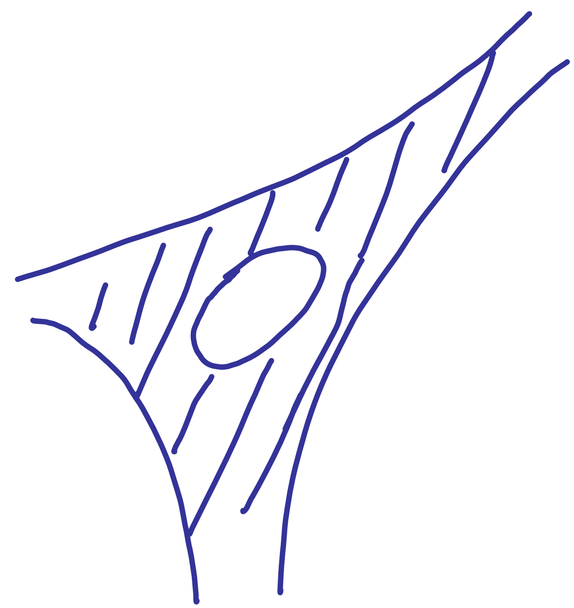
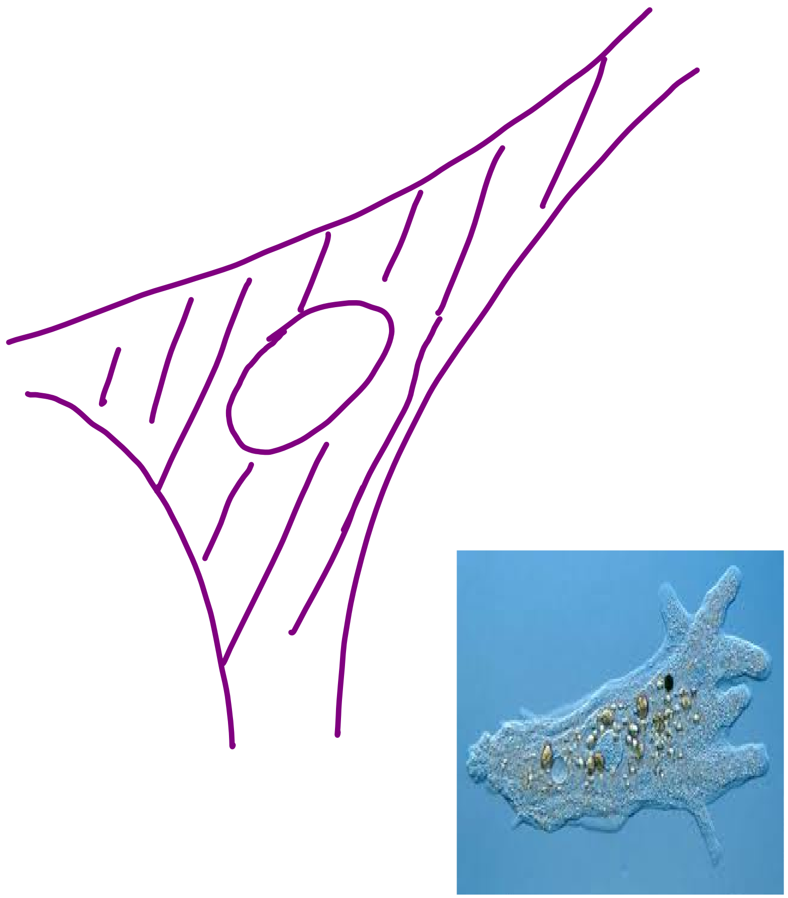
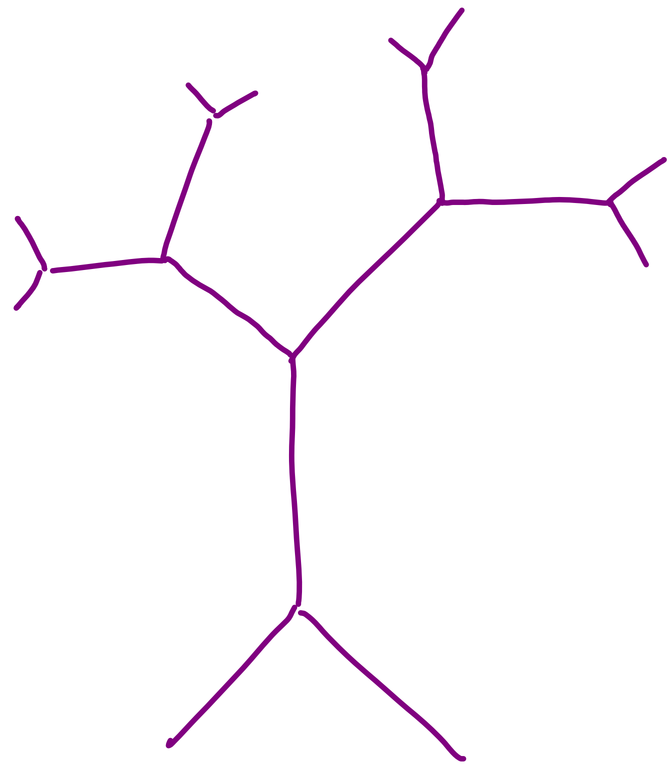
A short trip to tropical geometry



Siu-Cheong Lau
Boston University

Why "tropical": one of the founders, Imre Simon, is a Brazilian mathematician.
Another reason: the subject has a lot of "amoebas", "trees"...



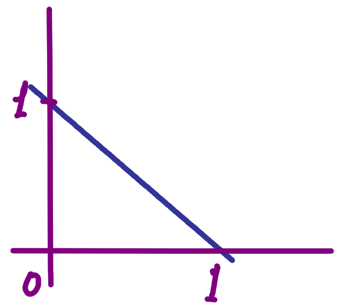


$$\text{Algebra} \xleftrightarrow{\text{Descartes}} \text{Geometry}$$

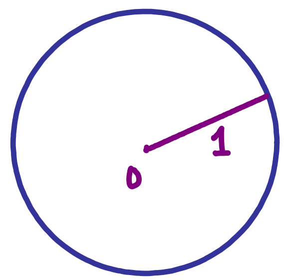
(1596 - 1650)

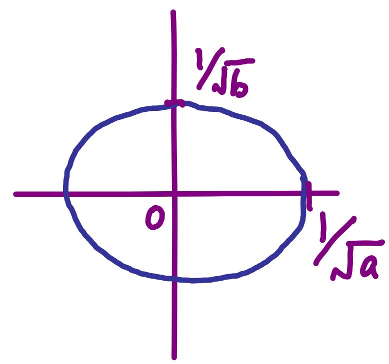
Cartesian coordinate system:
points in the plane <-> pairs of real numbers (x, y)

e.g. $x + y + 1 = 0.$

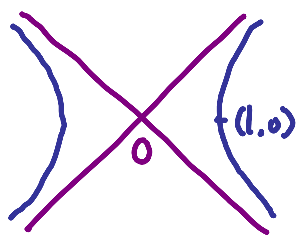


e.g. $ax^2 + by^2 = 1.$





e.g. $x^2 - y^2 = 1.$



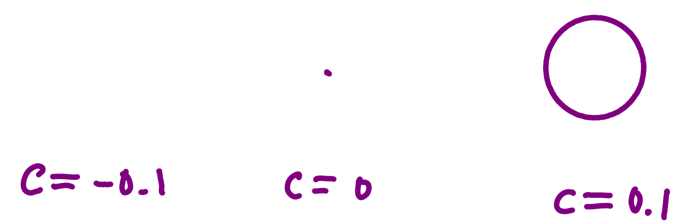
Real world, or complex world?

An equation over $\mathbb{R}$ may have no solution!

e.g. $x^2+y^2=-1$.

Also the solution set is not stable upon perturbation.

e.g. $x^2+y^2=c$.



It behaves much better over $\mathbb{C}$!

Imagine: have a number 'i' satisfies $i^2=-1$.

Complex number: $a+bi$. Have $+, \cdot, 1/(\cdot)$.

Euler

Then every equation has a solution (in complex numbers).

Unification!

Allowing complex numbers, all quadratic equations are "the same" by change of coordinates!

e.g. $x^2+y^2=-1 \xrightarrow{(x,y)\mapsto(ix,iy)} x^2+y^2=1.$

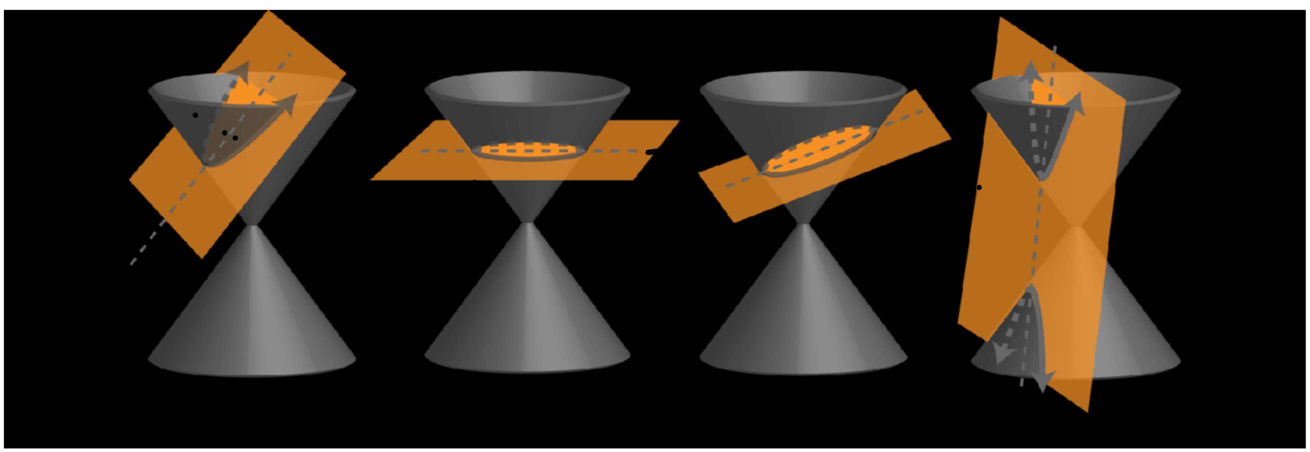
e.g. $x^2-y^2=1 \longrightarrow x^2+y^2=1.$

Hence all conics are uniformized to one shape!

Our real world
temporary; unstable
incomplete
apparent

Plato's "world of ideal forms"
eternal; stable
perfect
unseen

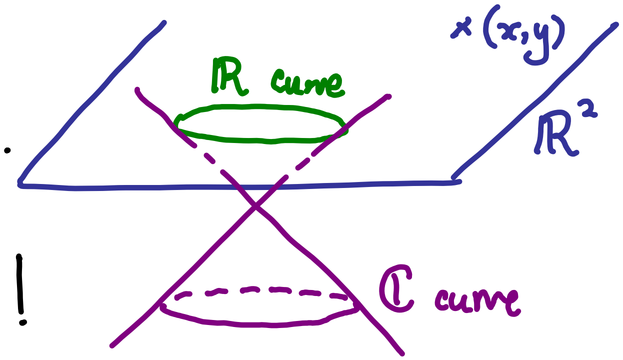
Plato



(From http://2012books.lardbucket.org)

Dimension is too HIGH!!!

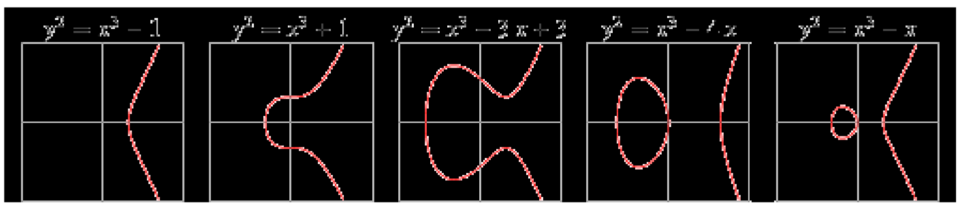
dim (Real plane) = 2. dim (Real curve) = 1.

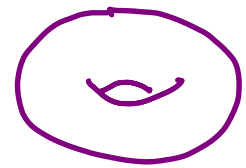


dim (Complex plane) = 4. dim (Complex curve) = 2!

$\ni (a+bi, c+di)$

Another famous example: Elliptic curve $y^2 = x^3 + ax + b$.

Real: 

$\xleftrightarrow{\text{Weierstrass}}$ Complex: 

From Wolfram http://mathworld.wolfram.com/EllipticCurve.html

Tropical method: TAKE LOG!

$$\underset{\text{real world}}{\mathbb{R}^2} \subset \underset{\text{complex world}}{\mathbb{C}^2} \xrightarrow{(\log|x|,\ \log|y|)} \underset{\text{tropical world}}{\mathbb{R}^2}$$

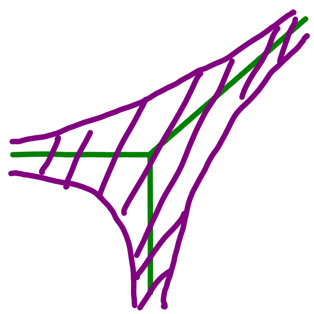
Consider the images of the curves under this map. The shapes look wild! (And hence not discovered by geometers before 20th century!)

This projection is more "stable" than taking real slices!

Image of curves under $\log|x|$

eg. $x = 1$.

eg. $x + y = 1$.



eg. $Ax^2 + By^2 + Cxy + Dx + Ey + F = 0$

If we rescale the base (moving the picture further and further away), we get the "spines" of the "amoebas"!

Log

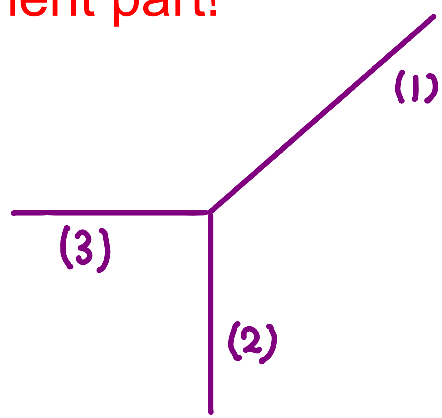
$$(x,y) \in \mathbb{C}^2 \longrightarrow \mathbb{R}^2 \ni (u,v) \qquad \begin{cases} u = \log_t |x| \\ v = \log_t |y|. \end{cases}$$

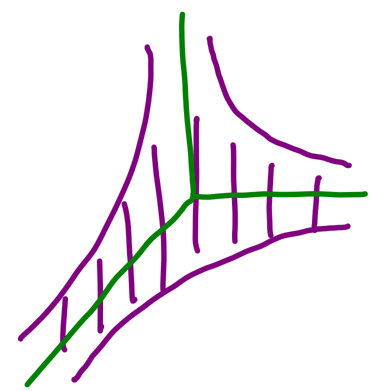
$$x = t^u e^{i\theta_1}, \; y = t^v e^{i\theta_2}.$$

Take t >> 0.

eg. x + y + 1 = 0 $\Longleftrightarrow$ $t^u e^{i\theta_1} + t^v e^{i\theta_2} + 1 = 0.$
LHS can be zero only when at least two of the terms are about of the same norm, and all the other terms are comparatively small.
Key: When t >> 0, can ignore the argument part!

Thus three possibilities:
(1) u = v and u, v > 0.
(2) u = 0 and v < 0.
(3) v = 0 and u < 0.





Extract the spine of the amoeba!

In other words, consider the function max {u,v,0} and take the place where the function is not "smooth".

$$x + y + 1 \longrightarrow \max\{u, v, 0\}.$$

Tropical Semiring $\mathbb{T} \triangleq (\mathbb{R} \cup \{-\infty\}, \hat{+}, *)$.

$$u \hat{+} v \triangleq \max\{u, v\} ; \qquad u * v \triangleq u + v.$$

$$\Big\updownarrow t \to \infty \qquad\qquad \|$$

$$u +_t v \triangleq \log_t (t^u + t^v) ; \qquad u *_t v \triangleq \log_t (t^u \cdot t^v) = u + v.$$

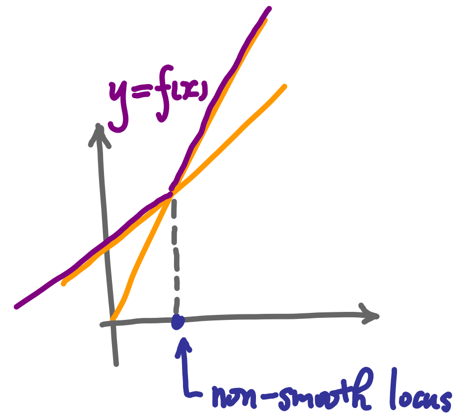
- ⋆ commutative: $x \hat{+} y = y \hat{+} x$.
- ⋆ distributive: $x * (y \hat{+} z) = x * y \hat{+} x * z$.
- ⋆ identity: $x \hat{+} (-\infty) = x$; $x * 0 = x$.

No additive inverse! e.g. $1 \hat{+} x = 0$ has no solution!

- $x \hat{+} x = x$.
- $0 * x = x$.
- $(x \hat{+} y)^n = x^n \hat{+} y^n$.

Tropical polynomial $f = \hat{\sum}_{(i,j)} a_{i,j}\, x^i y^j = \max_{(i,j)} \left(i\cdot x + j \cdot y + a_{i,j} \right)$.

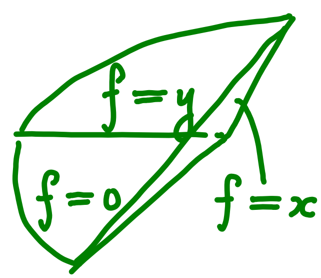
e.g. $f(x) = x^3 \hat{+} 2x = \max\{3x, x+2\}$.



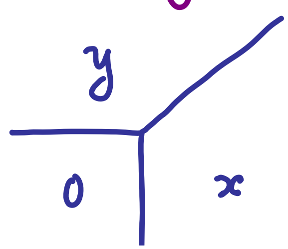
Tropical hypersurface defined by $f \triangleq$ non-smooth locus of f.

Note: f and kf defines the same hypersurface.

e.g. $x \hat{+} y \hat{+} 0 = \max\{x, y, 0\}$.



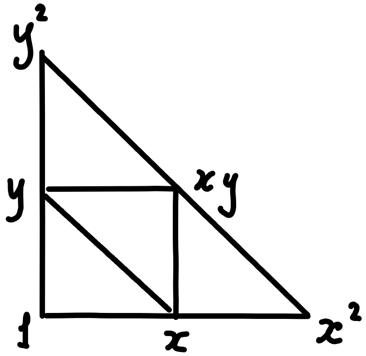
non-smooth locus :



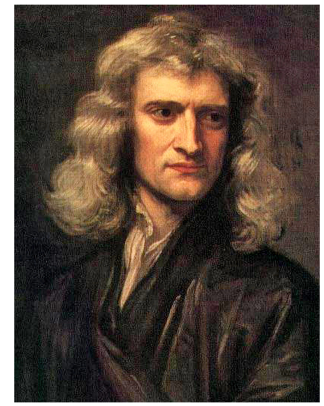
An easy way to draw tropical curve

e.g. $ax^2 \hat{+} bxy \hat{+} cy^2 \hat{+} dx \hat{+} ey \hat{+} f$.

take dual :



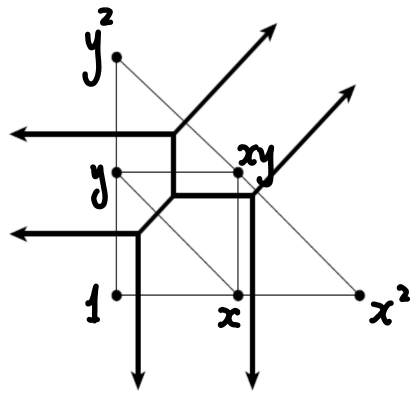
Newton polytope



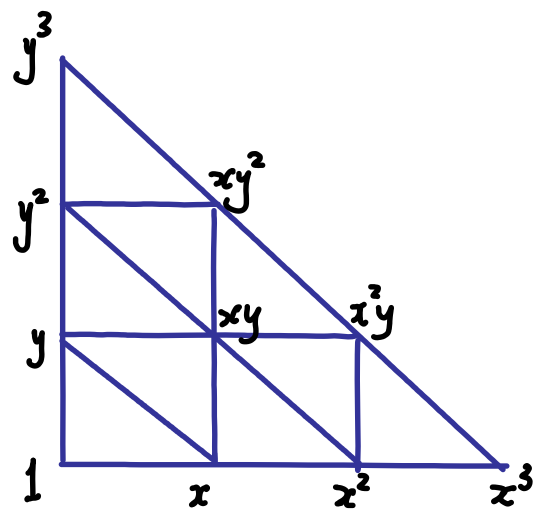
(1642-1727)

An easy way to draw tropical curve

e.g. $ax^2 \hat{+} bxy \hat{+} cy^2 \hat{+} dx \hat{+} ey \hat{+} f$.

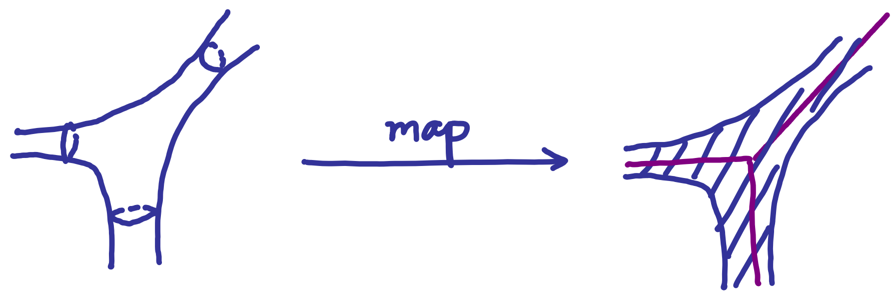


e.g. deg 3.

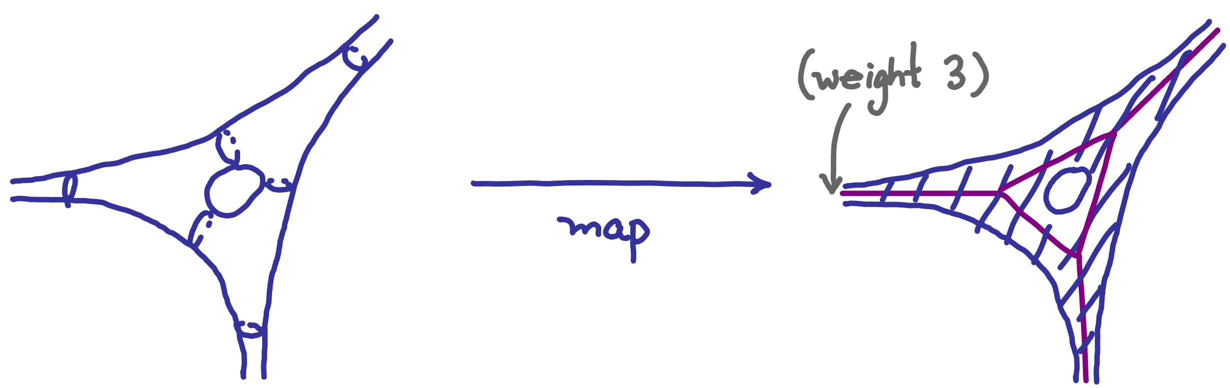


Complex vs tropical curve

Tropicalization:
$$2x+3y+1 \longrightarrow x \hat{+} y \hat{+} 0$$

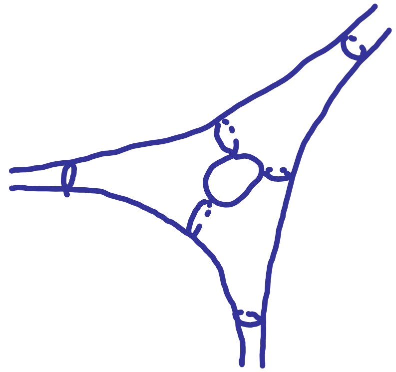


$$x^3+y^3+1+t\,xy \longrightarrow x^3 \hat{+} y^3 \hat{+} 0 \hat{+} 1 * xy$$



Conclusion

$\mathbb{C}$ world



Plato

section

projection

$\mathbb{R}$ world

$\mathbb{T}$ world