SYZ Mirror symmetry for toric Calabi-Yau manifolds

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Outline

§ 0. Background (Hori-Iqbal-Vafa mirror for toric Calabi-Yau manifolds)

§ 1. SYZ construction of mirrors.

§ 2. Mirror map = SYZ map.

§ 3. Computing open GW invariants.
What is mirror symmetry?
Duality for Calabi-Yau manifolds

<table>
<thead>
<tr>
<th>$(X, g, J, \Omega, \omega)$</th>
<th>$(\tilde{X}, \tilde{g}, \tilde{J}, \tilde{\Omega}, \tilde{\omega})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$ (Symplectic geometry)</td>
<td>$\tilde{J}$ (Complex geometry)</td>
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</tbody>
</table>
Main aspects of mirror symmetry

- **Mirror map**: \( M_{\text{Käh.}} \xrightarrow{\Phi_{\text{mir}}} \hat{M}_{\text{cpx.}} \)

\( (X, \omega) \quad (\hat{X}, \hat{\Omega}) \)

- **Enumerative prediction**
  (Candelas et. al.)

- **Homological mirror symmetry**
  (Kontsevich)

- **A-Yukawa coupling**

\[ \int_X \hat{\Omega} \wedge \nabla^3 \hat{\Omega} \]

\( \text{Fuk}(X, \omega) \equiv \mathcal{D}\text{Coh}(\hat{X}, \hat{\Omega}) \quad \text{s.t} \)

\[ \# \]

\[ \# \]

\[ \# \]

\[ \# \]
Why?
Strominger-Yau-Zaslow Approach

- Mirror symmetry is T-duality

Leung-Yau-Zaslow: \( \checkmark \) without singular fibers.

How about singular fibers?
How about singular fibers?
Toric Calabi-Yau manifolds

Ex. $\mathbb{C}^2$.

Ex. $K_{\mathbb{P}^1}$.

$X$ toric and $K_X \sim 0$. 

$\Rightarrow$ Background.
Importance of toric Calabi-Yau manifolds

- Rich physical structures.
  Mirror symmetry, Topological vertex, Large $N$-duality

- Local models of Lagrangian torus fibrations on compact Calabi-Yau manifolds. (Gross; Castano-Matessi)

- Singular fibers are present.
Mirror symmetry for toric C.Y.

Hori-Iqbal-Vafa: \( (\dim_{\mathbb{C}} X = n) \times \) is

\[ \left\{ (u,v,\vec{z}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_i \bar{\varphi}_i \bar{z}_i \right\}. \]

\[ \mathcal{M}_{\text{Käh.}} \xrightarrow{\Phi_{\text{mir}}} \mathcal{M}_{\text{cpx.}} \]
is obtained by solving Picard-Fuchs equations.
Questions:

1. Why such \( \times \) ?
\[
\left\{ (u,v,z) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{n-1} : uv = \sum_i \tilde{g}_i \tilde{z}_i \right\}
\]

2. Geometric meaning of \( \Phi_{\text{min}} \)?

Answer by SYZ approach.
Theorem 1. (Chan-L.-Leung) \( (X, \omega) \): toric Calabi-Yau \( n \)-fold.

Construction by T-duality with quantum corrections gives

\[
X = \left\{ (u, v, \bar{z}) \in \mathbb{C}^2 \times (\mathbb{C}^*)^{n-1} : uv = \sum_i \check{g}_i \bar{z}^i \right\}
\]

where \( \check{g}_i = (1 + \delta_i) g_i \); \( \delta_i = \sum_{\alpha \neq 0} n_{\beta i + \alpha} g_{\alpha} \).

Kähler parameters

Open G.W. invariants
T-duality with quantum corrections

1. T-duality away from singular fibers.

\[ X_0 \text{ (symplectic)} \quad \text{vs.} \quad \hat{X}_0 \text{ (complex)} \]
T-duality with quantum corrections

2. Fourier transform of open GW potentials

\[ I_i : \pi_1(F_r) \to \mathbb{R} \]

\[ I_i(\lambda) = \sum_{u, D_i} \exp(\text{Area}_w(u)) \quad u: \text{holomorphic discs bounded by } \lambda \]

\[ \tilde{X}_i = \sum_{\lambda} I_i(\lambda) e^{i(\lambda, \tilde{\theta})} \]

\[ X = \text{Spec}\langle \tilde{X}_1, ..., \tilde{X}_m \rangle \]
Wall-crossing (Auroux; Gross-Siebert)

\[ X = \mathbb{C}^2. \]

\[ \tilde{\mathbb{Z}}_1 = \mathbb{Z}_1. \]

\[ \tilde{\mathbb{Z}}_2 = \begin{cases} \mathbb{Z}_2 & \text{above} \\ \mathbb{Z}_2^{-1} & \text{below} \end{cases} \]

\[ \tilde{\mathbb{Z}}_3 = \begin{cases} \mathbb{Z}_2^{-1} & \text{above} \\ \mathbb{Z}_2^{-1} + \mathbb{Z}_2 & \text{below} \end{cases} \]

\[ \tilde{\mathbb{Z}}_2 \tilde{\mathbb{Z}}_3 = 1 + \mathbb{Z}_1. \]
§ 2. Mirror map = SYZ map

From Theorem 1, get

\[ \Phi_{\text{SYZ}} : M_{\text{Käh.}} \rightarrow \dot{M}_{\text{cpx.}} \]

\[ \dot{q}_i = (1 + \delta_i) q_i \]

open GW \quad \text{Kähler parameters}

Recall

\[ \Phi_{\text{mir}} : M_{\text{Käh.}} \rightarrow \dot{M}_{\text{cpx.}} \]
Conjecture. (Chan-L.-Leung)

$$\Phi_{\text{mir.}} \rightarrow \Phi_{\text{SYZ}}$$

for every toric Calabi-Yau manifold.

Geometric meaning of $\Phi_{\text{mir.}}$ (Answer to Question 2.)
Theorem 2. (L.-Leung-Wu):

For every toric Calabi-Yau surface,

$$\mathcal{H}_{\text{mir.}} = \mathcal{H}_{\text{SYZ}}.$$ 

- Proved by computing open GW invariants

$$\delta_i = \sum_{\alpha \neq 0} n_{\beta_i + \alpha} \phi_\alpha.$$
Open G.W. invariants.

\[ \times : \text{Kähler manifold.} \]

\[ L : \text{Lagrangian submanifold.} \]

\[ \beta \in \pi_2(X, L) \text{ a disk class.} \]

\[ \eta_\beta \triangleq \left( [M_1(\beta)], [\text{pt}] \right)_L \]

virtual fundamental class (Fukaya-Oh-Ohta-Ono)
$\eta_\beta$ is difficult to compute once obstruction theory is involved:

$[M_1(\beta)]$ not explicitly written down in general.
Some Known results

1. $X$ toric Fano;
   
   $L \subset X$ Lagrangian toric fiber.

   (Cho-Oh) $\eta_\beta = \begin{cases} 1 & \beta \text{ is a basic disk class} \\ 0 & \text{otherwise} \end{cases}$

2. $X = \mathbb{P}^2$ (Auroux; Fukaya-Oh-Ohta-Ono)
   
   (obstructed) $\eta_\beta = \begin{cases} 1 & \beta \text{ is basic, or } \beta = b + \alpha \\ 0 & \text{otherwise} \end{cases}$
Compute \( \eta_{\beta_i + \alpha} \) ?
Theorem 3 (Chan; L.-Leung-Wu):

$X$ is a toric Calabi-Yau.

$\eta_{\beta_i + \alpha} = \left< \text{pt.} \right>_{h + \alpha}^{X}$

provided that any rational curve in $X$ representing $\alpha$ is contained in $X$. 

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Theorem 4 (L.-Leung-Wu):

\[ X : \text{Toric Calabi-Yau surface.} \]

\[ \eta_{\beta, x + \alpha} = \begin{cases} 
  1 & \text{if } \alpha \text{ is admissible;} \\
  0 & \text{otherwise.} 
\end{cases} \]

Higher dimensional case?
$K3 \mathbb{P}^2, (\text{Chan-L.-Leung}):$ 

$$
\eta_{b+kl} = \begin{cases} 
1 & k = 0 \\
-2 & k = 1 \\
5 & k = 2 \\
-32 & k = 3 \\
286 & k = 4 \\
-3038 & k = 5 \\
& \vdots
\end{cases}
$$

(Use Hu & Gathmann; Chiang-Klemm-Yau-Zaslow)

$$
\Phi_{\text{SYZ}} (q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \ldots
$$

Graber-Zaslow:

$$
\Phi_{\text{mir}} (q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \ldots
$$