Open GW invariants and Seidel elements of toric manifolds

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Overview

Fukaya-Oh-Ohta-Ono
Cho-Oh
Open GW invariants

Seidel
Mcduff-Tolman
Seidel elements

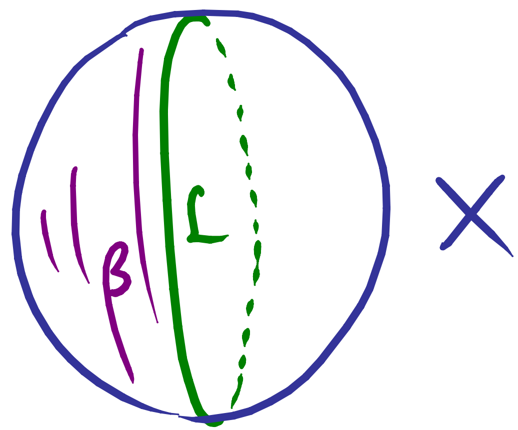
Chan-L.-Leung-Tseng

Gonzalez-Iritani

Mirror maps

for toric semi-Fano manifolds.

Open Gromov-Witten invariants



(X, ω) : Symplectic manifold.

$\cup$

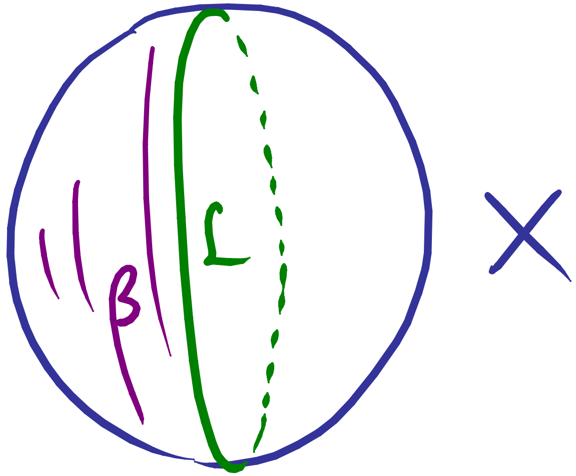
L : Lagrangian submanifold ; $\beta \in \pi_2(X, L)$.

Fukaya-Oh-Ohta-Ono :

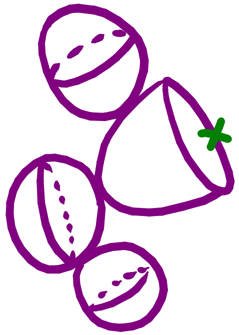
$$n_\beta \text{ `} = \text{' } \# \underbrace{\text{Pseudoholomorphic}}_{\text{Fix a complex structure } J \text{ compatible with } \omega} \text{ disks in class } \beta$$

Symplectic invariant if independent of J.

Open Gromov-Witten invariants



$$\mathcal{M}_1(\beta) \ \text{`='} \ \Big\{$$

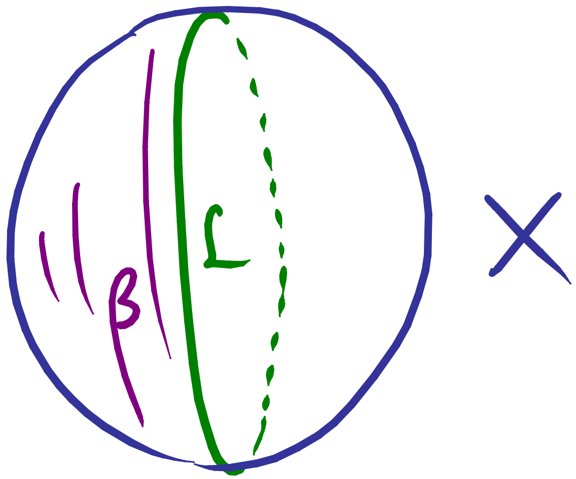


$$\xrightarrow{\text{stable}} (X, L) \Big\} \Big/ \mathrm{Aut}(\mathrm{Domain})$$

is compact, oriented, with no codimension$_{\mathbb{R}}$ 1 boundary in good conditions.

- L compact, oriented, spin.
- Any non-constant pseudoholomorphic $(\Delta, \partial\Delta) \longrightarrow (X, L)$ has Maslov index $\mu \geq 2$.
- $\mu(\beta) = 2$.

Open Gromov-Witten invariants



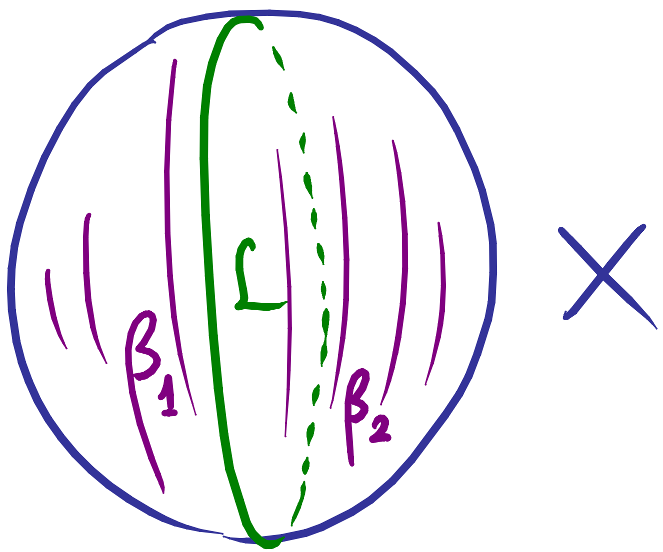
$$n_\beta \triangleq \int_{[\mathcal{M}_1(\beta)]_{\text{virt.}}} ev^* [pt]_L$$

'=' # pseudoholomorphic disks representing β passing through a generic point of L.

$\neq 0$ only when $\mu(\beta) = 2$.
(Maslov index)

Difficult to compute from definition!

Example: the sphere



$$\mu(\beta) = 2 \Rightarrow \beta = \beta_1 \text{ or } \beta_2.$$

$$n_{\beta_1} = n_{\beta_2} = 1.$$

Example: toric Fano manifold

Theorem (Cho-Oh):

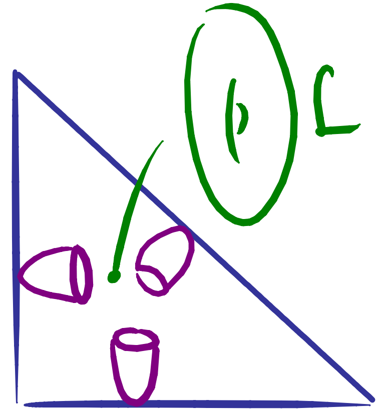
X : a toric Fano manifold.

L : a Lagrangian torus fiber.

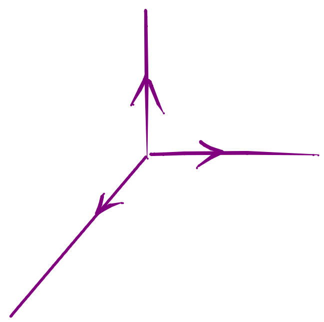
Then $\mu(\beta) = 2 \iff \beta$ is basic

$\iff n_\beta = 1.$

Obstruction of moduli is trivial.

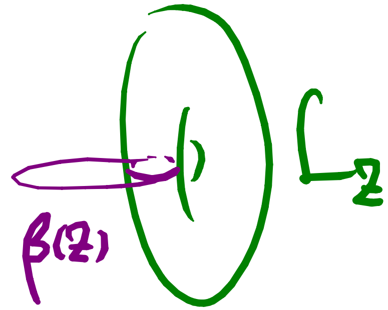


basic classes

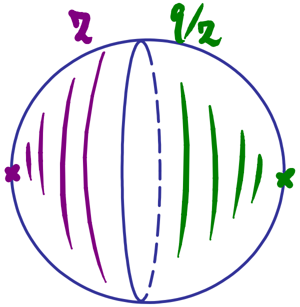


Generating function

$$W^{open} \underset{\substack{\text{Auroux} \\ \text{Chan-Leung} \\ \text{Cho-Oh} \\ \text{Fukaya-Oh-Ohta-Ono}}}{=} \sum_{\beta \in \pi_2(X,T)} n_\beta \, q^{\beta(Z)}$$



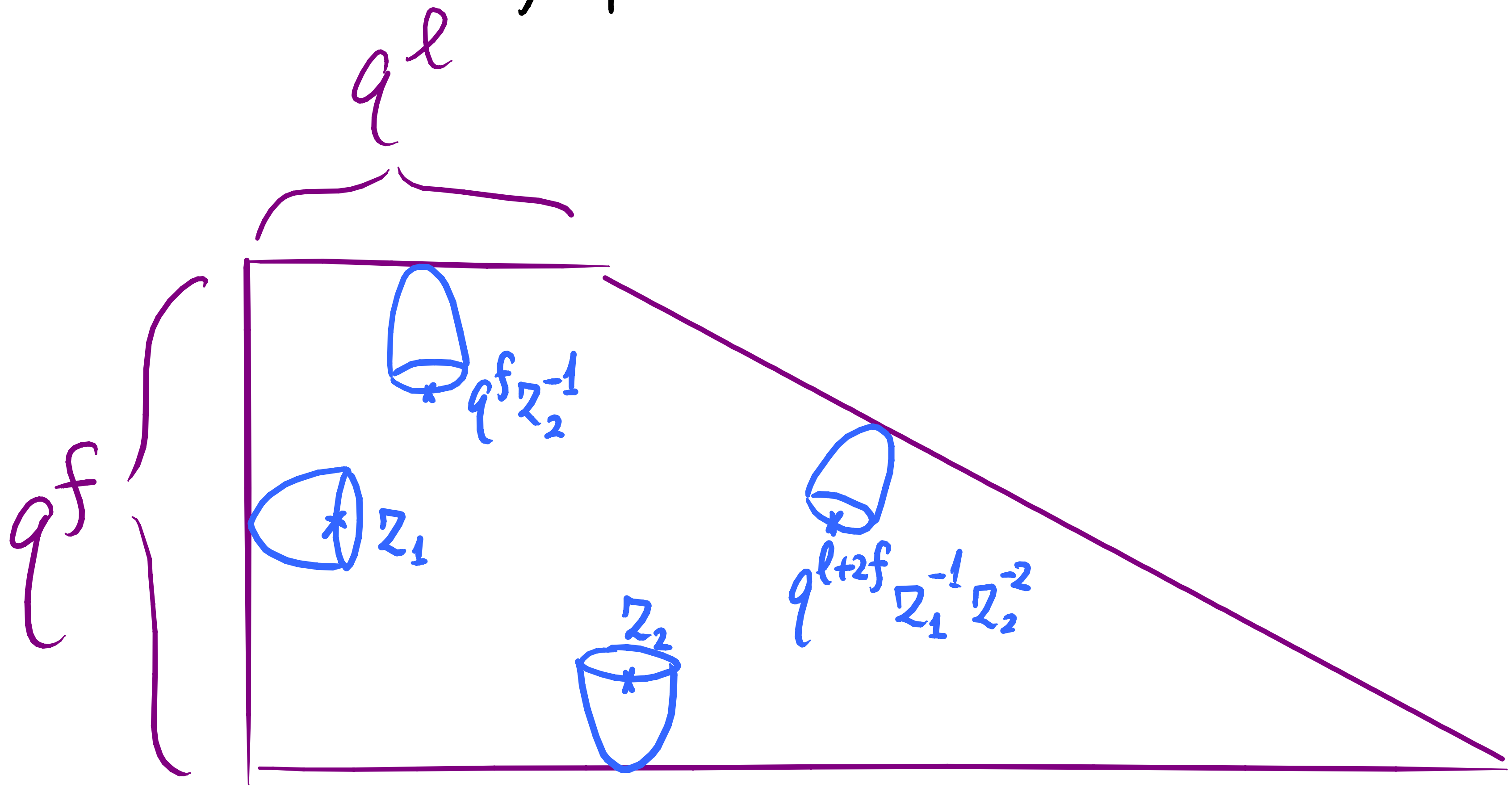
Z : parametrize (complexified) deformation of L.

e.g. $X =$ 

$W^{open} = z + \dfrac{q}{z}$ on $\mathbb{C}^\times$

An example: $\mathbb{F}_2$.

$\beta \in \pi_2(X, T)$ with $\mu(\beta) = 2$:



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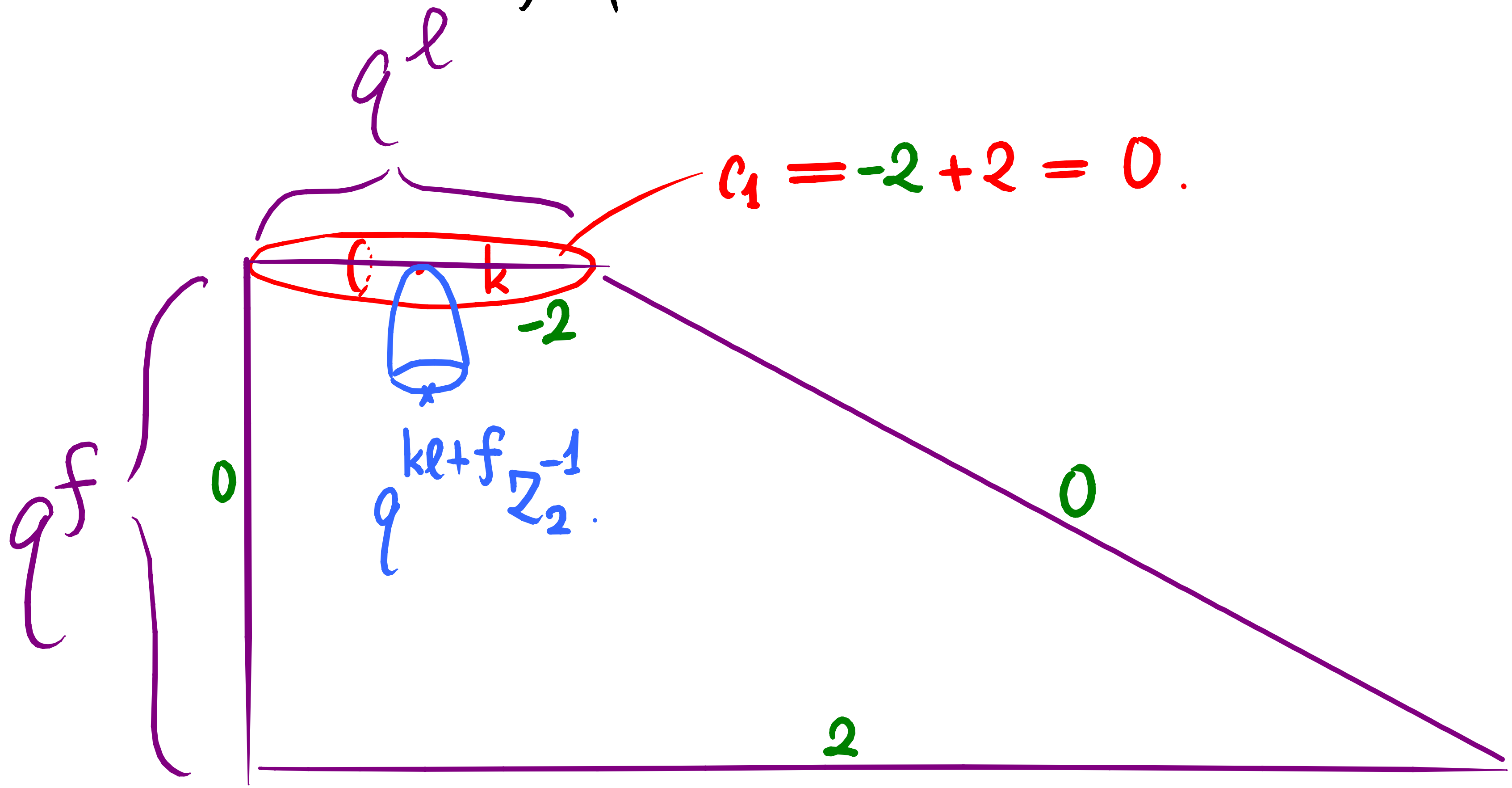
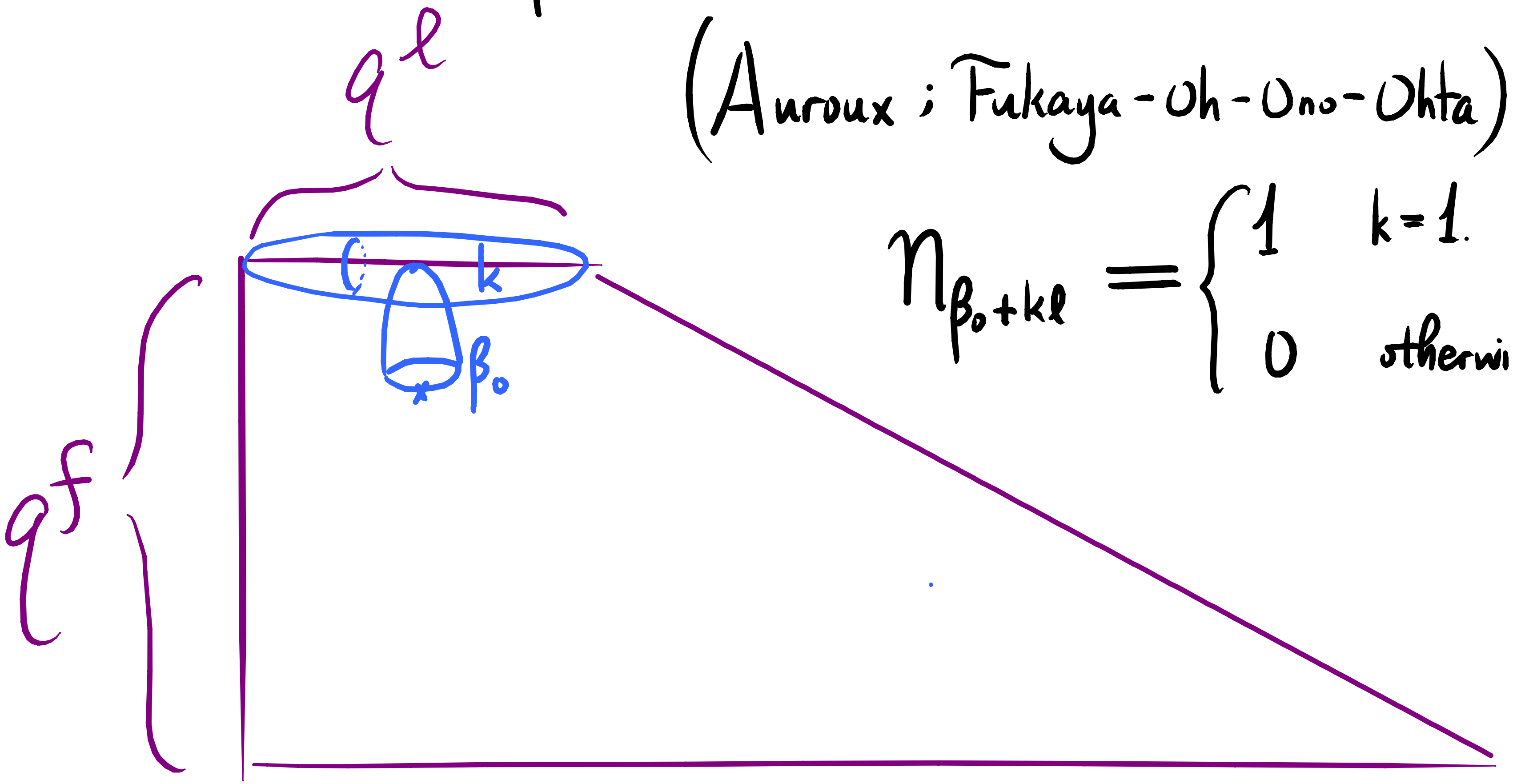


An example: $\mathbb{F}_2$.

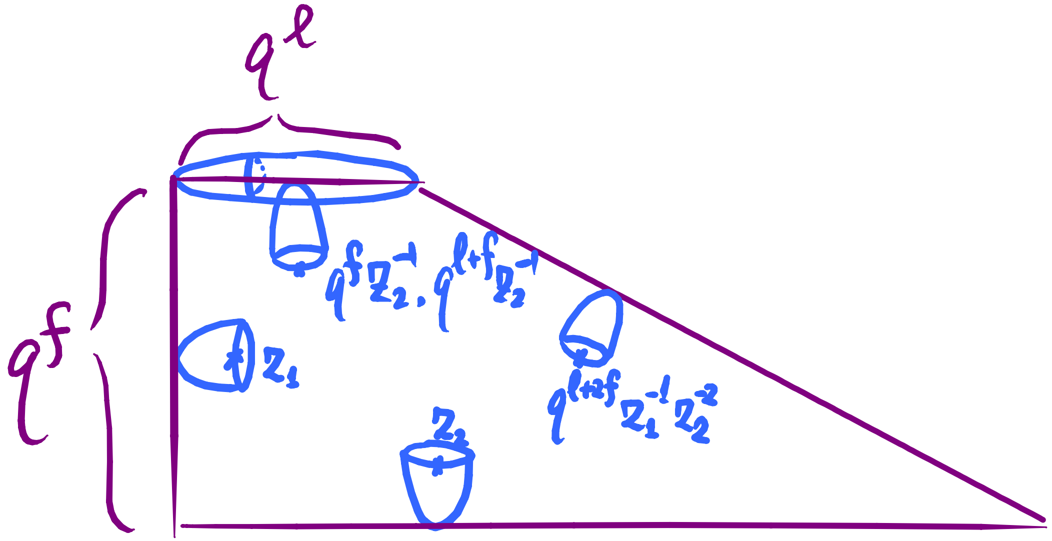
$\beta \in \pi_2(X,T)$ with $n_\beta \neq 0$:



(Auroux; Fukaya-Oh-Ono-Ohta)

$$n_{\beta_0+k\ell} = \begin{cases} 1 & k=1. \\ 0 & \text{otherwise.} \end{cases}$$

An example: $\mathbb{F}_2$.



$$W^{open} = \sum_{\beta \in \pi_2(X,T)} n_\beta Z_\beta$$

$$= Z_1 + Z_2 + q^f(1+q^\ell) Z_2^{-1} + q^{\ell+2f} Z_1^{-1} Z_2^{-2}$$

The mirror theorem (Fukaya-Oh-Ohta-Ono)

$$QH^{\cdot}(X) \xrightarrow{\cong} \mathrm{Jac}(W^{\mathrm{open}})$$

$$D_i \longmapsto \hat{D}_i \cdot W^{\mathrm{open}}$$

toric divisors

$$\overset{\text{divisor equation}}{=} \sum_{\beta \in \pi_2(X,L)} n_{1,1}(\beta; D_i)\, q^{\beta(Z)}.$$

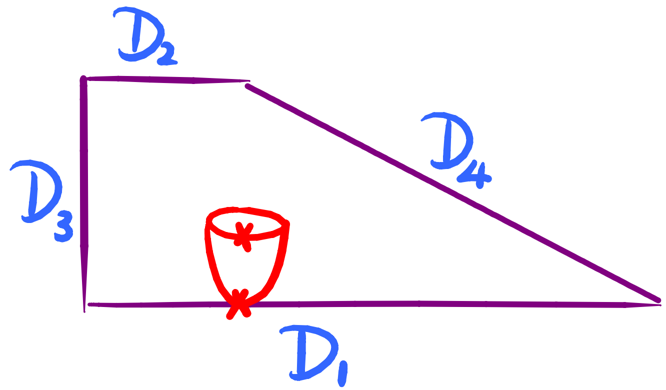
interior marked point

boundary marked point

Remark: Use Seidel elements in the proof.

The mirror theorem (Fukaya-Oh-Ohta-Ono)

e.g. $\mathbb{F}^2$.



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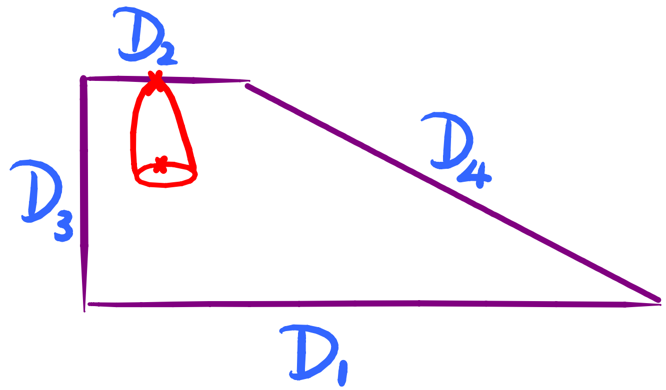

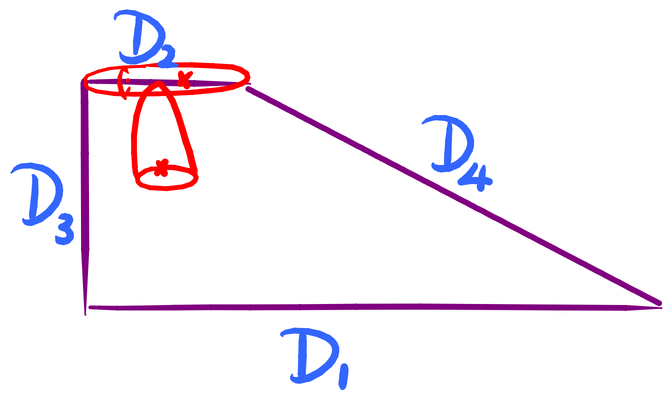

$$QH^{\cdot}(X) \xrightarrow{\cong} \mathrm{Jac}(W^{open})$$

$$D_1 \longmapsto z_2$$

$$D_2 \longmapsto q_2 z_2^{-1}(1-q_1).$$

The mirror theorem (Fukaya-Oh-Ohta-Ono)

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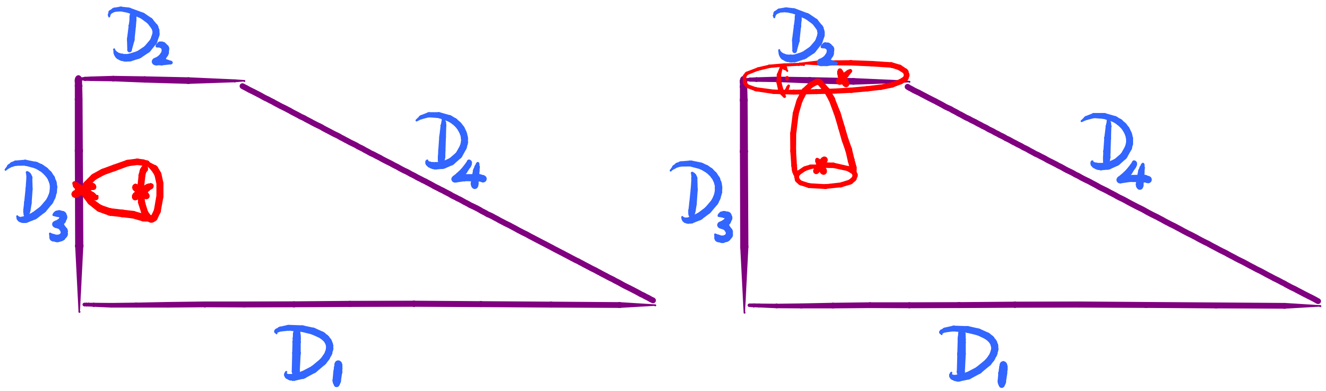

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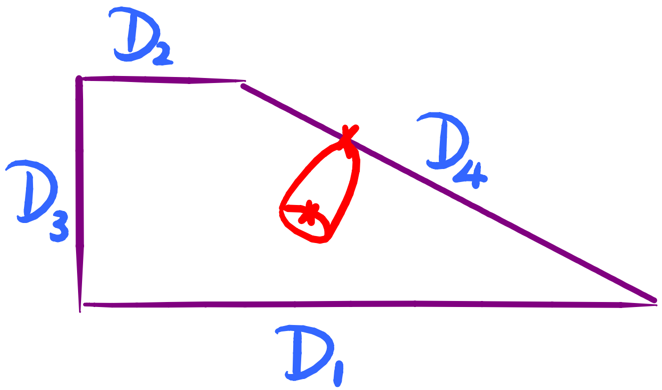
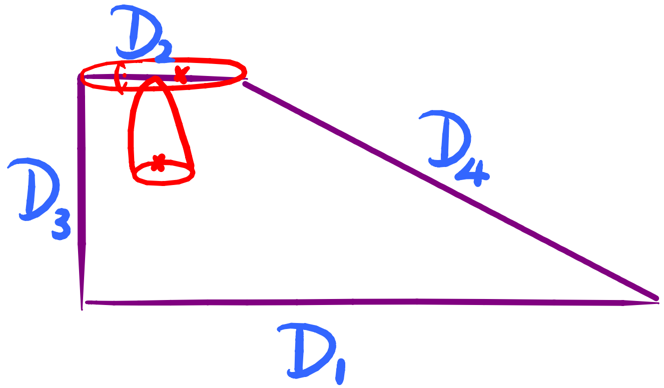




$$QH^{\cdot}(X) \xrightarrow{\cong} \mathrm{Jac}(W^{open})$$

$$D_1 \longmapsto z_2$$

$$D_2 \longmapsto q_2 z_2^{-1}(1-q_1).$$

$$D_3 \longmapsto z_1 + q_1 q_2 z_2^{-1}.$$

$$D_4 \longmapsto q_1 q_2^2 z_1^{-1} z_2^{-2} + q_1 q_2 z_2^{-1}.$$

Seidel elements

- Seidel, McDuff-Tolman:

$$\mathbb{C}^\times \curvearrowright X$$

basis of $H^\cdot(X)$.

$$\rightsquigarrow \quad S := \sum_{\alpha} \sum_{\beta \in \mathrm{NE}(E)^{\mathrm{sec}}_{\mathbb{Z}}} \langle \iota_* \phi_\alpha \rangle^E_{0,1,\beta} \, \phi^\alpha q^\beta \in (QH(X))^\times \curvearrowright QH(X)$$

where $X \xrightarrow{\iota} E = \left(X \times (\mathbb{C}^2 \times \{0\})\right) / \mathbb{C}^\times$

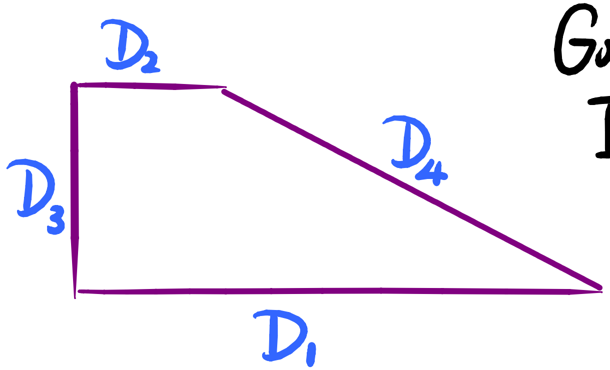
$$\begin{array}{c} E \\ \downarrow \\ \mathbb{P}^1 \end{array}$$

computed by J-function.

- X toric: Each ray of fan gives $\mathbb{C}^\times$-action $\longrightarrow S_i$.

Seidel elements under mirror isomorphism

e.g. $\mathbb{F}^2$.



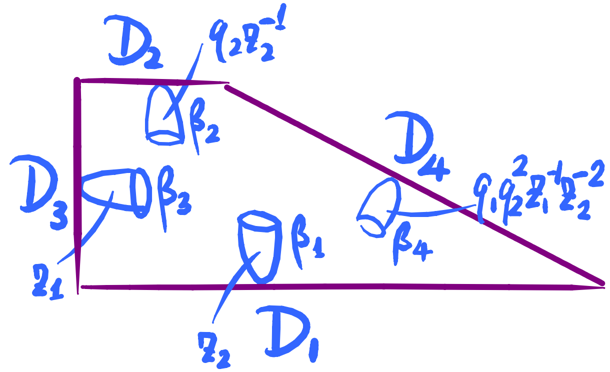
Gonzalez-Iritani
$$\begin{cases} S_1 = D_1. \\ S_2 = \dfrac{D_2}{1-q_1}. \\ S_3 = D_3 - \dfrac{q_1}{1-q_1} D_2 \\ S_4 = D_4 - \dfrac{q_1}{1-q_1} D_2. \end{cases}$$

$$QH^{\cdot}(X) \xrightarrow{\ \cong\ } \mathrm{Jac}(W^{open})$$

$$\begin{aligned} D_1 &\longmapsto z_2 \\ D_2 &\longmapsto q_2 z_2^{-1}(1-q_1). \\ D_3 &\longmapsto z_1 + q_1 q_2 z_2^{-1}. \\ D_4 &\longmapsto q_1 q_2^2 z_1^{-1} z_2^{-2} + q_1 q_2 z_2^{-1}. \end{aligned}$$

$$\therefore \begin{cases} S_1 \longmapsto z_2 \\ S_2 \longmapsto q_2 z_2^{-1} \\ S_3 \longmapsto z_1 \\ S_4 \longmapsto q_1 q_2^2 z_1^{-1} z_2^{-2} \end{cases}$$

Seidel elements under mirror isomorphism



Theorem (Chan-L.-Leung-Tseng):

Let X be a toric semi-Fano manifold.

$$QH^{\cdot}(X) \xrightarrow{\cong} \mathrm{Jac}(W^{\text{open}})$$

$$S_i \longmapsto q^{\beta_i(z)}$$

basis disk class corresponding to D_i.

assuming convergence of W^{open}.

Main ingredient:
- Open mirror symmetry
- Gonzalez-Iritani: Seidel elements in terms of mirror maps.

Hori-Vafa mirrors of compact toric manifolds

no disk counting involved

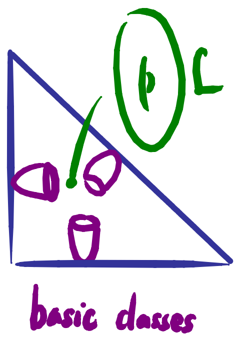
explicit monomial attached to β_i

$$W^{HV} \triangleq \sum_i Z_{\beta_i}(Z, \check{q})$$

complex parameters.

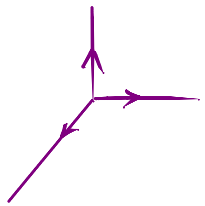
basic disk classes $\underset{1\text{-}1}{\longleftrightarrow}$ rays of fan.

q and $\check{q}$ are related by mirror maps.



$$W^{mirror}_q \triangleq W^{HV}_{\check{q}(q)}.$$

e.g. $\mathbb{P}^2$.



$$W^{HV} = Z_1 + Z_2 + \frac{\check{q}}{Z_1 Z_2}\ ; \quad \check{q} = q\ ; \quad W^{mirror} = Z_1 + Z_2 + \frac{q}{Z_1 Z_2}.$$

The mirror map

(Givental)

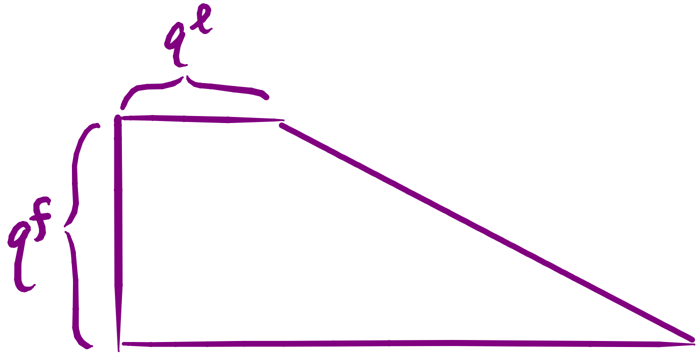
When X is a toric manifold with $-K_X$ nef,

Mirror map $q(\check{q})$ is given explicitly by $\frac{1}{z}$-coefficient of

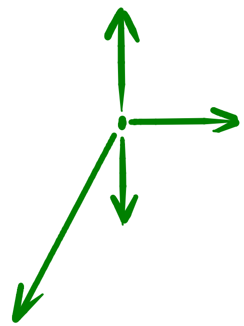
$$I(\hat{q}, z) = z\, \mathbf{e}^{(\mathbf{p}_1 \log \hat{q}_1 + \ldots + \mathbf{p}_l \log \hat{q}_l)/z} \sum_{d \in H_2^{\mathrm{eff}}(X)} \hat{q}^d \prod_i \frac{\prod_{m=-\infty}^{0} (D_i + mz)}{\prod_{m=-\infty}^{D_i \cdot d} (D_i + mz)}\,.$$

$\check{q}(q)$ is the inverse mirror map.

An example: $\mathbb{F}_2$.



fan:



$$W^{HV}_{\check{q}^{\ell},\check{q}^{f}} = z_1 + z_2 + \check{q}^{f} z_2^{-1} + \check{q}^{\ell+2f} z_1^{-1} z_2^{-2}.$$

where (mirror map)
$$\begin{cases} \check{q}^{f} = q^{f}(1+q^{\ell}); \\ \check{q}^{\ell} = q^{\ell}(1+q^{\ell})^{-2}. \end{cases}$$

$$\therefore W^{\text{mirror}}_{q^{\ell},q^{f}} = z_1 + z_2 + q^{f}(1+q^{\ell}) z_2^{-1} + q^{\ell+2f} z_1^{-1} z_2^{-2}$$
$$= W^{\text{open}}_{q^{\ell},q^{f}}.$$

Open mirror symmetry

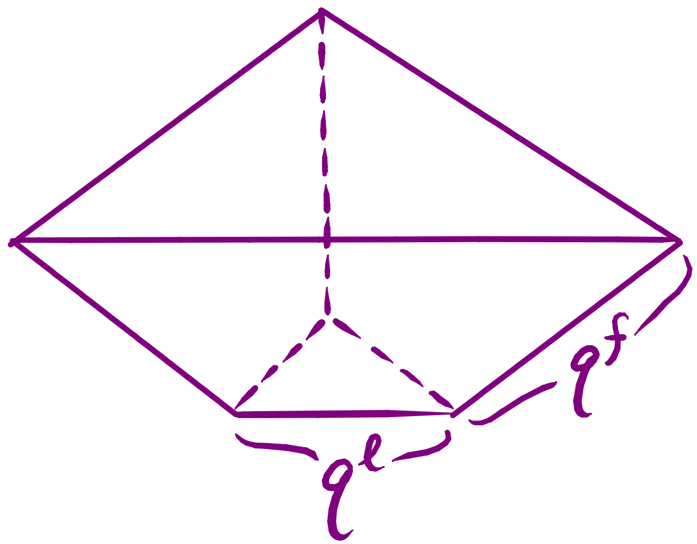
Theorem (Chan-L.-Leung-Tseng):

X: Compact toric manifold with $-K_X$ nef.

$$W^{\text{open}} = W^{\text{mirror}}$$

assuming convergence of LHS.

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.



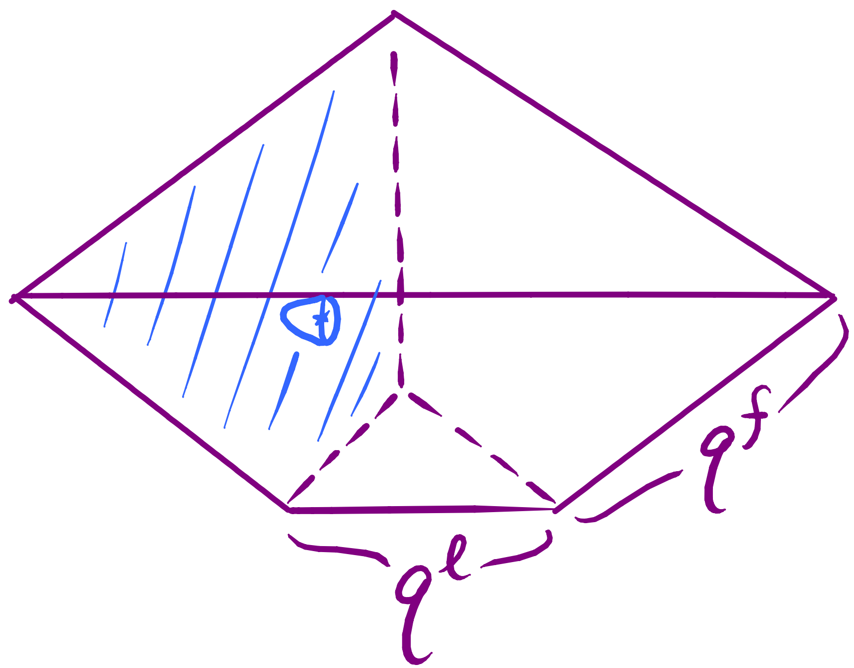
$$W^{\text{mirror}}_{\check{q}^\ell, \check{q}^f} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^\ell z_3^3 z_1^{-1} z_2^{-1}$$

where $\log q^f(\check{q}) = \log \check{q}^f + \sum_{k=1}^{\infty} \check{q}^{k\ell} \frac{(3k-1)!}{(k!)^3}$;

$$\log q^\ell(\check{q}) = \log q^\ell - 3 \sum_{k=1}^{\infty} \check{q}^{k\ell} \frac{(3k-1)!}{(k!)^3}.$$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

$\beta \in \pi_2(X,T)$ s.t. $\mu(\beta) = 2$:



Z_1

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

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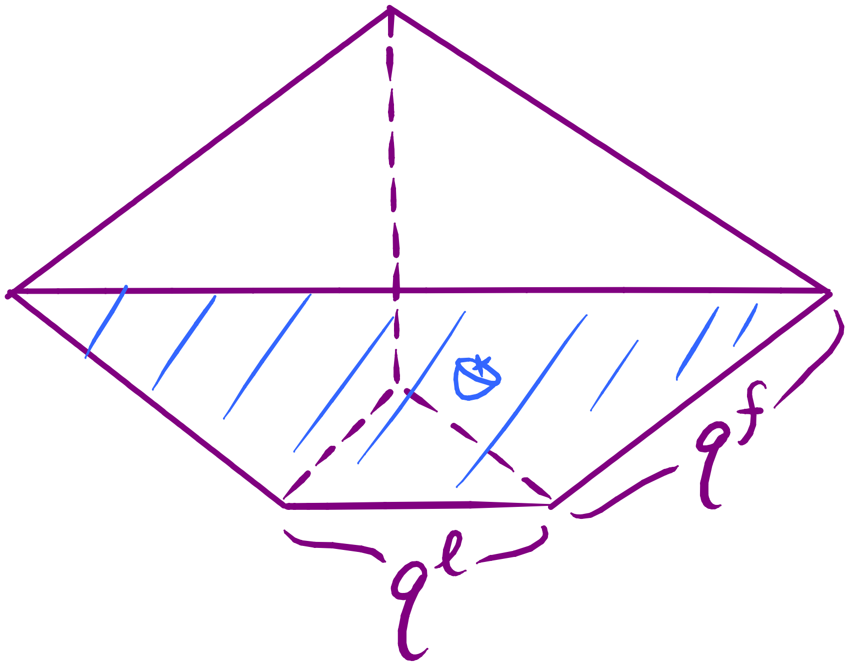

$\mathbb{Z}_2$

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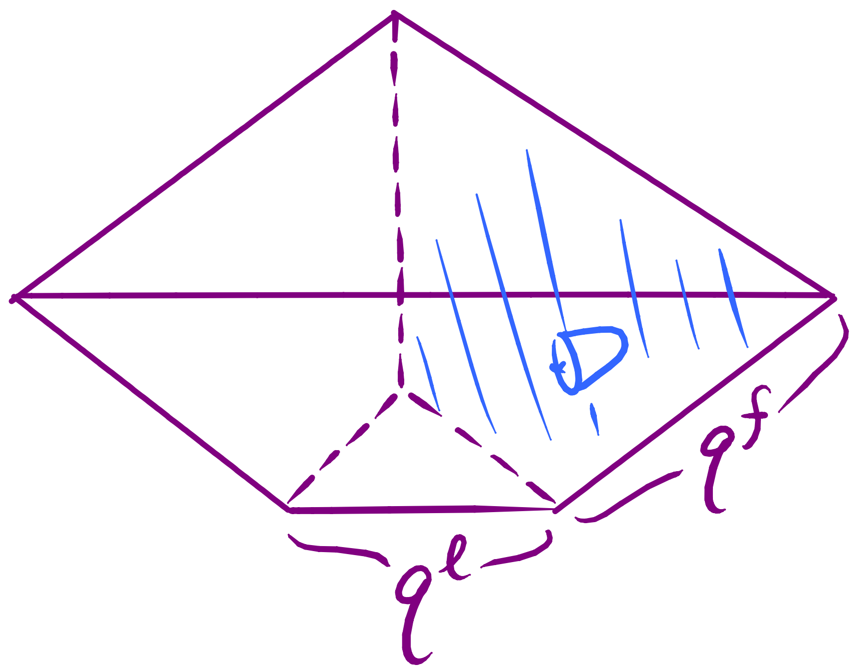

$q^\ell z_3^3 z_1^{-1} z_2^{-1}$

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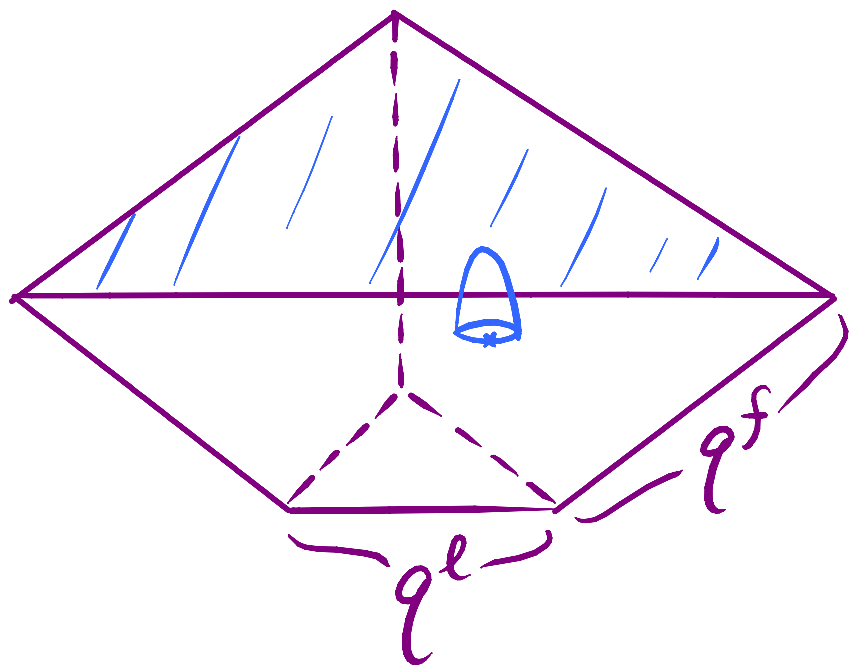

$q^f z_3^{-1}$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

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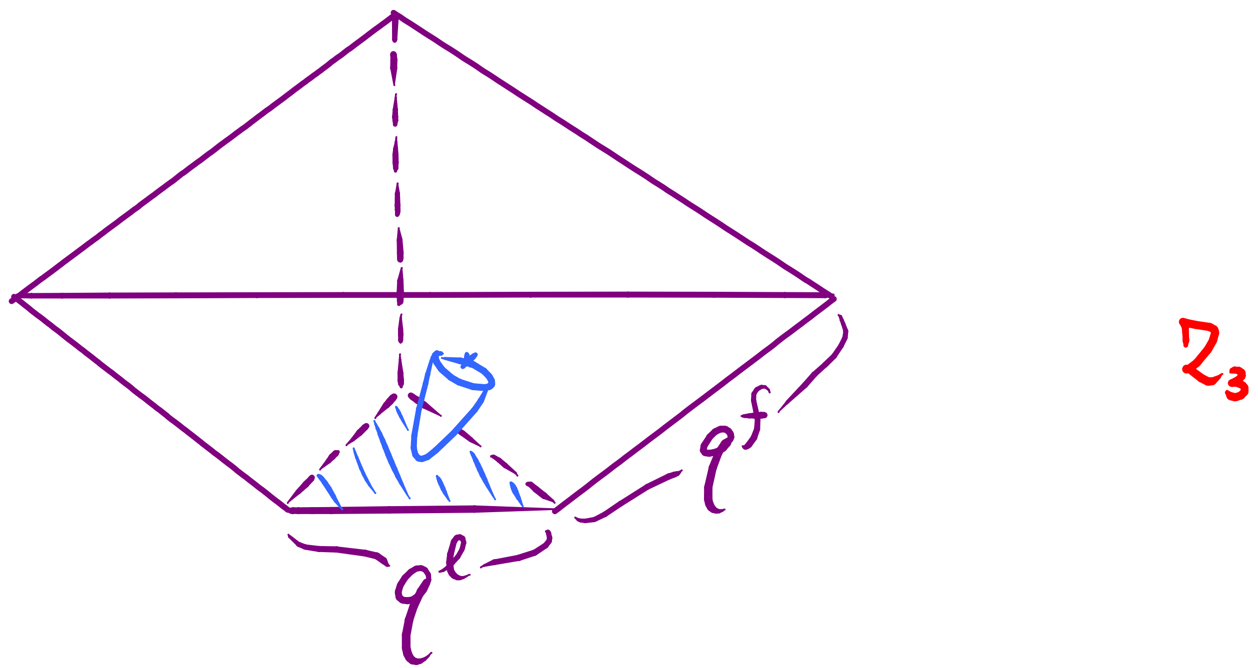
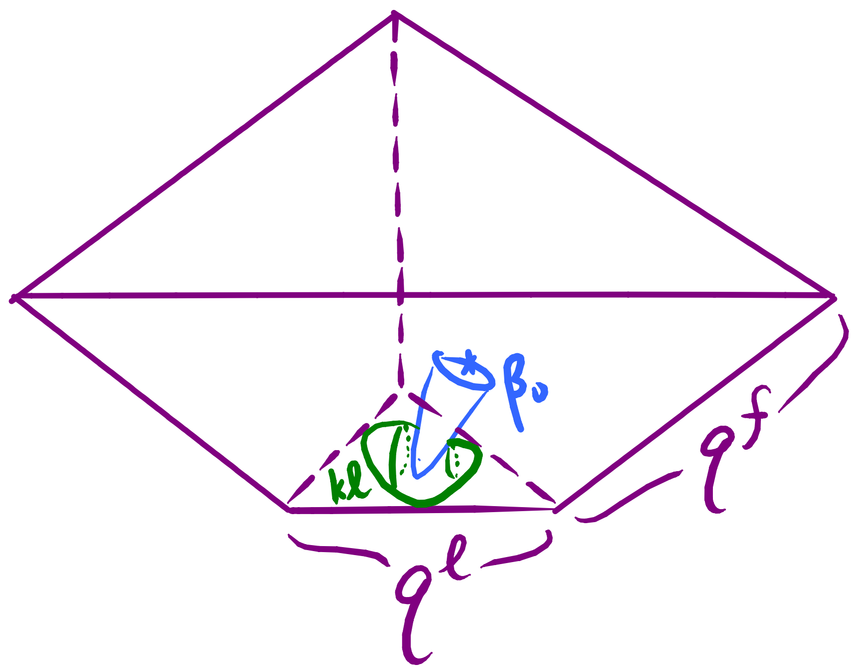


3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

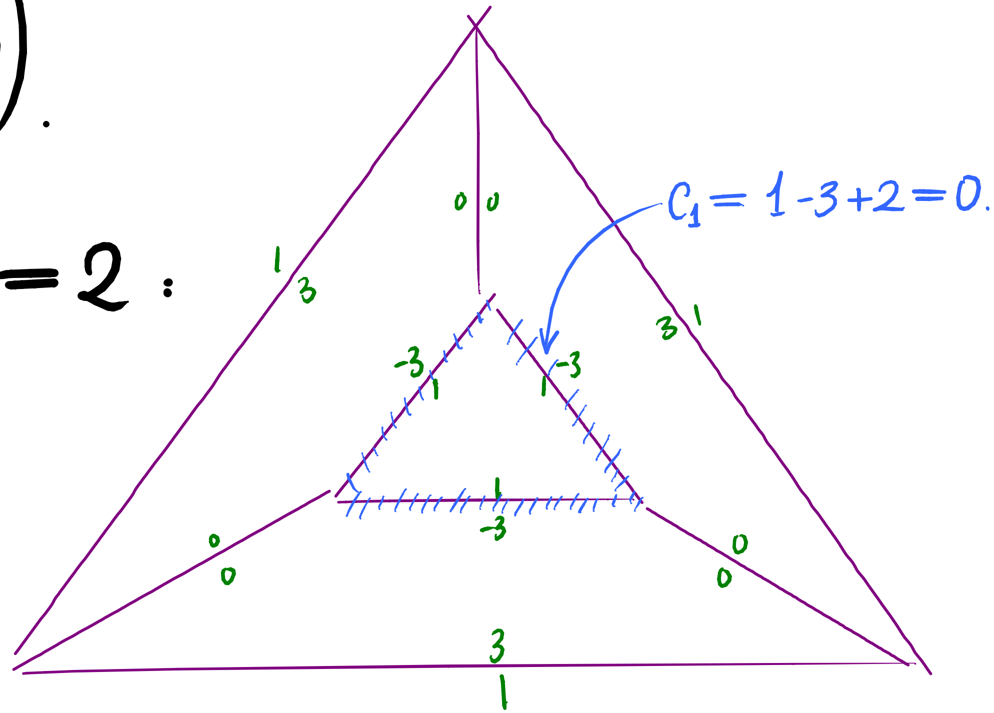
$\beta \in \pi_2(X,T)$ s.t. $\mu(\beta)=2$:

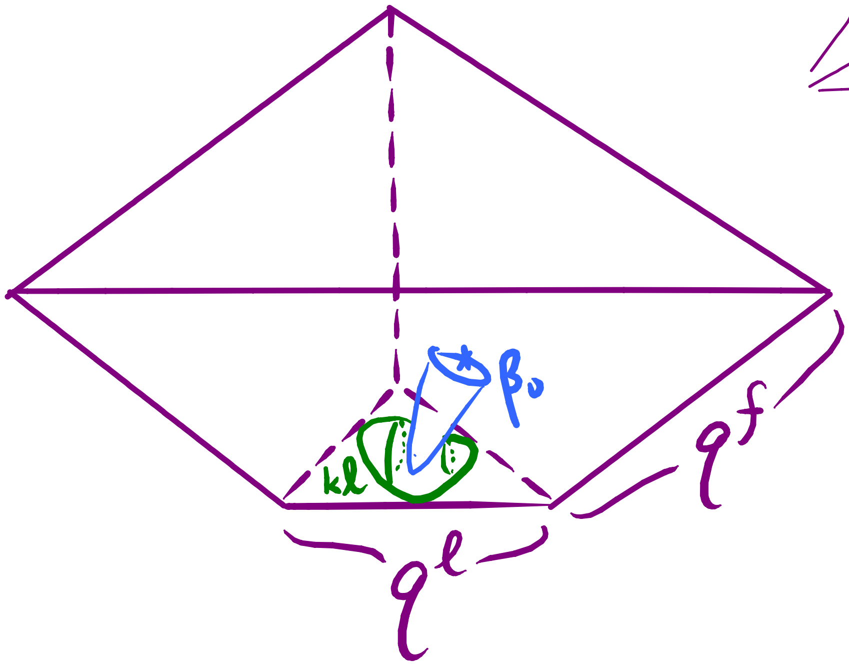


$q^{kl}\,\mathbb{Z}_3$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right)$.

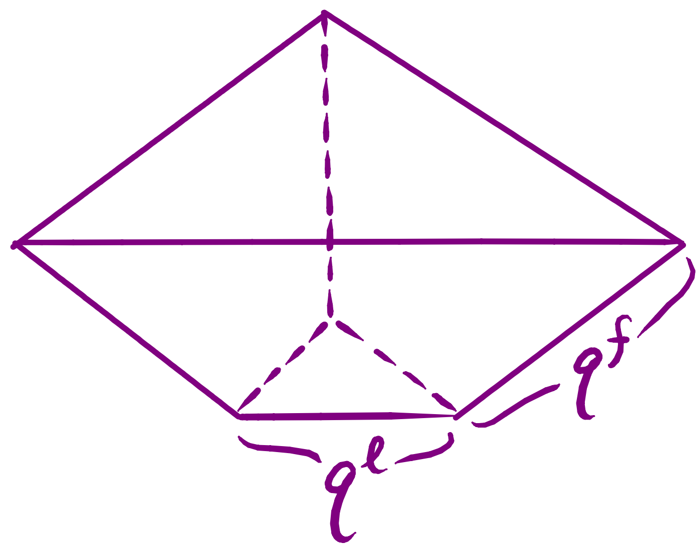
$\beta \in \pi_2(X, T)$ s.t. $\mu(\beta) = 2$:





$q^{kl} \, Z_3$

3D example: $\mathbb{P}\left(K_{\mathbb{P}^2} \oplus \mathcal{O}\right).$



$$W^{open}$$

$$\|$$

$$\sum n_\beta q^{\beta(z)} = z_1 + z_2 + \left(1 + \sum n_{\beta_0 + k\ell} q^{k\ell}\right) z_3 + q^f z_3^{-1} + q^\ell z_3^3 z_1^{-1} z_2^{-1}.$$

By change of coordinates,

$$W^{open} = z_1 + z_2 + z_3 + q^f \left(1 + \sum n_{\beta_0 + k\ell} q^{k\ell}\right) z_3^{-1} + \frac{q^\ell}{\left(1 + \sum n_{\beta_0 + k\ell} q^{k\ell}\right)^3} z_3^3 z_1^{-1} z_2^{-1}.$$

Enumerative meaning of $W^{\text{open}} = W^{\text{mirror}}$

$$W^{\text{mirror}} = z_1 + z_2 + z_3 + \check{q}^f z_3^{-1} + \check{q}^\ell z_3^3 z_1^{-1} z_2^{-1}$$

$$W^{\text{open}} = z_1 + z_2 + z_3 + q^f \left(1 + \sum n_{\beta_0+k\ell}\, q^{k\ell}\right) z_3^{-1} + \frac{q^\ell}{\left(1+\sum n_{\beta_0+k\ell}\, q^{k\ell}\right)^3} z_3^3 z_1^{-1} z_2^{-1}.$$

$$\text{Equality} \Leftrightarrow \underbrace{\check{q}^f}_{\text{Explicit in terms of } q} = q^f\left(1 + \sum_{k=1}^{\infty} n_{\beta_0+k\ell}\, q^{k\ell}\right).$$

Get $n_{\beta_0+k\ell}$.

k	0	1	2	3	4	5	6
$n_{\beta_0+k\ell}$	1	-2	5	-32	286	-3038	35870

$$W^{\text{open}} = W^{\text{mirror}}$$

$$\operatorname{Jac}(W_q^{\text{open}}) \overset{\text{FOOO}}{\cong} QH^*(X) \overset{\text{Givental}}{\cong} \operatorname{Jac}(W_q^{\text{mirror}})$$

$$\left[\frac{\partial}{\partial q} W_q^{\text{open}}\right] \longleftrightarrow \left[\frac{\partial}{\partial q} W_q^{\text{mirror}}\right]$$

$$\Longrightarrow \quad W_q^{\text{open}} = W_q^{\text{mirror}}$$

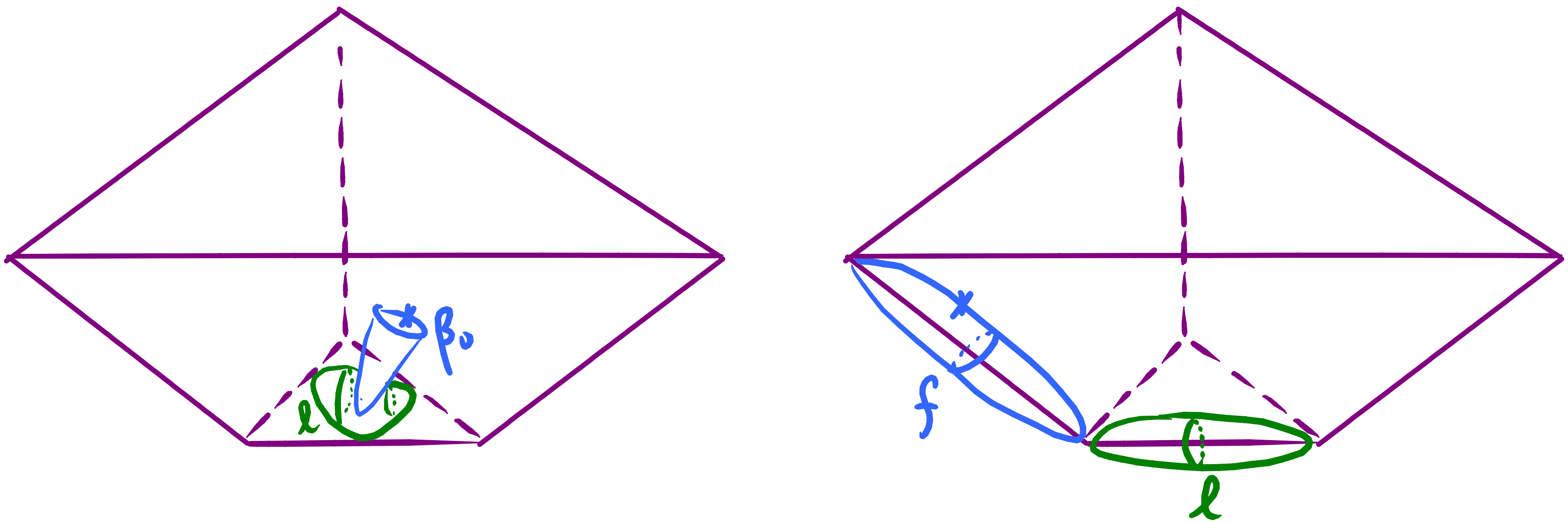
by use of semi-simplicity and universal unfolding.

Open-closed relation

Theorem: (Chan)

appear in J function.

$$n_{\beta_0+kl} = \langle [\mathbf{pt}] \rangle_{0,1,f+kl}.$$



Open-closed relation

$$J(q) \sim \sum_{\substack{\alpha,\\ d\in H_2^{\mathrm{eff}}(X)\setminus\{0\}}} \frac{q^d}{z} \sum_{k\geq 1} \left(\langle \phi_\alpha \psi^{k-1} \rangle_{0,1,d} \frac{\phi^\alpha}{z^k} \right) \in H^{\cdot}(X,\mathbb{C})[[\tfrac{1}{z}]].$$

$$H^0\text{-part of } \frac{1}{z^2}\text{-coefficient}$$

$$= q^f \left(1 + \sum_{k=1}^{\infty} \langle pt \rangle_{0,1,f+kl}\, q^{kl} \right)$$

$$= q^f \left(1 + \sum_{k=1}^{\infty} n_{\beta_0+kl}\, q^{kl} \right) \underset{\text{Want}}{=} \check{q}^f$$

Open-closed relation

Theorem: (Givental; Lian-Liu-Yau)

$$I(\check{q}(q)) = J(q).$$

$$\wr$$

$$\sum_{d \in H_2^{\mathrm{eff}}(X)\setminus\{0\}} \check{q}^d \prod_i \frac{\prod_{m=-\infty}^{0}(D_i + mz)}{\prod_{m=-\infty}^{D_i \cdot d}(D_i + mz)}$$

H^0-part of $\frac{1}{z^2}$-coefficient $= \check{q}^f$.

$\therefore\ q^f \left(1 + \sum_{k=1}^{\infty} n_{\beta_0 + kl}\, q^{kl}\right) = \check{q}^f.$

Seidel and Batyrev elements

Recall we want

$$QH^\cdot(X) \xrightarrow{\cong} \mathrm{Jac}(W^{open})$$

$$S_i \longmapsto q^{\beta_i(z)}.$$

Open mirror theorem: $W^{open} = W^{mirror} = \sum_{i=1}^{m} \check{q}^{\beta_i(z)}(q) = \sum_{i=1}^{m} (1+\delta_i(q))\, q^{\beta_i(z)}.$

Gonzalez - Iritani: $B_i = (1+\delta_i(q))\, S_i.$

$$B_i \longmapsto \check{q}^{\beta_i(z)}. \quad \#$$

Conclusion

- $W^{open} = W^{mirror}$.

- $QH^{\cdot}(X) \xrightarrow{\cong} Jac(W^{open})$

  $S_i \longmapsto q^{\beta_i(z)}$.