Generalized SYZ and Homological Mirror Symmetry

joint work with Cheol-Hyun Cho, Hansol Hong and Sang-Hyun Kim

Siu-Cheong Lau
Harvard University
Ultimate goal: to develop a unified geometric approach to construct and understand mirror symmetry

- Homological mirror symmetry (category)
- Genus-zero closed-string mirror symmetry (Frobenius structure)
- Higher-genus mirror symmetry (quantization)
- Global mirror symmetry (stability conditions)

- Non-commutative versions of mirror symmetry
- Mirror symmetry for non-Kaehler Calabi-Yau manifolds
An overview to SYZ mirror symmetry
Strominger-Yau-Zaslow: Mirror symmetry is T-duality

Torus duality:
\[ V/\Lambda^* = T \quad | \quad V/\Lambda = T \]

Semi-flat mirror symmetry: (Leung-Yau-Zaslow)

\[ T^*V/\Lambda^* = V \times V^*/\Lambda^* \]

with canonical symplectic form

\[ TV/\Lambda = V \times V/\Lambda \]

with canonical complex structure
SYZ for compact toric manifolds

**Toric manifolds**

$\mathbb{P}^1$  

$\mathbb{P}^2$

$L =$ moment map fiber

$q = \exp(-\int_{\mathbb{P}^1} \omega)$.

**Landau-Ginzburg mirror**

$\mathbb{C}^x, W = z + \frac{q}{z}$

$\mathbb{C}^x, W = z_1 + z_2 + \frac{q}{z_1 z_2}$
For semi-Fano toric manifolds, sphere bubbling may occur for discs of Maslov index two.

Thm. (Chan-Lo-Leung-Tseng):

\[ 1 + \delta_i(q) = \exp\left(g_t(q(q))\right), \]

\[ g_t(q) := \sum_d \frac{(-1)^{(D_d - d)(-D_t \cdot d) - 1)!}{\prod_{p \cdot d}(D_p \cdot d)!} q^d \]

← hypergeometric series appearing in mirror map.

by using Seidel representation and mirror theorem [Givental, Lian-Liu-Yau]
Gross-Siebert program

- In general Lagrangian fibrations have singular fibers
  \[\Rightarrow\text{need quantum corrections by holomorphic discs emanated from singular fibers.}\]

- Gross-Siebert:
  use wall-crossing and scattering (Kontsevich-Soibelman)
  \[\Rightarrow\text{Reconstruction of mirror by tropical geometry.}\]
SYZ for toric Calabi-Yau manifolds using symplectic geometry

Construct the SYZ mirrors for all toric Calabi-Yau manifolds (Chan-L.-Leung) by wall-crossing of open GW invariants (Auroux).
Gross-Siebert conjecture

(normalized) slab functions = generating functions of open GW invariants and they produce the inverse mirror map.

e.g. $K_{1p^2}$.

midor map: $q = \dot{q} e^{3 \sum_{k=1}^{\infty} \frac{(3k-1)!}{(k!)^3} \dot{q}^k}$

slab function: $(1-2q+5q^2-32q^3+\ldots) + x + y + qx^{-1} y^{-1} = \sum_{k>0} n_{\beta_0 + kl} q^k = \frac{q}{\left( \sum_{k>0} n_{\beta_0 + kl} q^k \right)^3}.$
Theorem (Chan-Cho-L.-Leung):

(inverse) mirror map = \text{SYZ map}.

(\text{generating function of open GW invariants})

for toric Calabi-Yau manifolds.

Theorem (L.):

slab functions = \text{SYZ map}

for toric Calabi-Yau manifolds.
Insights of Seidel and Sheridan

Seidel:
Construct an immersed Lagrangian in pair of pants and use it to prove homological mirror symmetry for Riemann surface of genus 2.

Sheridan; Ueda:
Generalize Seidel's construction to higher dimensions and prove homological mirror symmetry for Fermat Calabi-Yau.
Cho-Hong-L.: introduced a generalized version of SYZ mirror symmetry based on (immersed) Lagrangian Floer theory

[Fukaya-Oh-Ohta-Ono]
[Seidel]
[Akaho-Joyce]
Immersed Lagrangian Floer theory

Choose $\tilde{L} \rightarrow (X, \omega)$ Lagrangian immersion.

(infinitesimal)

deformation space: $H^i(\tilde{L} \times \tilde{L})$. e.g. $H^0(\tilde{L}) \oplus \text{Span}\{x, x\}$. 

obstruction term: $m^b_0 = \sum_{k \geq 0} m_k(b, \ldots, b)$

Weakly unobstructed: $m^b_0 = W(b) \cdot 1_L$. 

### SYZ vs. Generalized SYZ

<table>
<thead>
<tr>
<th>SYZ</th>
<th>Generalized SYZ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 Lagrangian torus fibration</td>
<td>immersed Lagrangian L</td>
</tr>
<tr>
<td>dual torus fibration</td>
<td>deformation space of L</td>
</tr>
<tr>
<td>counting discs</td>
<td>counting polygons</td>
</tr>
<tr>
<td>for quantum corrections</td>
<td>for quantum corrections</td>
</tr>
<tr>
<td>$m_0^L = W(z) \cdot 1_L$</td>
<td>$m_0^b = W(b) \cdot 1_L$</td>
</tr>
<tr>
<td>$W = \sum_{\rho} \eta_{\rho} q^\rho z^{a\rho}$</td>
<td>$W = \sum_{\rho} \eta_{\rho} q^\rho z^{a\rho}$</td>
</tr>
</tbody>
</table>
Mirror functor

[Fuk(M)] \exists A_\infty functor

\[
\begin{align*}
\text{Fuk}_0(M) & \longrightarrow M.F.(W) \\
\mathcal{U} & \longrightarrow m_1^{((L,b),U)} = \hat{\mathcal{U}} : \text{Span}((LbU)) \mathcal{C}
\end{align*}
\]

\[A_\infty - \text{relations} \Rightarrow \hat{\mathcal{U}}^2 = W(b).\]

Higher terms of the functor come from \( m_{k>1} \).
Homological mirror symmetry

Theorem (Cho-Hung-L.): Let \( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq 1 \).

The \( A^\infty \) functor

\[
\text{Fuk}(\mathbb{P}^1_{(a,b,c)}) \longrightarrow \text{MF}(W)
\]

induces an equivalence of triangulated categories

\[
\text{D}\text{Fuk}(\mathbb{P}^1_{(a,b,c)}) \overset{\sim}{\longrightarrow} \text{DMF}(W).
\]
Conclusion

- Choose a weakly unobstructed (immersed) Lagrangian \( L \subset X \).

\[
\hat{X} = \{ b : m_0^{(L, b)} \alpha 1 \}, \quad W \cong m_0^{(L, b)} / 1
\]

is the mirror localized to \( L \).

- \( \{ m > 0 \} \) gives an \( \mathcal{A}_\infty \) functor \( \text{Fuk}(X) \to \text{MF}(W) \).

- Can produce explicit matrix factorizations for toric mirror \( W \).
Example: Elliptic curve quotient

equation curve with complex multiplication $e^{\frac{2\pi i}{3}}$ with flat metric of total area $t$.

$$E \rightarrow \mathbb{Z}_3 = \mathbb{P}^1_{3,3,3}$$
• Fix a generic point $p$ in $\mathcal{L}$ (marked as $\times$ below)

• Count the number of polygons bounded by $\mathcal{L}$ passing thru $p$

with vertices being $\Delta, \Delta, \square$.
• Label the vertices by $\Delta, \Delta, \Box$.

• Record the labels of vertices of each triangle by a monomial.
\[ q_\alpha = e^{-\frac{\alpha}{24}}. \quad (\alpha \text{ stands for the minimal triangle}) \]

Record \[ \exp(-\text{area of each triangle}) \text{ in terms of } q_\alpha. \]
Fix an orientation of $\mathcal{L}$ (which is $\mathbb{Z}_3$-invariant).

Sign of each triangle: $(-1)^{\#\text{reversed edges}} (-1)^{\#\text{marked points on boundary}}$.

(due to spin structure by Seidel)
\[ W = \sum \eta_\beta q^\beta \bar{z}^{\alpha \beta} \]

\[ = -q_\alpha xyz - q_\alpha x^3 - q_\alpha y^3 + q_\alpha z^3 + \ldots \]
\[ W = -q_x y z - q_x^9 x^3 - q_y^9 y^3 + q_z^9 z^3 + \ldots \]

\[ = \phi \cdot (-x^3 - y^3 + z^3) - \psi \cdot xyz \]

where \[ \phi = \sum_{k=0}^{\infty} (-1)^{3k+1} (2k + 1) q_x^{3(12k^2 + 12k + 3)} \]

\[ \psi = -q_x + \sum_{k=1}^{\infty} \left( (-1)^{3k+1} (6k + 1) q_x^{6k+1} + (-1)^{3k} (6k - 1) q_x^{6k-1} \right) \cdot \]

\[ \sim x^3 + y^3 + z^3 - \frac{4}{\phi} xyz. \]

change of coordinates in \((x,y,z)\)

\[ \underline{\text{SYZ map}} \]
Consider \{ x^3 + y^3 + z^3 + \sigma xyz = 0 \} \subset \mathbb{P}^2.

A family of elliptic curves mirror to \( E \).

Flat coordinate is \( q(\sigma) = \exp\left( -\frac{\overline{\Pi}_B(\sigma)}{\overline{\Pi}_A(\sigma)} \right) \), where \( \overline{\Pi}_A, \overline{\Pi}_B \) satisfies Picard-Fuchs equation 
\[ u'' + \frac{3\sigma^2}{\sigma^3 + 27} u' + \frac{\sigma}{\sigma^3 + 27} u = 0. \]

\( q(\sigma) \) is called the mirror map.
**Theorem (Cho-Hong-L.):**

\[- \frac{\frac{\phi(q_\alpha)}{q_\alpha}}{\phi(q_\alpha)} = \sigma(q = q^8_\alpha)\]

where $\sigma(q)$ is the inverse mirror map.

$\Rightarrow$ Coefficients of $\sigma(q)$ are integers.
Transform of Seidel Lagrangian $L$

$$L \xrightarrow{\text{Cho-Hong-L.}} \left( \sum_{i=1}^{3} x_i X_i + \sum_{i=1}^{3} (x_i^2 + x_{i-1} x_{i+1}) Z x_i \right)^\wedge \mathbb{C}^3.$$ (need non-trivial change of coordinates)

split generates $DFuk(X)$

[AF000]

split generates $DMF(W)$

[Duclerkhoff]
Orbifolding

\[ E \quad \Downarrow \quad E/\mathbb{Z}_3 = \mathbb{P}^1_{3.3.3} \]

\[ (\mathbb{C}^3/\mathbb{Z}_3, W) \sim \{ W = 0 \} \subset \mathbb{P}^2 \]

\[ (\mathbb{C}^3, W = \phi(x^3 + y^3 + z^3) - 4xyz) \]
$$W = -q_q x y^2 + q_y x^2 + \phi \cdot (y^4 + z^4) + \psi \cdot y^2 z^2.$$
$W = q_\alpha xy z + q_\alpha x^2 + p_1 z^6 + p_2 y^3 + p_3 y^2 z^2 + p_4 y z^4$

where $p_1, ..., p_4$ are explicit series in $q_\alpha$. 
The construction works for general $P_{(a,b,c)}^{1}$ $\forall a,b,c \geq 1$:

$$W = -q_x xyz + (-1)^a \ q_x^3 x^a + (-1)^b \ q_y^3 y^b + (-1)^c \ q_z^3 z^c + \ldots$$

which is a formal power series.

[Cho-Hung-Kim-L.]:

Construct an algorithm to compute $W$ $\forall a,b,c$.

When $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1$,

$$P_{(a,b,c)}^{1} = \left( \sum_{g \geq 1} \right) / G.$$ $W(x,y,z,q)$ is an infinite series.

Thus: $W$ is convergent in a neighborhood of $(x,y,z,q) = 0$.

SYZ map is convergent even for general-type case!
\( P_{(a,b,c)} \) is quotient of space-forms

<table>
<thead>
<tr>
<th>Spherical</th>
<th>Planar</th>
<th>hyperbolic</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>( D )</td>
<td>( \tilde{E}_6 )</td>
</tr>
<tr>
<td>( a = 1 )</td>
<td>( (2,2,k) )</td>
<td>( (3,3,3) )</td>
</tr>
<tr>
<td>( k = 3,4,5 )</td>
<td>( (2.3,k) )</td>
<td>( (2.4.4) )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( (2.3.6) )</td>
<td>( (2.3.6) )</td>
</tr>
</tbody>
</table>

\[ \frac{1}{a} + \frac{1}{b} + \frac{1}{c} < 1. \]

- \( W \) is a polynomial
- \( W \) is a polynomial with series coefficients
- \( W \) is a convergent series

We compute \( W \) in all cases.
A hyperbolic example

\[ W = -q_\alpha x y^2 + q_\alpha^{12} (x^4 + y^4 + z^4) + q_\alpha^{34} x^2 y^2 + \ldots \]

\[ D \]

\[ \mathbb{P}^1_{(4,4,4)} \]
Modularity of open Gromov-Witten invariants

\[ [L.-Zhou] \quad \text{(Similar for matrix factorizations minor to L)} \]

\[ E/\mathbb{Z}_3 : W = \phi \cdot (x^3+y^3+z^3) + \psi \cdot xyz \quad \text{where} \]
\[ \phi = \eta(q^3)^3, \quad \psi = \eta(q^{1/3})^3 + 3\eta(q^3)^3 \]
are \( \Gamma(3) \)-modular forms.

\[ E/\mathbb{Z}_4 : W = \phi \cdot (x^4+y^4) + \psi \cdot y^2z^2 \quad \text{where} \]
\[ \phi = -\frac{E_2(q^{1/2})}{24} + \frac{E_2(q)}{8} - \frac{1}{12} E_2(q^2) \]
\[ \psi = \frac{1}{4} + \frac{E_2(q)}{4} - \frac{E_2(q^2)}{2} \quad \text{are } \Gamma(4) \)-modular forms.

\[ \eta = q^{1/24} \prod_{n=1}^{\infty} (1-q^n). \]

\[ \sum_{d \mid d} \sigma_1(d) q^d \]

\[ E_2 = 1 - 24 \sum_{d=1}^{\infty} \sigma_1(d) q^d \]
Global mirror symmetry

\[ \text{Fuk}(X) \xrightarrow{\text{mirror functor}} \text{MF}^{\text{graded}}(W) \]

\[ M_{\text{Käh.}}^{\text{global}} \xrightarrow{\text{SYZ map}} M_{\text{cpx.}}^{\text{global}} \]

\[ \mathcal{H}/T(N) \]

\[ \mathcal{H}/T(N) \]