SYZ Mirror symmetry for toric Calabi-Yau manifolds

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Outline

§0. Background (Hori–Iqbal–Vafa mirror for toric Calabi–Yau manifolds)

§1. SYZ construction of mirrors.

§2. Mirror map = SYZ map.

§3. Computing open GW invariants.

What is mirror symmetry?

- Duality for Calabi-Yau manifolds

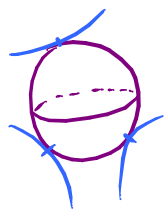
$(X, g, J, \Omega, \omega)$	$(\check{X}, \check{g}, \check{J}, \check{\Omega}, \check{\omega})$
ω (Symplectic geometry)	$\check{J}$ (Complex geometry)
J (Complex geometry)	$\check{\omega}$ (Symplectic geometry)

Main aspects of mirror symmetry

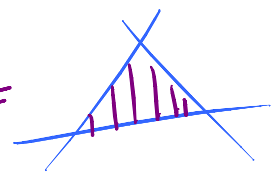
- Mirror map : $\mathcal{M}_{\text{Käh.}} \xrightarrow[\cong]{\Phi_{\text{mir}}} \check{\mathcal{M}}_{\text{cpx.}}$

  $(X, \omega) \longmapsto (\check{X}, \check{\Omega})$

- Enumerative prediction (Candales et. al.) : A-Yukaya coupling $=$ B-Yukaya coupling

  $\#$  $\qquad \int_{\check{X}} \check{\Omega} \wedge \nabla^3 \check{\Omega}$

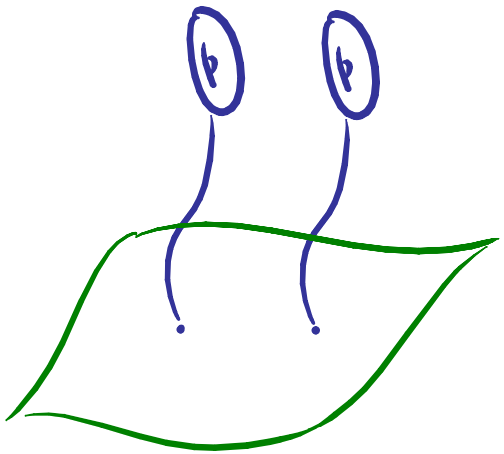
- Homological mirror symmetry (Kontsevich) : $\mathrm{Fuk}(X, \omega) \cong \mathrm{DCoh}(\check{X}, \check{\Omega})$

  $\#$  $\qquad$ s.t

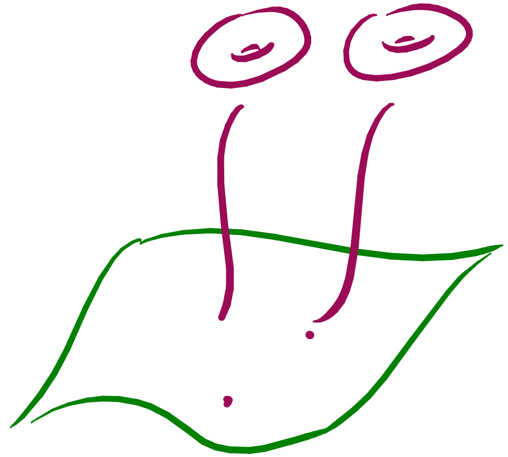
Why ?

Strominger-Yau-Zaslow Approach

- Mirror symmetry is T-duality



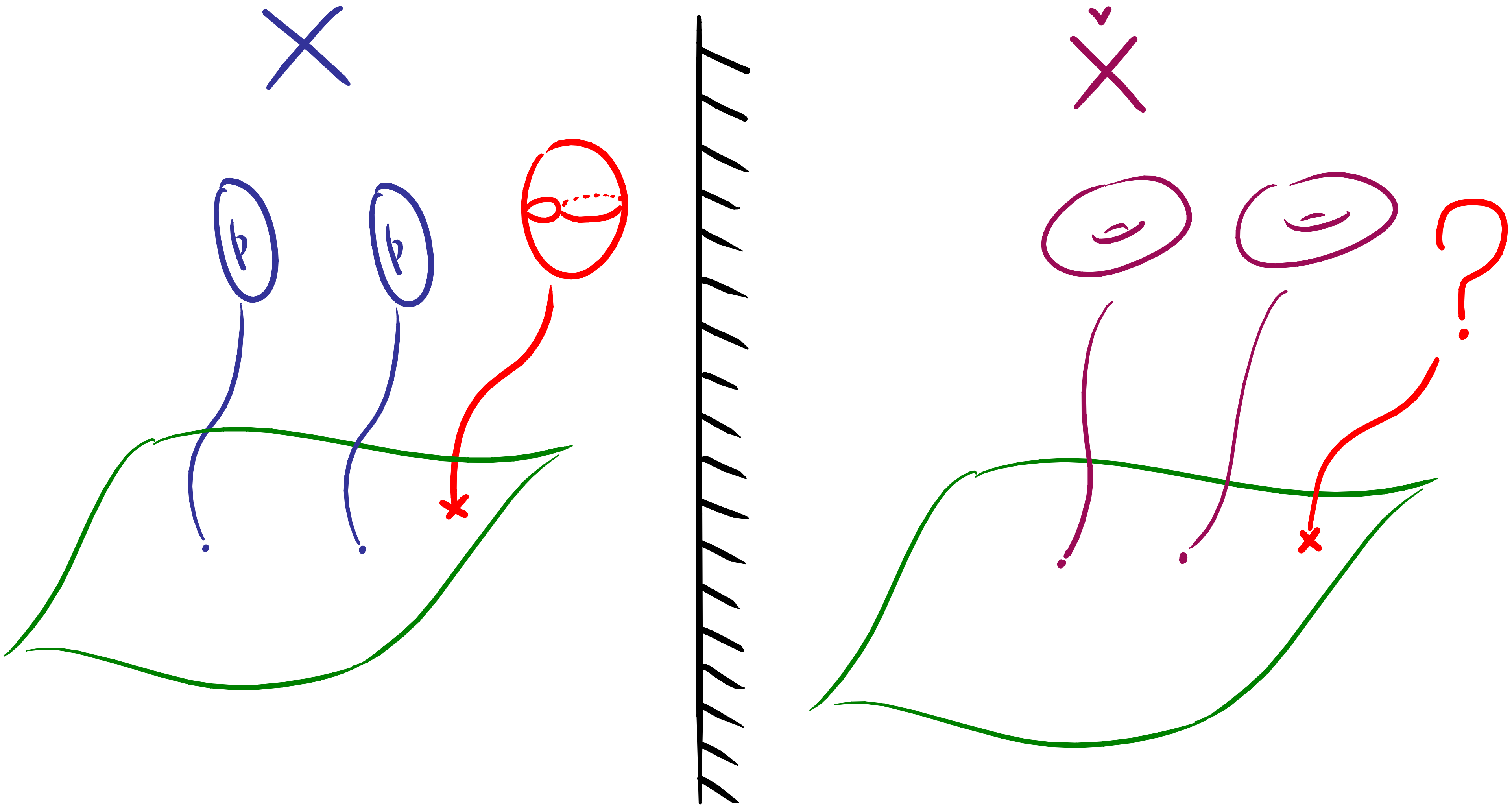
$$\begin{tikzcd} X \arrow[dr] & \text{dual torus bundles} & \check{X} \arrow[dl] \\ & B & \end{tikzcd}$$



Leung-Yau-Zaslow : ✓ without singular fibers.

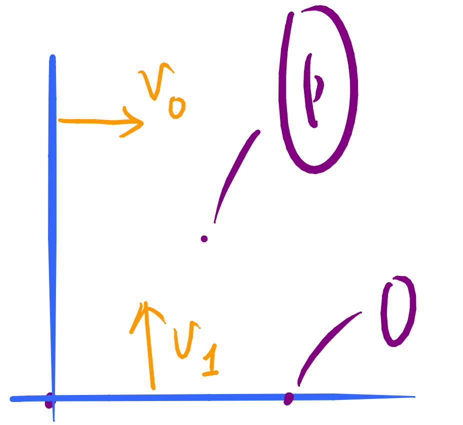
How about singular fibers?

How about singular fibers ?

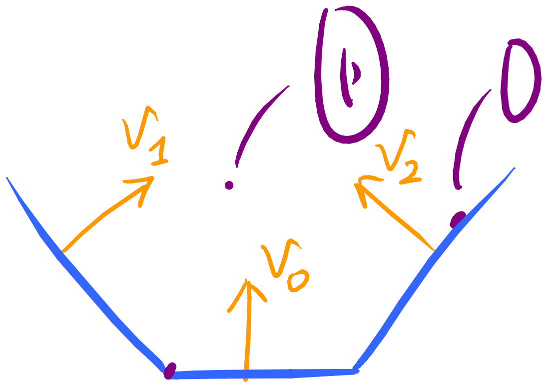


Toric Calabi-Yau manifolds

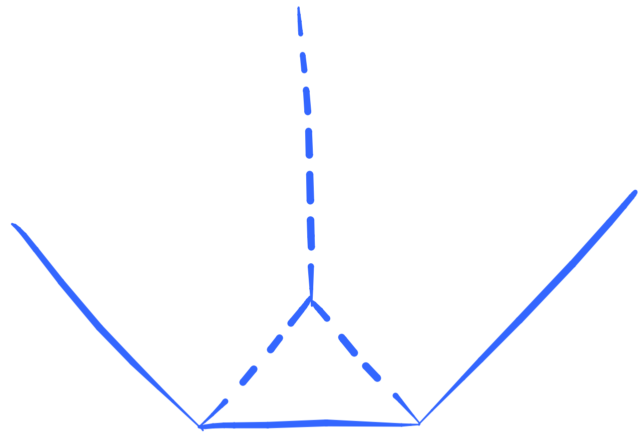
e.g. $\mathbb{C}^2$.



e.g. $K_{\mathbb{P}^1}$.



e.g. $K_{\mathbb{P}^2}$.



Def: X toric and $K_X \sim 0$.

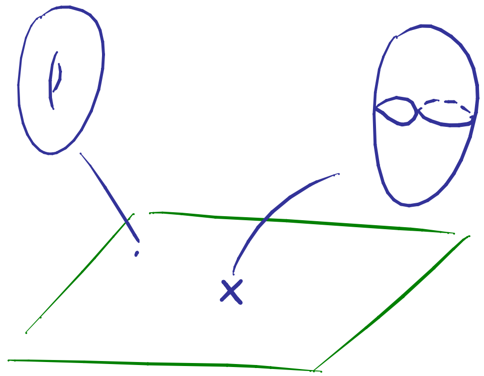
Importance of toric Calabi-Yau manifolds

- Rich physical structures.

  Mirror symmetry, Topological vertex, Large N-duality .....

- Local models of Lagrangian torus fibrations on compact Calabi-Yau manifolds. (Gross ; Castano-Matessi)

- Singular fibers are present.



Mirror symmetry for toric C.Y.

Hori-Iqbel-Vafa: $(\dim_{\mathbb{C}} X = n)$ $\check{X}$ is

$$\left\{ (u, v, \vec{z}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_i \check{q}_i \vec{z}^{\vec{v}_i} \right\}.$$

$$\begin{array}{ccc} \mathcal{M}_{\text{Käh.}} & \xrightarrow[\cong]{\Phi_{\text{mir}}} & \check{\mathcal{M}}_{\text{cpx.}} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ q & & \check{q} \end{array}$$

is obtained by solving Picard-Fuchs equations.

Questions :

1. Why such $\check{X}$

$$\check{X} = \left\{ (u, v, \vec{z}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_i \check{q}_i \vec{z}^{\vec{v}_i} \right\} ?$$

2. Geometric meaning of Φ_{mir} ?

Answer by SYZ approach.

Theorem 1. (Chan-L.-Leung)

(X, ω): toric Calabi-Yau n-fold.

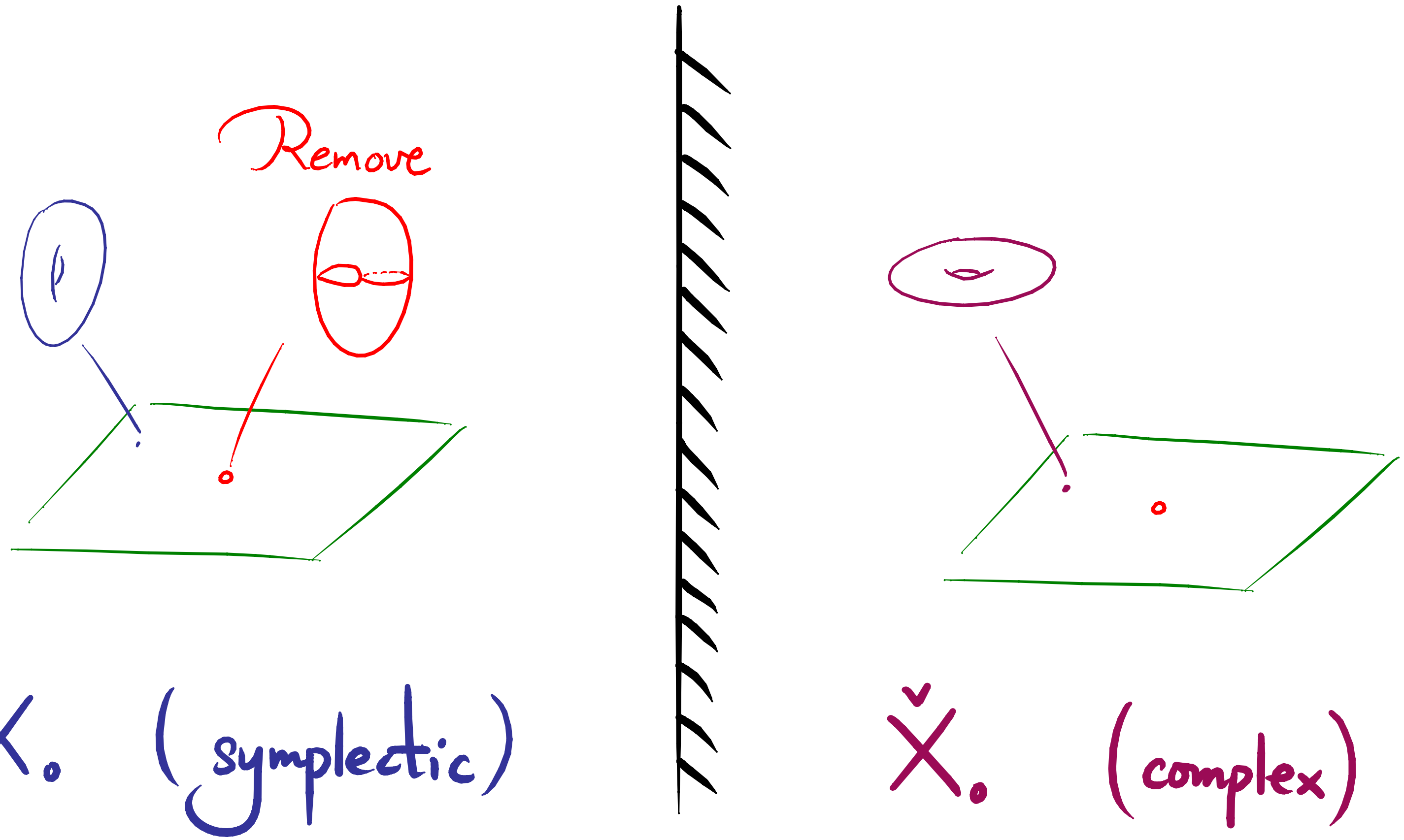
Construction by T-duality with quantum corrections gives

$$\check{X} = \left\{ (u, v, \vec{z}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = \sum_i \check{q}_i \vec{z}^{\,\vec{v}_i} \right\}$$

where $\check{q}_i = (1 + \delta_i) \underbrace{q_i}_{\text{Kähler parameters}}$; $\delta_i = \sum_{\alpha \neq 0} \underbrace{n_{\beta_i + \alpha}}_{\text{Open G.W. invariants}} q^\alpha$.

T-duality with quantum corrections

1. T-duality away from singular fibers.



T-duality with quantum corrections

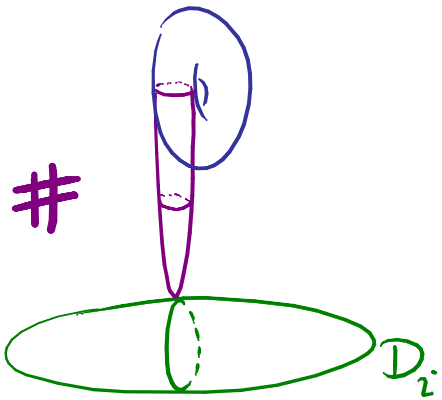
2. Fourier transform of open GW potentials

$$I_i : \pi_1(F_r) \longrightarrow \mathbb{R}$$

$$I_i(\lambda) \triangleq \sum (u, D_i) \exp\left(-\mathrm{Area}_\omega(u)\right)$$

u : holomorphic disks bounded by λ

Count #

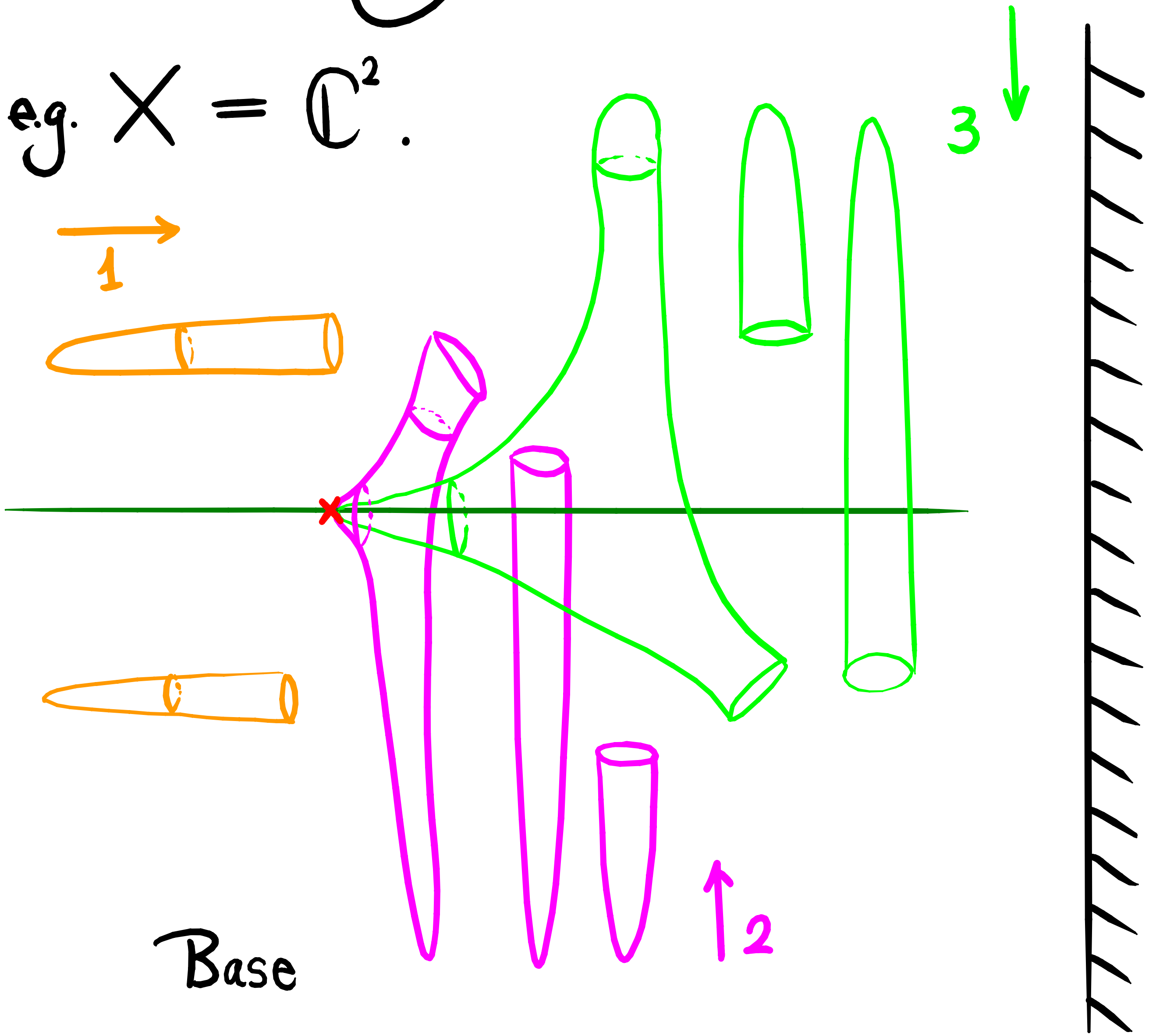


$$\tilde{z}_i \triangleq \sum_{\lambda} I_i(\lambda)\, e^{i(\lambda, \check{\theta})}$$

$$\check{X} \triangleq \mathrm{Spec}\langle \tilde{z}_1, \dots, \tilde{z}_m \rangle$$

Wall-crossing (Auroux ; Gross-Siebert)

e.g. $X = \mathbb{C}^2$.



$\tilde{z}_1 = z_1$.

$\tilde{z}_2 = \begin{cases} z_2 + z_1 z_2 & \text{above} \\ z_2 & \text{below} \end{cases}$

$\tilde{z}_3 = \begin{cases} z_2^{-1} & \text{above} \\ z_2^{-1} + z_1 z_2^{-1} & \text{below} \end{cases}$

$\tilde{z}_2 \tilde{z}_3 = 1 + z_1$.

SYZ map

From Theorem 1, get

$$\Phi_{SYZ} : \mathcal{M}_{\text{Käh.}} \longrightarrow \check{\mathcal{M}}_{\text{cpx.}}$$

$$\check{q}_i = (1 + \underbrace{\delta_i}_{\text{open GW}}) \, \underbrace{q_i}_{\text{Kähler parameters}}$$

Recall

$$\Phi_{mir} : \mathcal{M}_{\text{Käh.}} \longrightarrow \check{\mathcal{M}}_{\text{cpx.}}$$

Conjecture. (Chan - L. - Leung)

$$\Phi_{\text{mir.}} = \Phi_{SYZ}$$

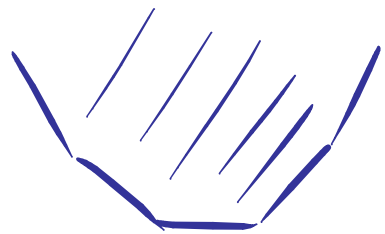
for every toric Calabi-Yau manifold.

$\longrightarrow$ Geometric meaning of $\Phi_{\text{mir.}}$ (Answer to Question 2.)

Theorem 2. (L.-Leung-Wu):

For every toric Calabi-Yau surface,

$$\Phi_{\text{mir.}} = \Phi_{\text{SYZ}}.$$



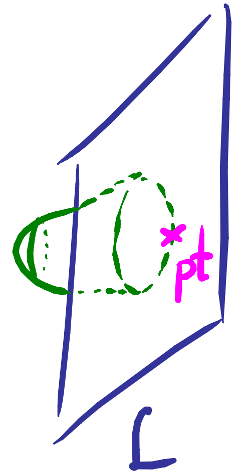
- Proved by computing open GW invariants to find $\delta_i = \sum_{\alpha \neq 0} n_{\beta_i + \alpha} q^{\alpha}$.

Open G.W. invariants.

X : Kähler manifold.

L : Lagrangian submanifold.

$\beta \in \pi_2(X, L)$ a disk class.



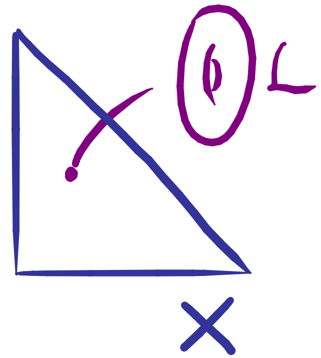
$$n_\beta \triangleq \Big(\underbrace{[\mathcal{M}_1(\beta)]}_{\text{virtual fundamental class (Fukaya-Oh-Ohta-Ono)}}, [pt] \Big)_L$$

n_β is difficult to compute

once obstruction theory is involved:

$[\mathcal{M}_1(\beta)]$ not explicitly written down in general.

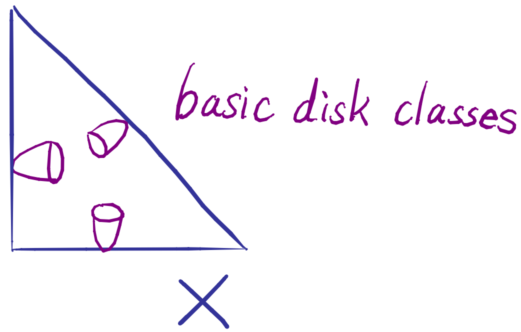
Some Known results



1. X toric Fano;

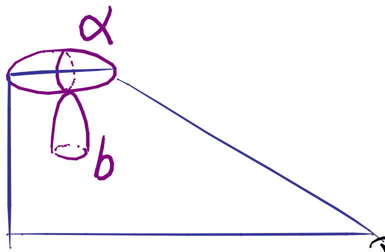
   $L \subset X$ Lagrangian toric fiber.

(Cho-Oh)
$$n_\beta = \begin{cases} 1 & \beta \text{ is a basic disk class;} \\ 0 & \text{otherwise.} \end{cases}$$

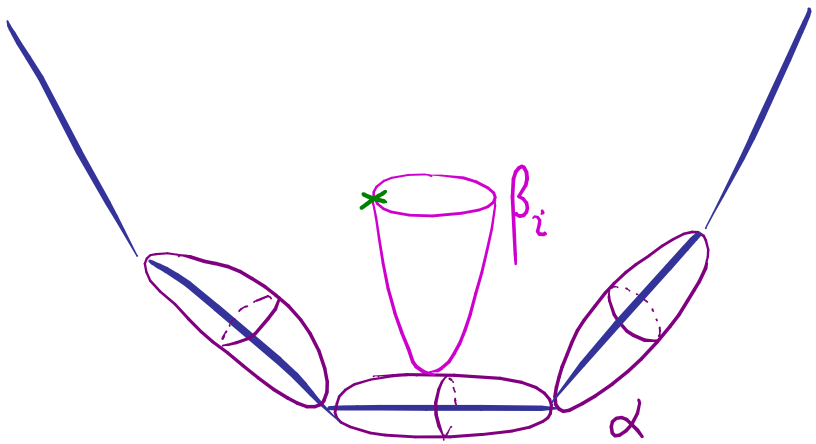


2. $X = \mathbb{F}_2$. (Auroux; Fukaya-Oh-Ohta-Ono)

(obstructed)
$$n_\beta = \begin{cases} 1 & \beta \text{ is basic, or } \beta = b + \alpha. \\ 0 & \text{otherwise} \end{cases}$$



Compute $n_{\beta_i+\alpha}$?

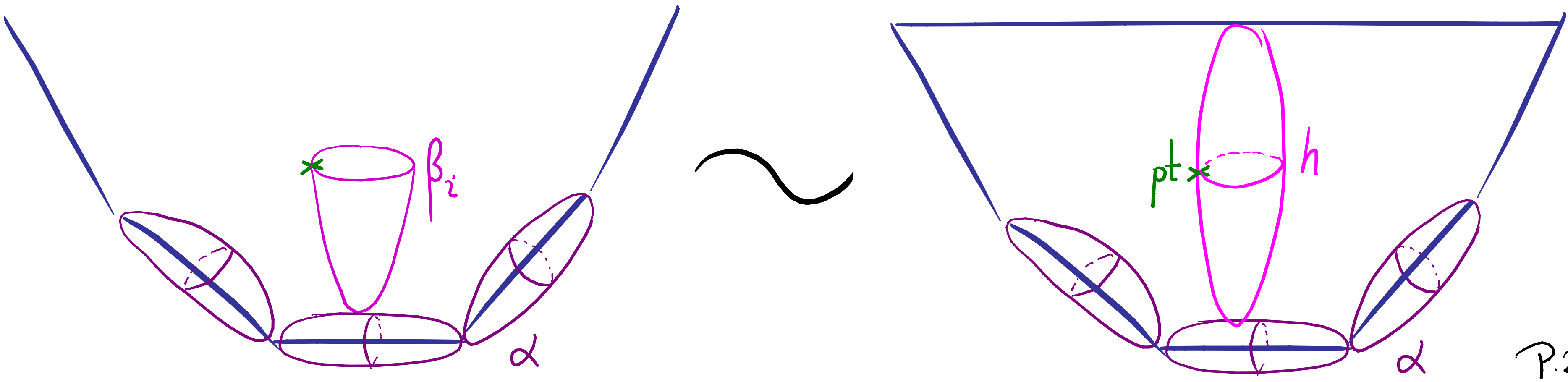


Theorem 3 (Chan; L.-Leung-Wu):

X : toric Calabi-Yau.

$$n_{\beta_i+\alpha} = \langle pt. \rangle^{\bar{X}}_{h+\alpha}$$

provided that any rational curve in $\bar{X}$ representing α is contained in X.



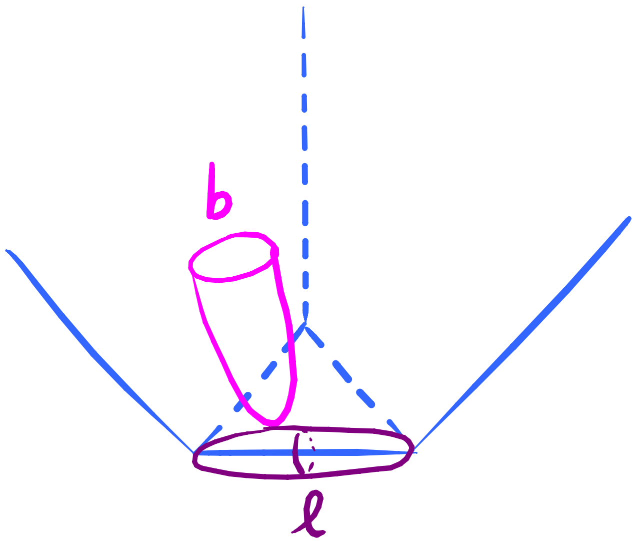
Theorem 4 (L.-Leung-Wu):

X : Toric Calabi-Yau surface.

$$n_{\beta_i+\alpha} = \begin{cases} 1 & \alpha \text{ is } \underbrace{\text{admissible}}_{\text{some explicit combinatorial condition (Bryan-Leung)}} ; \\ 0 & \text{otherwise.} \end{cases}$$

Higher dimensional case ?

$K\mathbb{P}^2$. (Chan-L.-Leung):



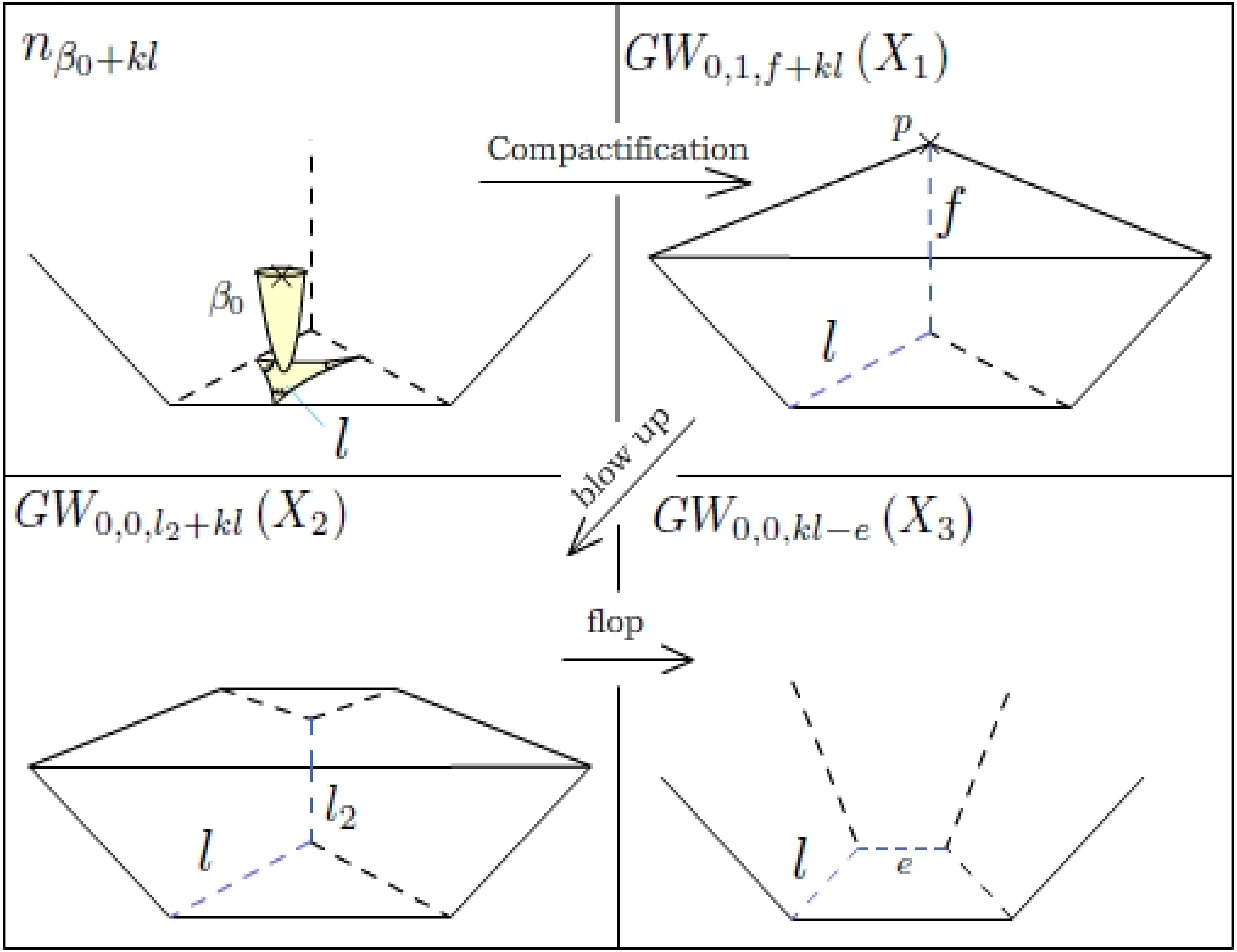
$$n_{b+k\ell} = \begin{cases} 1 & k=0 \\ -2 & k=1 \\ 5 & k=2 \\ -32 & k=3 \\ 286 & k=4 \\ -3038 & k=5 \\ \vdots & \vdots \end{cases}$$

(Use Hu & Gathmann; Chiang-Klemm-Yau-Zaslow)

$$\Phi_{SYZ}(q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots$$

Graber-Zaslow:

$$\Phi_{mir}(q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \dots$$



n_{β_0+kl}
β_0
l
$GW_{0,1,f+kl}(X_1)$
p
f
l
Compactification
blow up
$GW_{0,0,l_2+kl}(X_2)$
l_2
l
flop
$GW_{0,0,kl-e}(X_3)$
l
e

A1