic curve E by $q = q_{\mathrm{d}}^8$.

By direct computation, the critical locus of W is the zero section $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ instead of a point. The Frobenius structure on the universal deformation space of W is unclear since Saito's theory is not yet known for non-isolated singularities. Nevertheless, we can consider the mirror elliptic curve family to obtain the flat coordinate for marginal deformations, and compare it with the generalized SYZ map ψ/ϕ.

To be more precise, W descends to the quotient of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ by $\mathbb{Z}_2$, which is the total space of canonical line bundle $K_{\mathbb{P}^1\times\mathbb{P}^1}$. The critical locus of W in $K_{\mathbb{P}^1\times\mathbb{P}^1}$ is the elliptic curve $\{W = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$ which is the mirror of E, where $(x : z, y : w)$ are the standard homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. It can also be embedded into $\mathbb{P}^3$ via Segre embedding

$$x_1 = xy, x_2 = xw, x_3 = zw, x_4 = zy.$$

Then the mirror of E is the elliptic curve given as the complete intersection

$$\{x_1x_3 = x_2x_4\} \cap \{\phi(q_d)(x_1^2 + x_2^2 + x_3^2 + x_4^2) + \psi(q_d)x_1x_3 = 0\} \subset \mathbb{P}^3.$$

The j-invariant of the elliptic curve family

$$\{((xy)^2 + (xw)^2 + (zy)^2 + (zw)^2) + \sigma xyzw = 0\} \subset \mathbb{P}^1 \times \mathbb{P}^1$$

can be obtained by using the algorithm provided in [Con96], which is

$$j(\sigma) = \frac{\left(\sigma^4 - 16\sigma^2 + 256\right)^3}{\sigma^4\left(\sigma^2 - 16\right)^2}\,. \tag{3.45}$$

Comparing this with the j-invariant for the E_5 elliptic curve family discussed in Section 2.3, we are led to

$$\sigma = 2 \cdot \frac{1 + \alpha^{\frac{1}{2}}}{\alpha^{\frac{1}{4}}}\,, \tag{3.46}$$

where α is the Hauptmodul for $\Gamma_0(4)$.

Now we consider the generalized SYZ map ψ/ϕ. We can rewrite $\phi(q_d)$ and $\psi(q_d)$ in terms of η-products as follows. Using the computations used in deriving (3.17), we find

$$\psi(q_d) + 4\phi(q_d) = \frac{\eta(q_d^4)^8}{\eta(q_d^2)^4}\,. \tag{3.47}$$

This identity implies that

$$\psi(q_d) - 4\phi(q_d) = \frac{\eta(q_d^4)^8}{\eta(-q_d^2)^4} = \frac{\eta(q_d^8)^4\eta(q_d^2)^4}{\eta(q_d^4)^4}\,. \tag{3.48}$$

Solving for $\phi(q_d), \psi(q_d)$ from the above two identities, we obtain

$$\phi(q_d) = \frac{\eta(q_d^8)^2\eta(q_d^{16})^4}{\eta(q_d^4)^2}\,, \quad \psi(q_d) = \frac{\eta(q_d^8)^{14}}{\eta(q_d^4)^6\eta(q_d^{16})^4}\,. \tag{3.49}$$

Now using the η-expansions of the modular forms for $\Gamma_0(4)$ in Table 1, we get (recall $q_d = q^{\frac{1}{8}}$)

$$\phi = \frac{1}{2^3}A_4(q^{\frac{1}{2}})^{\frac{1}{2}}C_4(q^{\frac{1}{2}})^{\frac{3}{2}}\,, \quad \psi = \frac{1}{2}A_4(q^{\frac{1}{2}})^{\frac{1}{2}}C_4(q^{\frac{1}{2}})^{\frac{1}{2}}\,. \tag{3.50}$$

Since $\Gamma_0(4)$ is isomorphic to $\Gamma(2)$ via $\tau \mapsto 2\tau$, we know that if $f(\tau)$ is a modular form for $\Gamma_0(4)$, then $f(\frac{\tau}{2})$ is so for $\Gamma(2)$. This tells that ϕ, ψ are modular forms for $\Gamma(2)$.

It follows that the generalized SYZ map is

$$\frac{\psi(q_d(q))}{\phi(q_d(q))} = 4\frac{A_4(q^{\frac{1}{2}})}{C_4(q^{\frac{1}{2}})} = \frac{\eta(q)^{12}}{\eta(q^2)^8\eta(q^{\frac{1}{2}})^4}\,. \tag{3.51}$$

Using the η-expansions of the modular forms for $\Gamma_0(4)$ in Table 1, we see that the generalized SYZ map in (3.51) produced by Lagrangian Floer theory is identical to the modular function given by (3.46). As a result, the generalized SYZ map equals to the inverse mirror map for $\mathbb{P}^1_{2,2,2,2}$.

4 Modularity of matrix factorizations

In [CHL13], an A_∞ functor was constructed from the Fukaya category of Lagrangian branes in a symplectic manifold X to the category of matrix factorizations of the open Gromov-Witten potential W. The construction of W was reviewed in the beginning of Section 3. For $W \in R = \mathbb{C}[z_1, \ldots, z_n]$, a matrix factorization is simply an odd endomorphism δ on a $\mathbb{Z}_2$-graded R-module $M = M_0 \oplus M_1$ which satisfies $\delta^2 = W \cdot \mathrm{Id}$. Such a functor is motivated from the celebrated homological mirror symmetry conjecture [Kon95].

Let us review very briefly the functor in the object level. Given a spin oriented Lagrangian L which intersects the reference Lagrangian $\mathcal{L}$ (fixed in the beginning of Section 3) transversely, define $M = \oplus_p R \cdot p$ where the sum is over all intersection points $p \in L \cap \mathcal{L}$, and $R \cdot p$ has odd (or even) degree if p has odd (or even) degree. Then δ is defined to be $m_1^{(b,0)}$ (which automatically has odd degree), which is roughly speaking counting pseudo-holomorphic strips with one side bounded by $(\mathcal{L}, b)$ and another side bounded by L. Since the formal deformation b is assumed to be weakly unobstructed, it follows from the A_∞ relation

$$(m_1^{(b,0)})^2 = m_2(m_0^b, \cdot) = m_2(m_0^b, \cdot) = m_2(W(b)\mathbb{1}_{\tilde{\mathcal{L}}}, \cdot) = W(b) \cdot \mathrm{Id} \tag{4.1}$$

that δ is a matrix factorization.

In particular, the Seidel Lagrangian of an elliptic $\mathbb{P}^1$ orbifold can be transformed to a matrix factorization of the open Gromov-Witten potential W. They are split generators of the derived Fukaya category and the derived category of matrix factorizations respectively. In this section, we study the modularity of the matrix factorizations constructed from the potential W for the elliptic orbifolds.

4.1 $(3, 3, 3)$ case

The matrix factorization mirror to the Seidel Lagrangian in $\mathbb{P}^1_{3,3,3}$ was computed in [CHKL14, Section 7.7]. In the following we check that their coefficients are modular forms with possibly non-trivial multiplier systems.

Theorem 4.1. *The matrix factorization mirror to the Seidel Lagrangian in $\mathbb{P}^1_{3,3,3}$ is $M = (\bigwedge^* \mathbb{C}^3, \delta)$ where $\delta = (xX + yY + zZ) \wedge \cdot + w_x \iota_X + w_y \iota_Y + w_z \iota_Z$, and w_x, w_y, w_z are the following polynomials whose coefficients are modular forms:*

$$w_x = (-\eta(q^3)^3)x^2 + \left(-\frac{1}{3}\eta(q^{\frac{1}{3}})^3 + \eta(q^3)^3 - \frac{2}{3}\eta(q)\right) yz,$$

$$w_y = (-\eta(q^3)^3)y^2 + \left(-\frac{1}{3}\eta(q^{\frac{1}{3}})^3 + \eta(q^3)^3 + \frac{1}{3}\eta(q)\right) xz,$$

$$w_z = (-\eta(q^3)^3)z^2 + \left(-\frac{1}{3}\eta(q^{\frac{1}{3}})^3 + \eta(q^3)^3 + \frac{1}{3}\eta(q)\right) xy.$$

Proof. From the result of [CHKL14, Section 7.7], the matrix factorization is (M, δ) defined above where

$$\begin{aligned}
w_x =& x^2 \sum_{k=0}^{\infty} (-1)^{k+1} (2k+1) q_{\mathrm{d}}^{(3(2k+1))^2} \\
&+ yz \left(-q_{\mathrm{d}} + \sum_{k=1}^{\infty} (-1)^{k+1} \left((2k+1) q_{\mathrm{d}}^{(6k+1)^2} - (2k-1) q_{\mathrm{d}}^{(6k-1)^2} \right) \right), \\
w_y =& y^2 \sum_{k=0}^{\infty} (-1)^{k+1} (2k+1) q_{\mathrm{d}}^{(3(2k+1))^2} + xz \sum_{k=1}^{\infty} (-1)^{k+1} \left(2k q_{\mathrm{d}}^{(6k+1)^2} - 2k q_{\mathrm{d}}^{(6k-1)^2} \right), \\
w_z =& z^2 \sum_{k=0}^{\infty} (-1)^{k+1} (2k+1) q_{\mathrm{d}}^{(3(2k+1))^2} + xy \sum_{k=1}^{\infty} (-1)^{k+1} \left(2k q_{\mathrm{d}}^{(6k+1)^2} - 2k q_{\mathrm{d}}^{(6k-1)^2} \right).
\end{aligned}$$

The coefficient of x^2 in w_x (or that of y^2 in w_y, or that of z^2 in w_z) equals to $\phi(q_{\mathrm{d}}) = \mathbf{i}\partial_v \theta_1(q^3, 1)$. The coefficient of yz in w_x is

$$\begin{aligned}
& -q_{\mathrm{d}} + \sum_{k=1}^{\infty} (-1)^{k+1} \left((2k+1) q_{\mathrm{d}}^{(6k+1)^2} - (2k-1) q_{\mathrm{d}}^{(6k-1)^2} \right) \\
=& \sum_{k=-\infty}^{\infty} (-1)^{k+1} (2k+1) q_{\mathrm{d}}^{(6k+1)^2} \\
=& \frac{1}{3} \sum_{k=-\infty}^{\infty} (-1)^{k+1} (6k+3) q_{\mathrm{d}}^{(6k+1)^2} \\
=& \frac{\psi(q_{\mathrm{d}})}{3} + \frac{2}{3} \sum_{k=-\infty}^{\infty} (-1)^{k+1} q_{\mathrm{d}}^{(6k+1)^2} \\
=& \frac{1}{3}\psi(q_{\mathrm{d}}) - \frac{2}{3}\eta(q_{\mathrm{d}}^{24}),
\end{aligned} \tag{4.2}$$

where we have used the identity that

$$\eta(q) = q^{\frac{1}{24}} \sum_{k=-\infty}^{\infty} (-1)^k q^{\frac{3k^2-k}{2}}.$$

Written in terms of the parameter q, this is

$$\frac{1}{3}\psi(q) - \frac{2}{3}\eta(q). \tag{4.3}$$

The coefficient of xz in w_y (or that of xy in w_z) is

$$\begin{aligned}
& \sum_{k=1}^{\infty} (-1)^{k+1} \left(2k q_{\mathrm{d}}^{(6k+1)^2} - 2k q_{\mathrm{d}}^{(6k-1)^2} \right) \\
=& \sum_{k=-\infty}^{\infty} (-1)^{k+1} (2k) \cdot q_{\mathrm{d}}^{(6k+1)^2} \\
=& \frac{1}{3} \sum_{k=-\infty}^{\infty} (-1)^{k+1} (6k+1) q_{\mathrm{d}}^{(6k+1)^2} - \frac{1}{3} \sum_{k=-\infty}^{\infty} (-1)^{k+1} q_{\mathrm{d}}^{(6k+1)^2} \\
=& \frac{1}{3}\psi(q_{\mathrm{d}}) + \frac{1}{3}\eta(q_{\mathrm{d}}^{24}).
\end{aligned} \tag{4.4}$$

Written in terms of the parameter q, this is

$$\frac{1}{3}\psi(q) + \frac{1}{3}\eta(q)\,. \tag{4.5}$$

All mentioned earlier in Section 3, both ϕ, ψ are modular forms with respect to $\Gamma(3)$, hence all the coefficients studied here are modular forms, and they have the explicit expressions as stated in the theorem. □

Remark 4.2. It is easy to check that $xw_x + yw_y + zw_z = W$ by straightforward calculation.

4.2 $(2,4,4)$ case

Theorem 4.3. *The matrix factorization mirror to the Seidel Lagrangian of $\mathbb{P}^1_{2,4,4}$ is $M = (\bigwedge^* \mathbb{C}^3, \delta)$ where*

$$\delta = (xX + yY + zZ) \wedge \cdot + w_x \iota_X + w_y \iota_Y + w_z \iota_Z,$$

and w_x, w_y, w_z are the following polynomials whose coefficients are modular forms (up to a multiple by a power of q):

$$\begin{aligned}
w_x &= q^{\frac{3}{16}} x - q^{\frac{1}{32}} yz,\\
w_y &= \left(\frac{1}{8} q^{-\frac{1}{8}} \cdot C_2^2(q)\right) y^3 + \left(\frac{q^{-\frac{1}{8}}}{8}\left(1 - A_2^2(q)\right)\right) yz^2,\\
w_z &= \left(\frac{1}{8} q^{-\frac{1}{8}} \cdot C_2^2(q)\right) z^3 + \left(\frac{q^{-\frac{1}{8}}}{8}\left(1 - A_2^2(q)\right)\right) y^2 z.
\end{aligned}$$

Proof. It is a direct computation as in [CHKL14, Section 7.7] that the mirror matrix factorization is (M, δ) defined above, where

$$\begin{aligned}
w_x =& q_{\mathrm{d}}^6 x - q_{\mathrm{d}} yz\,,\\
w_y =& \left(\sum_{0\le r} (2r+1) q_{\mathrm{d}}^{16(2r+1)^2-4} + \sum_{0\le r<s} (2r+2s+2) q_{\mathrm{d}}^{16(2r+1)(2s+1)-4}\right) y^3\\
&+ \left(\sum_{r\ge 1, s\ge 1} \left(-(2r+2s-1) q_{\mathrm{d}}^{16(2r-1)2s-4} + 2r q_{\mathrm{d}}^{64rs-4}\right)\right) yz^2\,,\\
w_z =& \left(\sum_{0\le r} (2r+1) q_{\mathrm{d}}^{16(2r+1)^2-4} + \sum_{0\le r<s} (2r+2s+2) q_{\mathrm{d}}^{16(2r+1)(2s+1)-4}\right) z^3\\
&+ \left(\sum_{r\ge 1, s\ge 1} \left(-(2r+2s-1) q_{\mathrm{d}}^{16(2r-1)2s-4} + 2s q_{\mathrm{d}}^{64rs-4}\right)\right) y^2 z\,.
\end{aligned}$$

The coefficient of y^3 in w_y (or that of z^3 in w_z) is nothing but d_y studied in Section 3, while the coefficient of yz^2 in w_y (or that of $y^2 z$ in w_z) is $d_{yz}/2$. They have been shown to be modular forms with respect to $\Gamma(4)$ in the previous section. □

4.3 $(2,3,6)$ case

Similarly, we can directly compute the matrix factorization mirror to the Seidel Lagrangian of $\mathbb{P}^1_{2,3,6}$. The result is $(M = \bigwedge^* \mathbb{C}^3, \delta)$, where

$$\delta = (xX + yY + zZ) \wedge \cdot + w_x \iota_X + w_y \iota_Y + w_z \iota_Z,$$

and w_x, w_y, w_z are defined by

$$\begin{aligned}
w_x =& q_{\mathrm{d}}^6 x - q_{\mathrm{d}} yz, \\
w_y =& c_y(q_{\mathrm{d}}) y^2 + yz^2 \sum_{a,b \geq 0} \left((-1)^b (2a + 4b + 5) q_{\mathrm{d}}^{48A(a+b,a,0,0)-4} + (2b+2) q_{\mathrm{d}}^{48A(a+b,a,b,0)-4} \right) \\
&+ z^4 \sum_{a,b \geq 0, n \geq a+b} (-1)^{n-a-b} (2n - 2a + 2) q_{\mathrm{d}}^{48A(n,a,b,0)-17}, \\
w_z =& c_z(q_{\mathrm{d}}) z^5 + y^2 z \sum_{a,b \geq 0} \left((-1)^b (2a + 2b + 3) q_{\mathrm{d}}^{48A(a+b,a,0,0)-4} + (2a+2) q_{\mathrm{d}}^{48A(a+b,a,b,0)-4} \right) \\
&+ yz^3 \sum_{a,b \geq 0, n \geq a+b} (-1)^{n-a-b} (4n - 2b + 5) q_{\mathrm{d}}^{48A(n,a,b,0)-17},
\end{aligned}$$

and $A(n,a,b,c)$, c_y and c_z are given in (3.30), (3.31) and (3.34) respectively.

The sum of coefficients for the yz^2, y^2z terms of (M, δ) gives the one for y^2z^2 in W, similarly for z^4, yz^3 terms. Recall that

$$q_{\mathrm{d}}^{48} = q\,, \quad A(n,a,b,c) = \binom{n+2}{2} - \binom{a+1}{2} - \binom{b+1}{2} - \binom{c+1}{2}.$$

By pulling out q_{d}^{-4} for the first parts in the yz^2, y^2z terms, we get

$$\sum_{a,b \geq 0} (-1)^b (4b + 2a + 5) q^{\frac{1}{2}(b+1))b+1+2a+1)}\,, \tag{4.6}$$

$$\sum_{a,b \geq 0} (-1)^b (2b + 2a + 3) q^{\frac{1}{2}(b+1)(b+1+2a+1)}\,. \tag{4.7}$$

The following quantity is easily computed:

$$\sum_{a,b \geq 0} (-1)^b (2a+1) q^{\frac{1}{2}(b+1)(b+1+2a+1)} = \frac{1}{24}(1 - E_2(q))\,. \tag{4.8}$$

More precisely, we have

$$\begin{aligned}
&\sum_{a,b\geq 0}(-1)^b(2a+1)q^{\frac{1}{2}(b+1)(b+1+2a+1)}\\
=&\sum_{\substack{k\geq 1,l\geq k,\\ l=k+\text{odd}}}(-1)^{k-1}(l-k)q^{\frac{1}{2}kl}\\
=&\sum_{\substack{k\geq 1,l\geq k,\\ l=k+\text{odd}}}((-1)^k k+(-1)^l l)q^{\frac{1}{2}kl}\\
=&\sum_{\substack{k,l\geq 1,\\ l=k+\text{odd}}}(-1)^k k q^{\frac{1}{2}kl}\\
=&\sum_{\substack{k,l\geq 1,\\ k=\text{odd},\\ l=\text{even}}}(-1)^k k q^{\frac{1}{2}kl}+\sum_{\substack{k,l\geq 1,\\ k=\text{even},\\ l=\text{odd}}}(-1)^k k q^{\frac{1}{2}kl}\\
=&-\sum_{\substack{k,l\geq 1,\\ k=\text{odd},\\ l=\text{even}}}k q^{\frac{1}{2}kl}+\sum_{\substack{k,l\geq 1,\\ k=\text{even},\\ l=\text{odd}}}k q^{\frac{1}{2}kl}\,.
\end{aligned}$$

Then the statement follows from the summations we computed in Section 3.2.

Comparing (4.6), (4.7) with (4.8), we can see what is left is to calculate

$$\sum_{a,b\geq 0}(-1)^b(b+1)q^{\frac{1}{2}(b+1)(b+1+2a+1)}\,.$$

This can be simplified further as follows (changing the variable $b+1$ to k)

$$\sum_{k\geq 1,a\geq 0}(-1)^{k-1}kq^{\frac{1}{2}k(k+2a+1)}=\sum_{k\geq 1}(-1)^{k-1}k\frac{q^{\frac{1}{2}(k^2+k)}}{1-q^k}\,. \tag{4.9}$$

It is related to the derivative of the Appell function of level one[7]. The other terms involving $2b+2, 2a+2$ in the yz^2, y^2z terms can be calculated due to symmetry and the result for W, both are equal to $q^{-\frac{1}{12}}(1-E_2(q))/12$. For the coefficient of z^4 in w_y and that of yz^3 in w_z, we need to compute (by pulling out q_{d}^{-17}, using $q_{\mathrm{d}}^{48}=q$ and defining $k=n-a-b$)

$$\sum_{k,a,b\geq 0}(-1)^k(2k+2b+2)q^{1+a+b+ab+\frac{3k}{2}+ak+bk+\frac{k^2}{2}}\,,$$

$$\sum_{k,a,b\geq 0}(-1)^k(4k+4a+2b+5)q^{1+a+b+ab+\frac{3k}{2}+ak+bk+\frac{k^2}{2}}\,.$$

Taking the difference of the above two formulas, and simplifying a little further, we are left with

$$\sum_{k,a,b\geq 0}(-1)^k(2a+1)q^{1+a+b+ab+\frac{3k}{2}+ak+bk+\frac{k^2}{2}}\,, \tag{4.10}$$

$$\sum_{k,a,b\geq 0}(-1)^k(2k+1)q^{1+a+b+ab+\frac{3k}{2}+ak+bk+\frac{k^2}{2}}\,. \tag{4.11}$$

[7]See Section 7 of [DMZ12] for related discussion on this series.

We expect that all the quantities in (4.9), (4.10), (4.11) are quasi-modular forms (up to a multiple of a power of q) for $\Gamma(6)$ with possibly non-trivial multiplier systems. This would then imply that the coefficients in the matrix factorization (M, δ) for the $(2,3,6)$ case are modular. However, we are not able to prove this at this moment.[8]

5 Mirror symmetry over global moduli

In Section 3 and Section 4 we proved that the potential W and the matrix factorization M are modular for some modular group Γ which depends on the geometry, hence they extend automatically to be sections of holomorphic line bundles on the modular curves $\Gamma\backslash\mathcal{H}^*$. The proof is based on straightforward calculations. In this section we explain why modularity is expected from the point of view of global mirror symmetry.

5.1 LG/CY correspondence

It is well-known that the elliptic curve is self-mirror. This simple important fact can be obtained using group action and LG/CY correspondence as follows.

Given a symplectic torus E, we equip it with the complex structure with an automorphism group G, where $G = \mathbb{Z}_3, \mathbb{Z}_4$ or $\mathbb{Z}_6$. Then $E/G = \mathbb{P}^1_{3,3,3}, \mathbb{P}^1_{2,4,4}$ or $\mathbb{P}^1_{2,3,6}$ respectively. By the mirror construction [CHL13] which is briefly explained in the beginning of Section 3, the Landau-Ginzburg mirror is the open Gromov-Witten potential W defined on $\mathbb{C}^3$ whose explicit expressions are given in Theorem 3.2, 3.5 or 3.8 respectively. The potential W is invariant under the action of the dual group $\check{G} \cong G$, and the mirror of E is given by $(\mathbb{C}^3/\check{G}, W)$ [Sei11, CHL13]. By LG/CY correspondence [Orl09], the complex geometry (so-called the B-model) of $(\mathbb{C}^3/\check{G}, W)$ is equivalent to that of the elliptic curve $\check{E} = \{W = 0\} \subset W\mathbb{P}^2$, where $W\mathbb{P}^2$ is the weighted projective space $(\mathbb{C}^3 - \{0\})/\mathbb{C}^\times$ and the $\mathbb{C}^\times$ action has weights $(1,1,1)$, $(1,2,2)$ and $(2,3,6)$ respectively. This gives an explanation, which is different from the usual SYZ approach, of why the elliptic curve is self-mirror.

$$\begin{tikzcd}
\text{A-model} \arrow[r, leftrightarrow, "\text{mirror symmetry}"] & \text{B-model} \\
E/G \arrow[r, leftrightarrow, "\text{mirror}"] \arrow[dr, "\text{Open Gromov-Witten potential}"] & \check{E} = \{W = 0\} \subset W\mathbb{P}^2 \\
& (\mathbb{C}^3/\check{G}, W) \arrow[u, "\text{LG/CY correspondence}"']
\end{tikzcd}$$

Figure 2: *Chain of dualities*

[8] We are kindly informed by Kathrin Bringmann and Larry Rolen in a private communication that these summations are nice objects which are related to mock modular forms.

The moduli space of complex structures on $\check{E}$ is the (compactified) upper half plane quotient by $\mathrm{SL}(2,\mathbb{Z})$. By global mirror symmetry, the Kähler moduli of E is also the upper half plane quotient by $\mathrm{SL}(2,\mathbb{Z})$ (this can also be seen from considering the moduli space of Bridgeland stability conditions [Bri07]). The global mirror map in this case is simply given by the identity map.

On the other hand, the mirror elliptic curve family under consideration is given by the equation $W = 0$, which is *not* the universal family over the moduli stack $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^*$ of complex structures of the mirror elliptic curve. This elliptic curve family is essentially (up to reparametrization and base change, as shown in Section 3) the elliptic curve families of type E_n reviewed in Section 2. Note that the base change would also alter the modular group for which the parameter σ in the elliptic curve family in (1.1) is a Hauptmodul. Since the parameter for the base of the family $W = 0$ is a modular function for certain modular group, one would expect that the coefficients in the equation $W = 0$, as functions on the modular curve, are related to modular forms. For example, in the $\mathbb{P}^1_{3,3,3}$ case, the equation $W = 0$ defines the universal family of elliptic curves over the modular curve $\Gamma(3)\backslash\mathcal{H}^*$, and the parameters ϕ, ψ are modular forms for $\Gamma(3)$. The big picture is illustrated in Figure 2.

Now in order to see more clearly why it is the modular subgroup Γ instead of the full modular group $\mathrm{SL}(2,\mathbb{Z})$ that enters the picture, the main point is as follows. We have fixed the Seidel Lagrangian $\mathcal{L} \subset E/G$ to define the open Gromov-Witten potential. The Lagrangian $\mathcal{L}$ lifts to r copies of Lagrangians $L_1, \ldots, L_r$ in E, where $r = 3, 4, 6$ respectively. Thus the A-side moduli under consideration is *the Kähler structure together with the markings by these r Lagrangians*. By homological mirror symmetry, the corresponding B-side moduli for the mirror is the complex structure on $\check{E}$ together with the coherent sheaves mirror to $L_1, \ldots, L_r$. In the next subsection, we show that these sheaves give rise to a cyclic subgroup of order r of the group of r-torsion points on $\check{E}$. Thus the moduli space is given by the modular curve $X_\Gamma = \Gamma\backslash\mathcal{H}^*$ instead of $\mathrm{SL}(2,\mathbb{Z})\backslash\mathcal{H}^*$.

5.2 T-duality

It is a standard fact that the modular curve $\Gamma_0(r)\backslash\mathcal{H}^*$ is the (coarse) moduli space of pairs (E, H), where E is an elliptic curve and $H < E_r$ is a cyclic subgroup of order r of the group of r-torsion points on E.

For simplicity, we focus on $\mathbb{P}^1_{3,3,3}$, and the other two cases are similar. The Seidel Lagrangian in $\mathbb{P}^1_{3,3,3}$ lifts to three Lagrangian cycles in the elliptic curve E_ρ with its automorphism group generated by the cube root of unity $\rho = \exp(2\pi i/3)$. They are denoted as $\{L, \rho L, \rho^2 L\}$, with

$$[L] = A + 2B\,, \quad [\rho L] = -2A - B\,, \quad [\rho^2 L] = A - B\,, \tag{5.1}$$

where $A, B \in H_1(E_\rho, \mathbb{Z})$ are the generators corresponding to the lattice points 1 and ρ which give rise to the elliptic curve E_ρ, respectively.

We will use T-duality to transform $\{L, \rho L, \rho^2 L\}$ to coherent sheaves on the mirror elliptic curve $\check{E}_\rho$. T-duality and homological mirror symmetry for elliptic curves was well-studied, see for instance [PZ98], and we include it here for completeness of the paper.

To avoid dealing with multi-sections, we consider the double cover $\tilde{E}_\rho$ of the elliptic curve E_ρ with its corresponding lattice generated by $2, \rho$. The Lagrangians $L_1 = L, L_2 =$

$\rho L, L_3 = \rho^2 L$ lifts to Lagrangians $\tilde{L}_1, \tilde{L}_2, \tilde{L}_3$ in the double cover. Take the generators of $H_1(\tilde{E}_\rho)$ to be $\tilde{A}, B$ corresponding to the lattice points $2, \rho$. Then we have

$$[\tilde{L}_1] = \tilde{A} + 4B, [\tilde{L}_2] = \tilde{A} + B, [\tilde{L}_3] = \tilde{A} - 2B\,. \tag{5.2}$$

The intersections are

$$\tilde{L}_1 \cap \tilde{L}_2 = -3, \tilde{L}_2 \cap \tilde{L}_3 = -3, \tilde{L}_3 \cap \tilde{L}_1 = 6\,. \tag{5.3}$$

Let $s = \tilde{L}_1$ and $f = \tilde{B} = -3B$. We then have

$$\tilde{L}_1 = s, \tilde{L}_2 = s + f, \tilde{L}_3 = s + 2f\,. \tag{5.4}$$

Consider the elliptic curve C whose lattice is generated by $2 + 4\rho, -3\rho$. Now s and f can be regarded as a section and a fiber of a Lagrangian fibration on this elliptic curve. By T-duality, they are mirror to the following sheaves on the mirror curve $\check{C}$: $\mathcal{O}_1 = \mathcal{O}, \mathcal{O}_2 = \mathcal{O}(D), \mathcal{O}_3 = \mathcal{O}(2D)$ where D is the divisor of degree 1 corresponding to the fiber class f (equipped with trivial flat connection).

The action which takes a Lagrangian section s to $s + f$ corresponds to tensoring $\mathcal{O}(D)$ in the mirror curve $\check{C}$. The relation $\rho^3 = 1$ says the mirror $\mathbb{Z}_3$ action permutes $\mathcal{O}, \mathcal{O}(D), \mathcal{O}(2D)$ cyclically. It follows that the sheaves give rise to a cyclic subgroup of order 3 of the group of 3-torsion points on the variety $\mathrm{Pic}^0(\check{C})$, which is isomorphic to the mirror elliptic curve $\check{C}$ itself.

To conclude, for the mirror side, we should consider the moduli space of complex structures of an elliptic curve decorated with a cyclic subgroup of order three of the group of 3-torsion points on the elliptic curve. Thus the global moduli is given by $\Gamma_0(3)\backslash\mathcal{H}^*$, and the open Gromov-Witten potential should be globally defined over $\Gamma_0(3)\backslash\mathcal{H}^*$. From previous sections we see that it is actually a global object over $\Gamma(3)\backslash\mathcal{H}^*$.

5.3 One more example

We now give one more example for which the global moduli space of Kähler structures can be identified with a modular curve and the generating functions of Gromov-Witten invariants are modular forms.

The mirror manifold of $K_{\mathbb{P}^2}$ is a non-compact Calabi-Yau 3-fold X given by [HV00]

$$\{uv = 1 + z + w + \alpha/zw\} \subseteq \mathbb{C}^2_{u,v} \times (\mathbb{C}^\times)^2_{z,w}\,, \tag{5.5}$$

and is a conic fibration over the base $(\mathbb{C}^\times)^2_{z,w}$. The flat coordinate, denoted by $t(\alpha)$, for the threefold X can be expressed in terms of the flat coordinate $\tau(\alpha)$ for the corresponding elliptic curve $\{1 + z + w + \alpha/zw = 0\} \subset (\mathbb{C}^\times)^2_{z,w}$ which is the discriminant locus of the conic fibration.

The idea is the following. On one hand, $\alpha(\tau)$ is automatically a modular form as it is the Hauptmodul for the modular curve $\Gamma_0(3)\backslash\mathcal{H}^*$ which parametrizes the elliptic curve family above, see [ASYZ14]. Thus it is a tautology that $\alpha(t(\tau))$ is a modular form. On the other hand, in the A-model on $K_{\mathbb{P}^2}$, we know that $\alpha(t)$ is a generating function of open Gromov-Witten invariants [CLL12]. Therefore, we know that the generating function of open Gromov-Witten invariants of $K_{\mathbb{P}^2}$ is a modular form defined over the complexified

Kähler moduli space, which under mirror symmetry is identified with the modular curve $\Gamma_0(3)\backslash\mathcal{H}^*$ parametrizing the mirror manifolds of $K_{\mathbb{P}^2}$.

The details are given as follows. The SYZ mirror Calabi-Yau 3-fold X for $K_{\mathbb{P}^2}$ is given by [CLL12]

$$w_1 w_2 = 1 + \delta(q) + z_1 + z_2 + \frac{q}{z_1 z_2}\,, \tag{5.6}$$

with

$$1 + \delta(q) = \sum_{k=0}^{\infty} n_k q^k\,, \tag{5.7}$$

where $q = q_t := \exp 2\pi i t$, t is the flat coordinate on the complexified Kähler moduli space of $K_{\mathbb{P}^2}$. Then the mirror curve is given by $1 + \delta(q) + z_1 + z_2 + \frac{q}{z_1 z_2} = 0$. A scaling on the coordinates shows that this curve is equivalent to

$$1 + z_1 + z_2 + \frac{z}{z_1 z_2} = 0\,, \quad z = \frac{q_t}{(1+\delta(q_t))^3}\,. \tag{5.8}$$

Now consider z as the complex structure modulus for the mirror curve. It is a standard fact that this elliptic curve family is 3-isogenous to the $\tilde{E}_6$ curve family in Section 2.3 and thus is parametrized by the modular curve $\Gamma_0(3)\backslash\mathcal{H}^*$. Furthermore, one has

$$z(\tau) = -\frac{\alpha(\tau)}{27} = -\frac{1}{27}\frac{\left(\frac{3\eta(3\tau)^3}{\eta(\tau)}\right)^3}{\left(\frac{3\eta(3\tau)^3}{\eta(\tau)}\right)^3 + \left(\frac{\eta(\tau)^3}{\eta(3\tau)}\right)^3}\,. \tag{5.9}$$

The relation between the modular variable $q_\tau := \exp 2\pi i \tau$ and the flat coordinate t is given by [MOY01, Sti06, Zho14],

$$q_\tau = (-q_t)\prod_{d\geq 1}(1-q_t^d)^{3d^2 n^{\mathrm{GV}}_{0,d}}\,, \quad q_t = (-q_\tau)\prod_{n\geq 1}(1-q_\tau^n)^{9n\chi_{-3}(n)}\,. \tag{5.10}$$

where $n^{\mathrm{GV}}_{0,d} = 3, -6, 27, -192, 1695\cdots$ are the genus 0 degree d Gopakumar-Vafa invariants [GV98a, GV98b, KKV99], and $\chi_{-3}(n)$ is the non-trivial Dirichlet character mod 3 (it takes the value $0, 1, -1$ on an integer $3k, 3k+1, 3k+2$, respectively). From the above formulas in (5.8), (5.9) for the same quantity z, one then has

$$1 + \delta(q_t) = (-27)^{\frac{1}{3}} q_t^{\frac{1}{3}} \alpha(q_\tau)^{-\frac{1}{3}} = (-27)^{\frac{1}{3}} q_t^{\frac{1}{3}} \alpha(q_\tau(q_t))^{-\frac{1}{3}}\,. \tag{5.11}$$

The first few constants $\{n_k\}_{k\geq 0} = \{1, -2, 5, -32, 286, -3038, 35870\cdots\}$ predicted by using this formula and (5.7) give exactly the open Gromov-Witten invariants computed by a direct counting as done in [CLL12]. That is, the generating function $1 + \delta(q_t)$, up to multiplication by the factor $q_t^{1/3}$, is a modular form in q_τ.

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