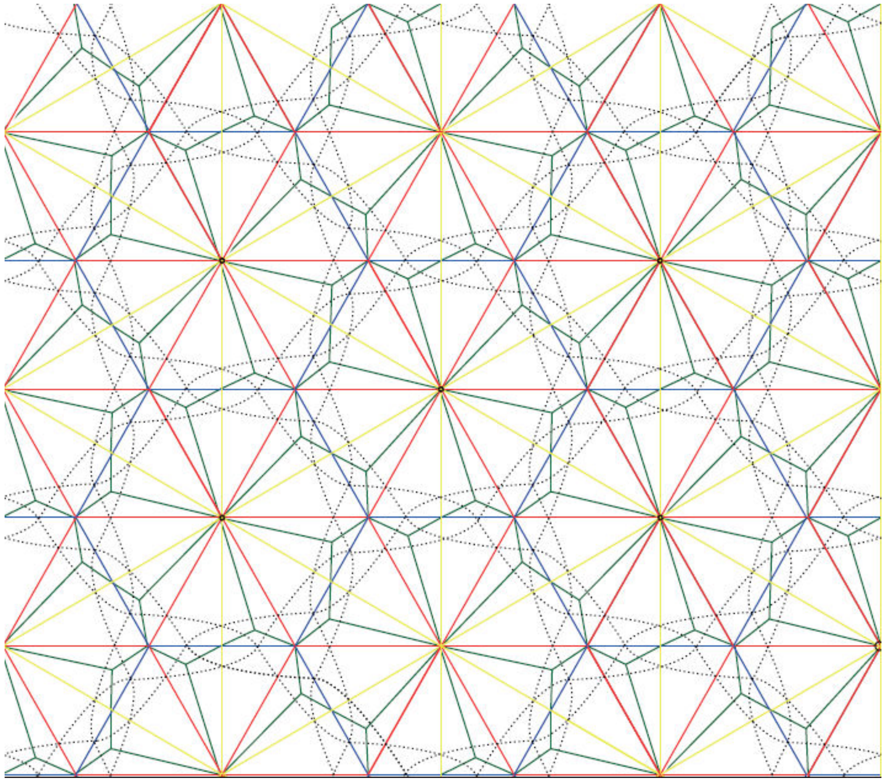
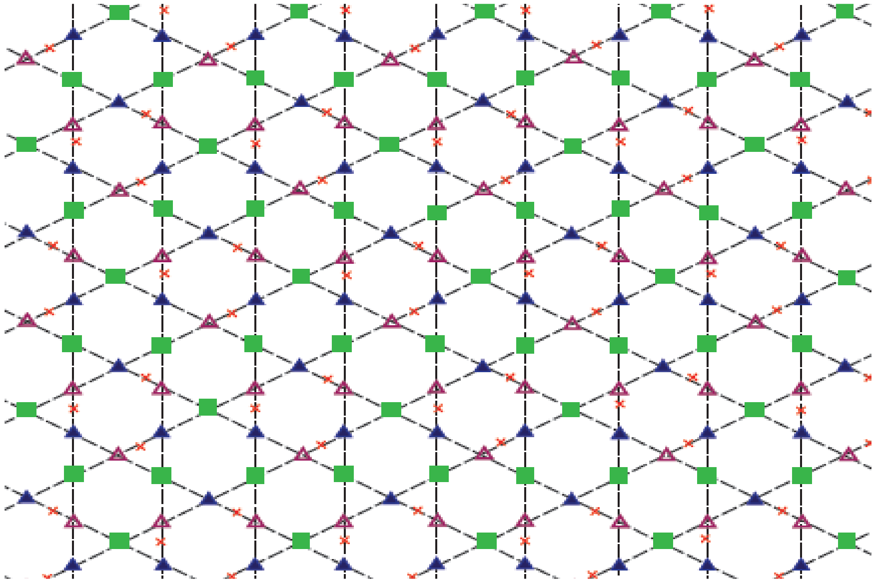
Generalized SYZ and Homological Mirror Symmetry



joint work with Cheol-Hyun Cho, Hansol Hong and Sang-Hyun Kim



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Ultimate goal: to develop a unified geometric approach to construct and understand mirror symmetry

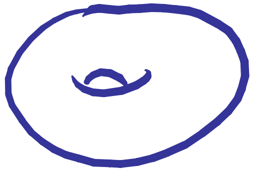
- Homological mirror symmetry (category)
- Genus-zero closed-string mirror symmetry (Frobenius structure)
- Higher-genus mirror symmetry (quantization)
- Global mirror symmetry (stability conditions)

- Non-commutative versions of mirror symmetry
- Mirror symmetry for non-Kaehler Calabi-Yau manifolds

An overview to SYZ mirror symmetry

Strominger-Yau-Zaslow: Mirror symmetry is T-duality

Torus duality:

 $V^*/\Lambda^* = T$ | $V/\Lambda = T$ 

Semi-flat mirror symmetry: (Leung-Yau-Zaslow)

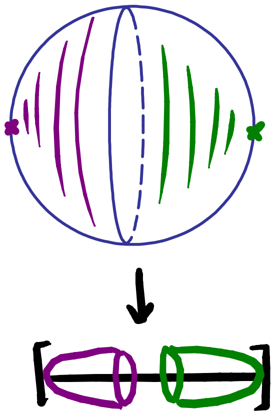
$T^*V/\Lambda^* = V \times V^*/\Lambda^*$	$TV/\Lambda = V \times V/\Lambda$
with canonical symplectic form	with canonical complex structure

SYZ for compact toric manifolds

(Cho-Oh, Chan-Leung, Auroux, Fukaya-Oh-Ohta-Ono)

toric manifolds

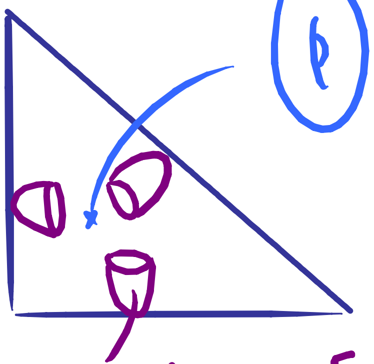
$\mathbb{P}^1$



Take L = moment map fiber

$q = \exp\left(-\int_{\mathbb{P}^1} \omega\right)$.

$\mathbb{P}^2$



L Lagrangian torus

basic discs [Cho-Oh]

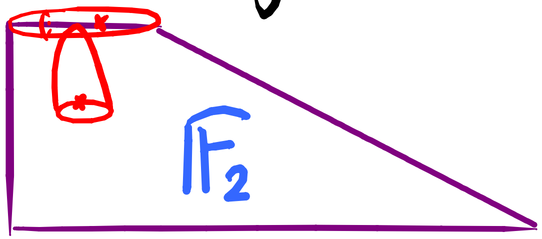
Landau Ginzburg mirror

$$\left(\mathbb{C}^\times, \ W = z + \frac{q}{z}\right)$$

$$\left(\mathbb{C}^\times, \ W = z_1 + z_2 + \frac{q}{z_1 z_2}\right)$$

For semi-Fano toric manifolds,

sphere bubbling may occur for discs of Maslov index two.



Thm. (Chan-L.-Leung-Tseng):

bubbling term

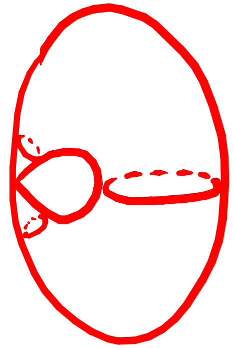
$$1 + \delta_l(q) = \exp(g_l(\check{q}(q))),$$

$$g_l(\check{q}) := \sum_d \frac{(-1)^{(D_l \cdot d)}(-(D_l \cdot d) - 1)!}{\prod_{p \neq l}(D_p \cdot d)!} \check{q}^d$$ ← hypergeometric series appearing in mirror map.

by using Seidel representation and mirror theorem [Givental, Lian-Liu-Yau].

Gross-Siebert program

- In general Lagrangian fibrations have singular fibers

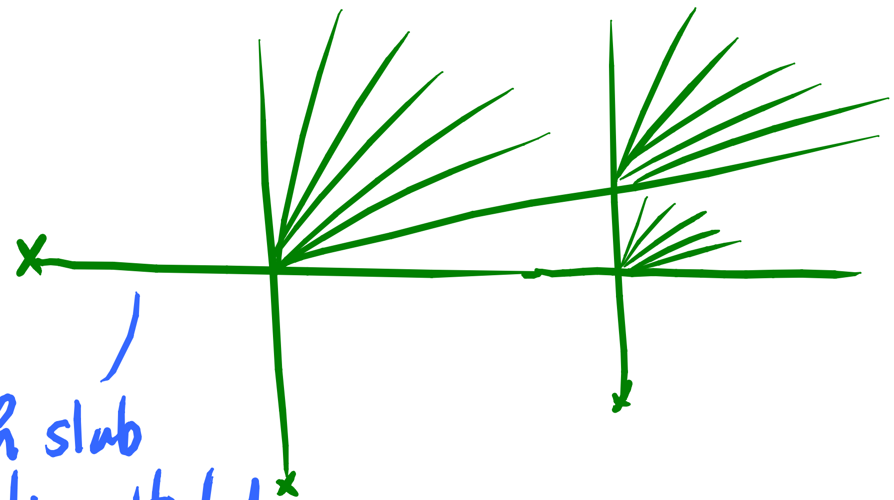


$\Rightarrow$ need quantum corrections by holomorphic discs emanated from singular fibers.

- Gross-Siebert:

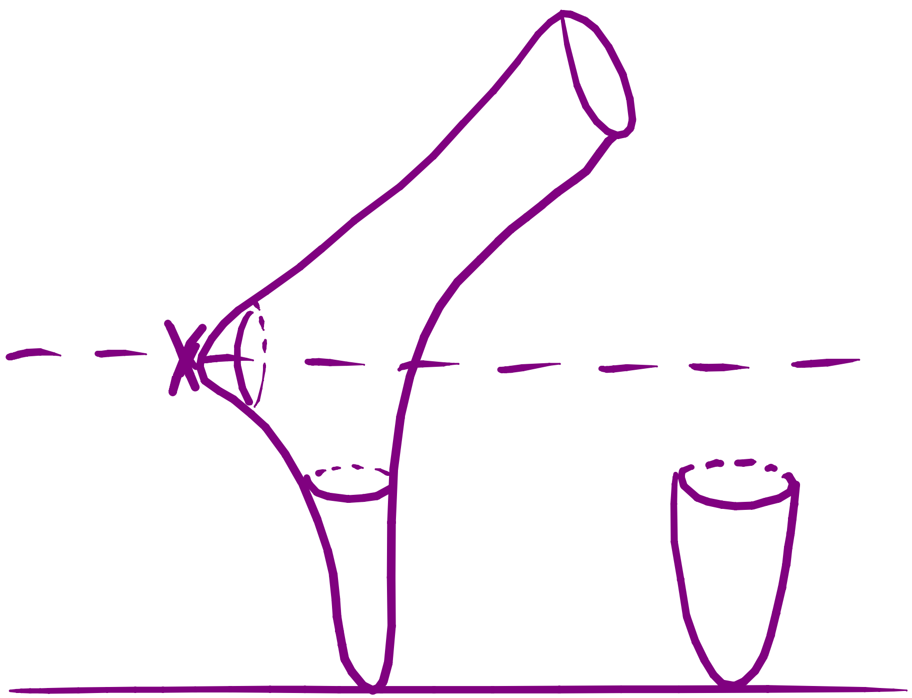
use wall-crossing and scattering (Kontsevich-Soibelman)

$\Rightarrow$ Reconstruction of mirror by tropical geometry.



SYZ for toric Calabi-Yau manifolds using symplectic geometry

Construct the SYZ mirrors for all toric Calabi-Yau manifolds (Chan-L.-Leung) by wall-crossing of open GW invariants (Auroux).



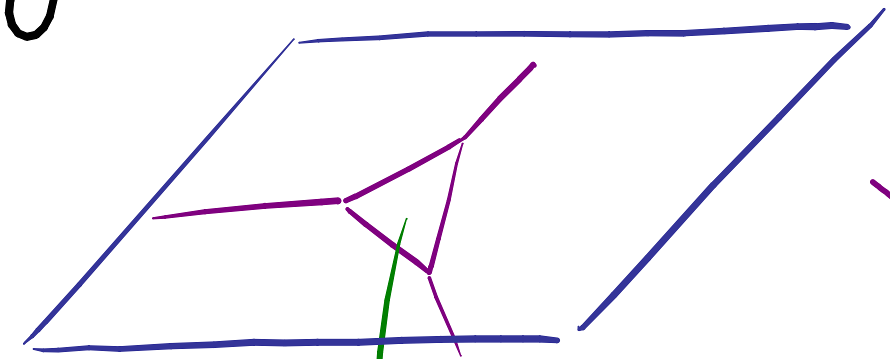
Gross-Siebert conjecture

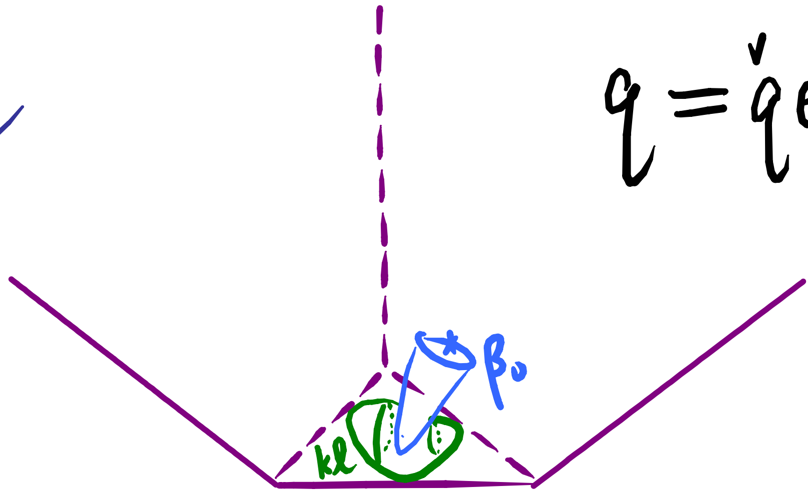
(normalized) slab functions = generating functions of open GW invariants and they produce the inverse mirror map.

e.g. $K_{\mathbb{P}^2}$.

mirror map:

$$q = \check{q} \exp\left(-3\sum_{k=1}^{\infty} \frac{(3k-1)!}{(k!)^3} \check{q}^k\right)$$





$$\check{q}(q) = \frac{q}{\left(\sum_{k\geq 0} n_{\beta_0+k\ell}\, q^k\right)^3}.$$

slab function: $(1-2q+5q^2-32q^3+\dots) + x + y + qx^{-1}y^{-1}$

$$(1-2q+5q^2-32q^3+\dots) = \sum_{k\geq 0} n_{\beta_0+k\ell}\, q^k$$

Theorem (Chan-Cho-C.-Leung):

$\text{(inverse) mirror map} = \underbrace{\text{SYZ map}}_{\text{(generating function of open GW invariants)}}.$

for toric Calabi-Yau manifolds.

Theorem (L.):

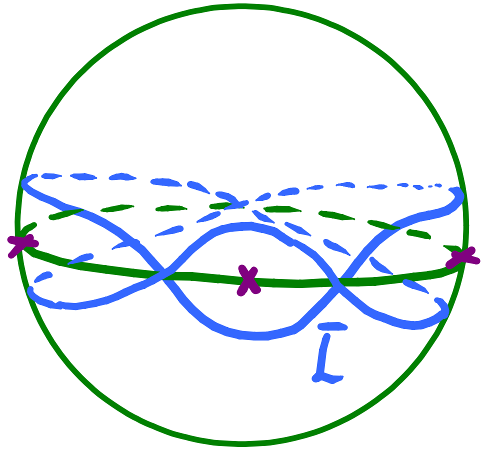
slab functions = SYZ map

for toric Calabi-Yau manifolds.

Insights of Seidel and Sheridan

Seidel:

Construct an immersed Lagrangian in pair of pants and use it to prove homological mirror symmetry for Riemann surface of genus 2.



Sheridan; Ueda:

Generalize Seidel's construction to higher dimensions and prove homological mirror symmetry for Fermat Calabi-Yau.

Cho-Hong-L.: introduced a generalized version of SYZ mirror symmetry based on (immersed) Lagrangian Floer theory

[Fukaya-Oh-Ohta-Ono]
[Seidel]
[Akaho-Joyce]

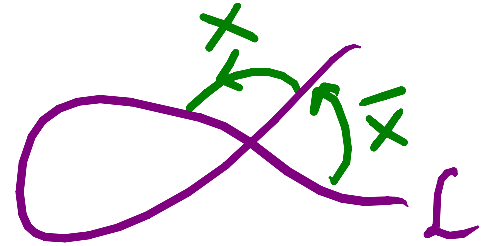
Immersed Lagrangian Floer theory

Choose $\tilde{L} \xrightarrow{\iota} (X, \omega)$ Lagrangian immersion.

(infinitestimal) deformation space: $H^{\bullet}(\tilde{L} \times_{\iota} \tilde{L})$.

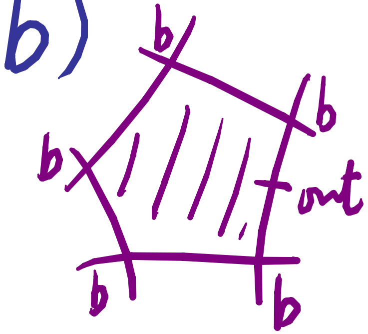
$b \in$

e.g.

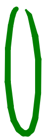
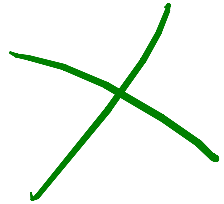
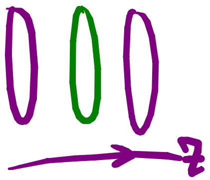
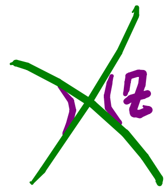
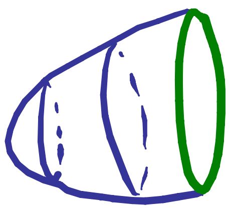
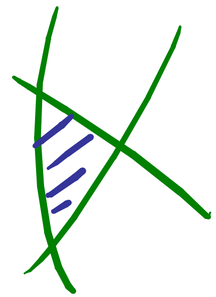


$H^{\bullet}(\tilde{L}) \oplus \mathrm{Span}\{X, \bar{X}\}$.

obstruction term: $m_0^b = \sum_{k \geq 0} m_k(b, \dots, b)$



Weakly unobstructed: $m_0^b = W(b) \cdot \mathbb{1}_L$.

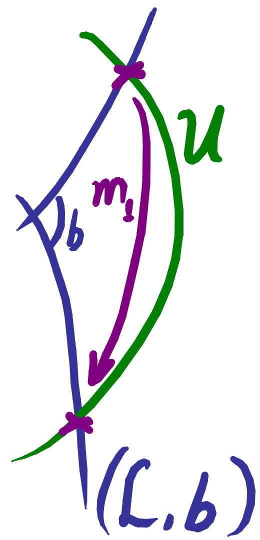
SYZ		Generalized SYZ	
	Lagrangian torus fibration	immersed Lagrangian L	
	dual torus fibration	deformation space of L	
	counting discs for quantum corrections $m_0^{L_z} = W(z) \cdot \mathbb{1}_{L_z}$. $W = \sum_\beta n_\beta q^\beta z^{\partial \beta}$.	counting polygons for quantum corrections $m_0^{b} = W(b) \cdot \mathbb{1}_{L_z}$ $W = \sum_\beta n_\beta q^\beta z^{\partial \beta}$.	

Mirror functor

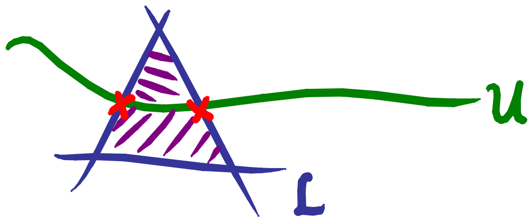
[Cho-Hong-L.] $\exists$ A_∞ functor (Yoneda embedding in the presence of m_0^b!)

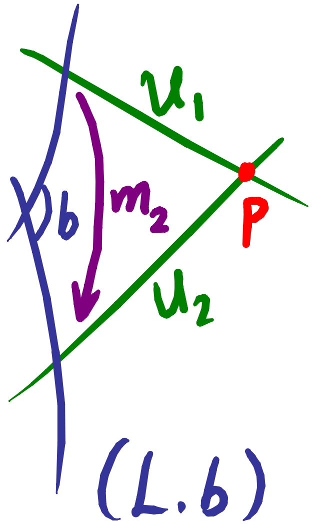
$$\begin{array}{ccc} \mathrm{Fuk}_0(M) & \longrightarrow & M.F.(W) \\ \rotatebox{90}{$\in$} & & \\ \mathcal{U} & \longmapsto & m_1^{((L,b),\mathcal{U})} \triangleq \check{\mathcal{U}} : \mathrm{Span}(L \cap \mathcal{U}) \circlearrowleft \end{array}$$

singularity theory of W



A_∞-relations $\Rightarrow$ $\check{\mathcal{U}}^2 = W(b)$.





Higher terms of the functor come from $m_{k>1}$.

Homological mirror symmetry

Theorem (Cho-Hong-L.): Let $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leqslant 1$.

The A^∞ functor

mirror map is explicitly involved

$$\mathrm{Fuk}(\mathbb{P}^1_{(a,b,c)}) \longrightarrow MF(W)$$

induces an equivalence of triangulated categories

$$D\mathrm{Fuk}(\mathbb{P}^1_{(a,b,c)}) \xrightarrow{\sim} DMF(W).$$

Conclusion

- Choose a weakly unobstructed (immersed) Lagrangian $L \subset X$.
- $\left(\check{X} = \{ b : m_0^{(L,b)} \propto \mathbb{1} \}, \; W \triangleq m_0^{(L,b)} / \mathbb{1} \right)$

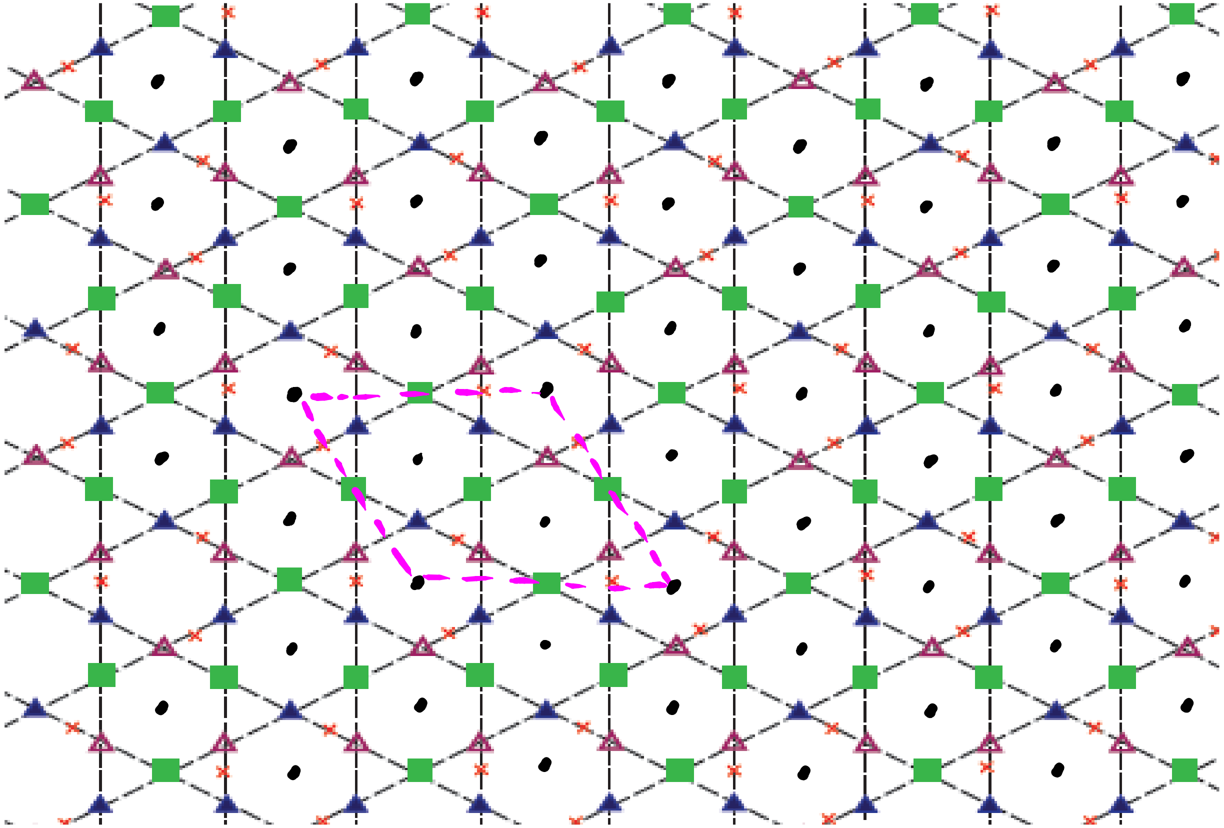
  is the mirror localized to L.
- $\{ m_{>0} \}$ gives A_∞ functor $\mathrm{Fuk}(X) \longrightarrow \mathrm{MF}(W)$.
- Can produce explicit matrix factorizations for toric mirror W.

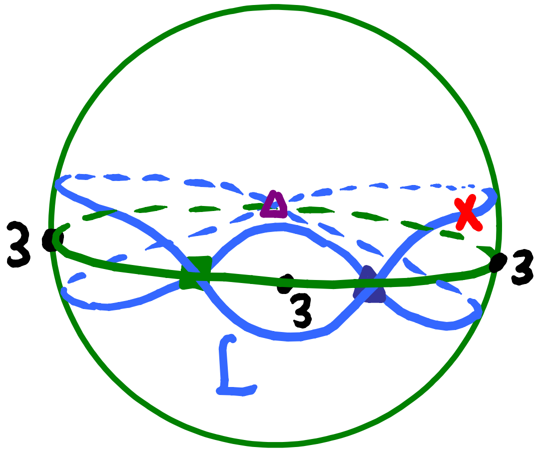
Example:
Elliptic curve quotient

elliptic curve with complex multiplication $e^{2\pi i/3}$

E with flat metric of total area t.

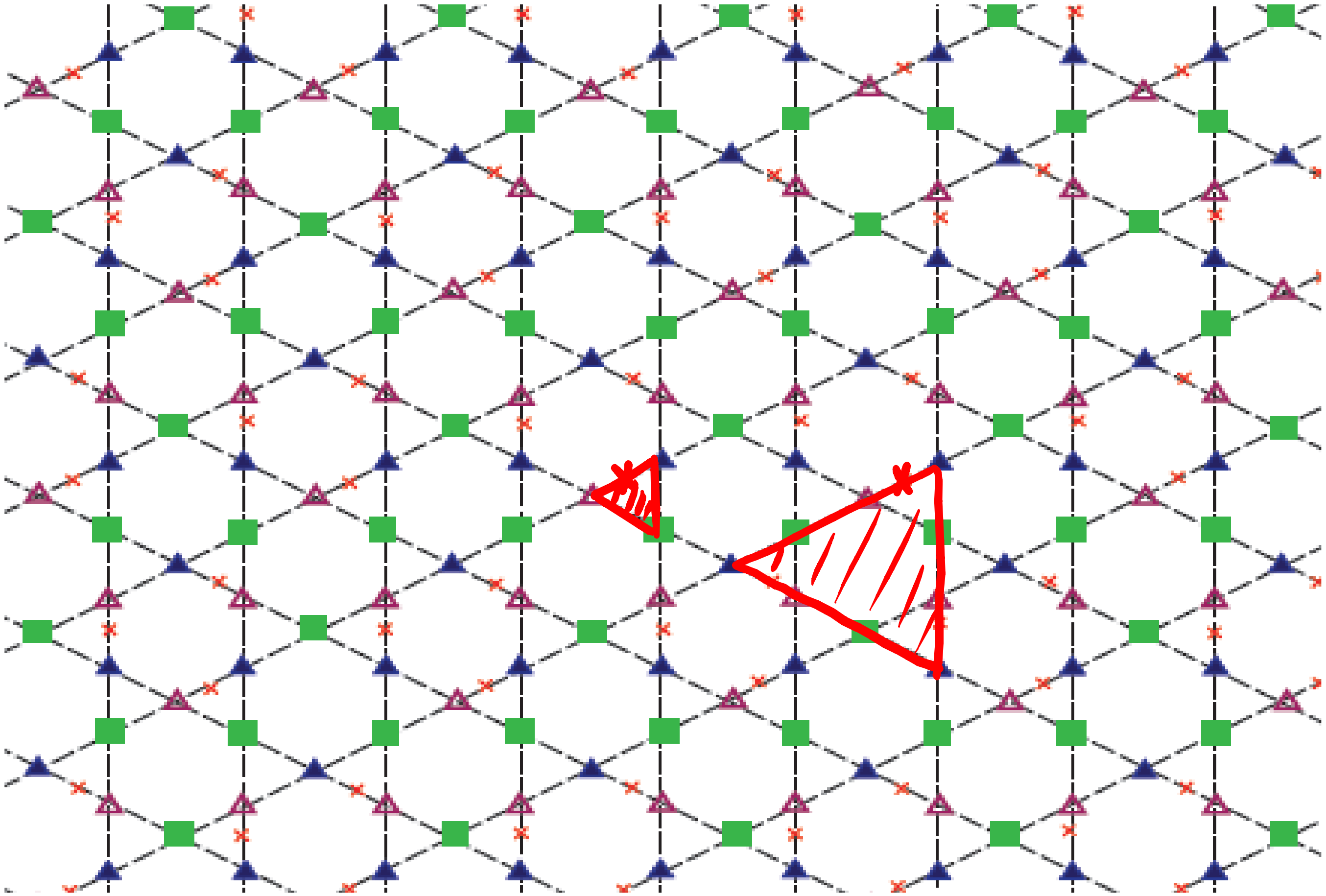
$$E \longrightarrow E/\mathbb{Z}_3 = \mathbb{P}^1_{3,3,3}$$



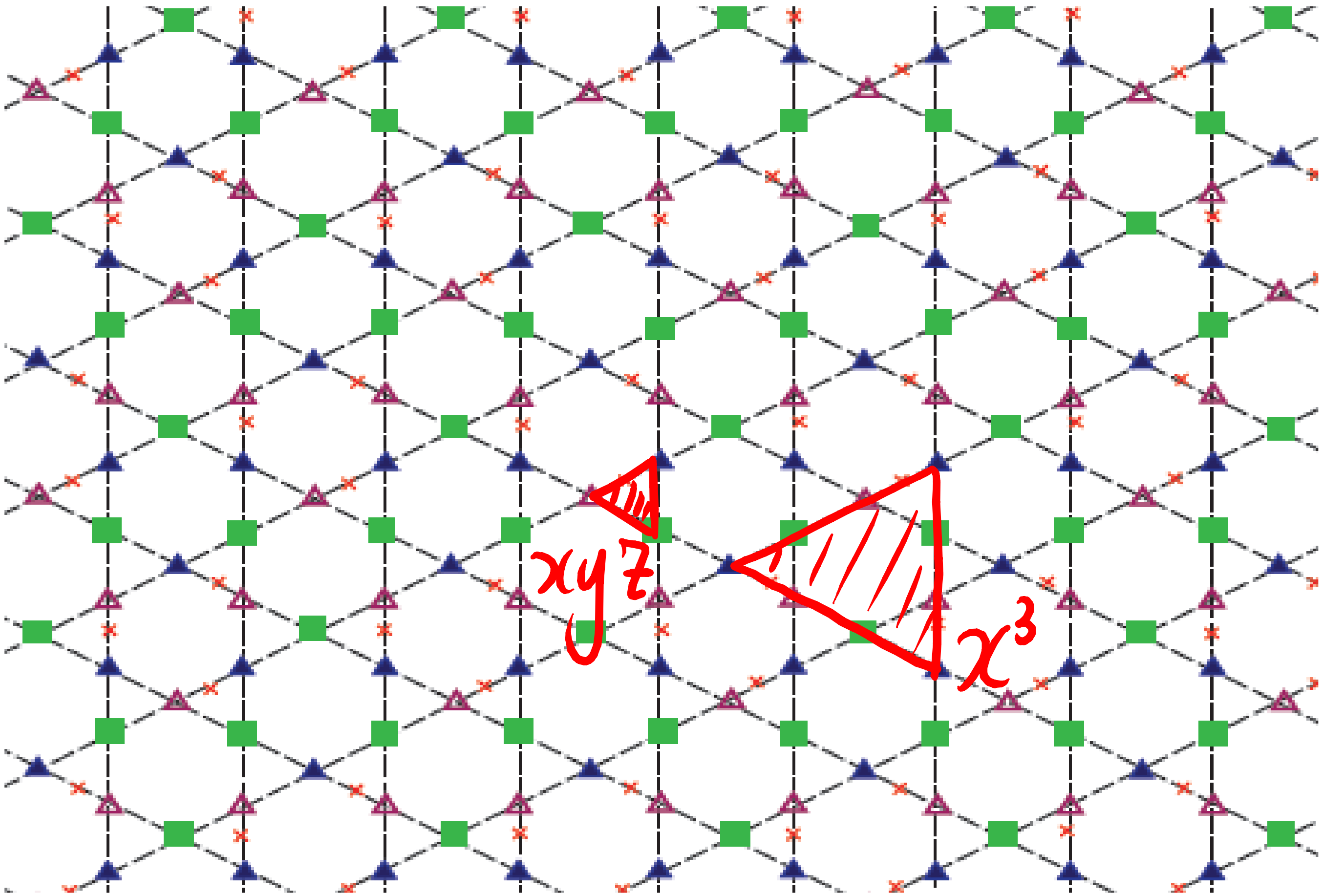


- Fix a generic point p in L (marked as × below)
- Count the number of polygons bounded by L passing thru p with vertices being ▲, △, ■.

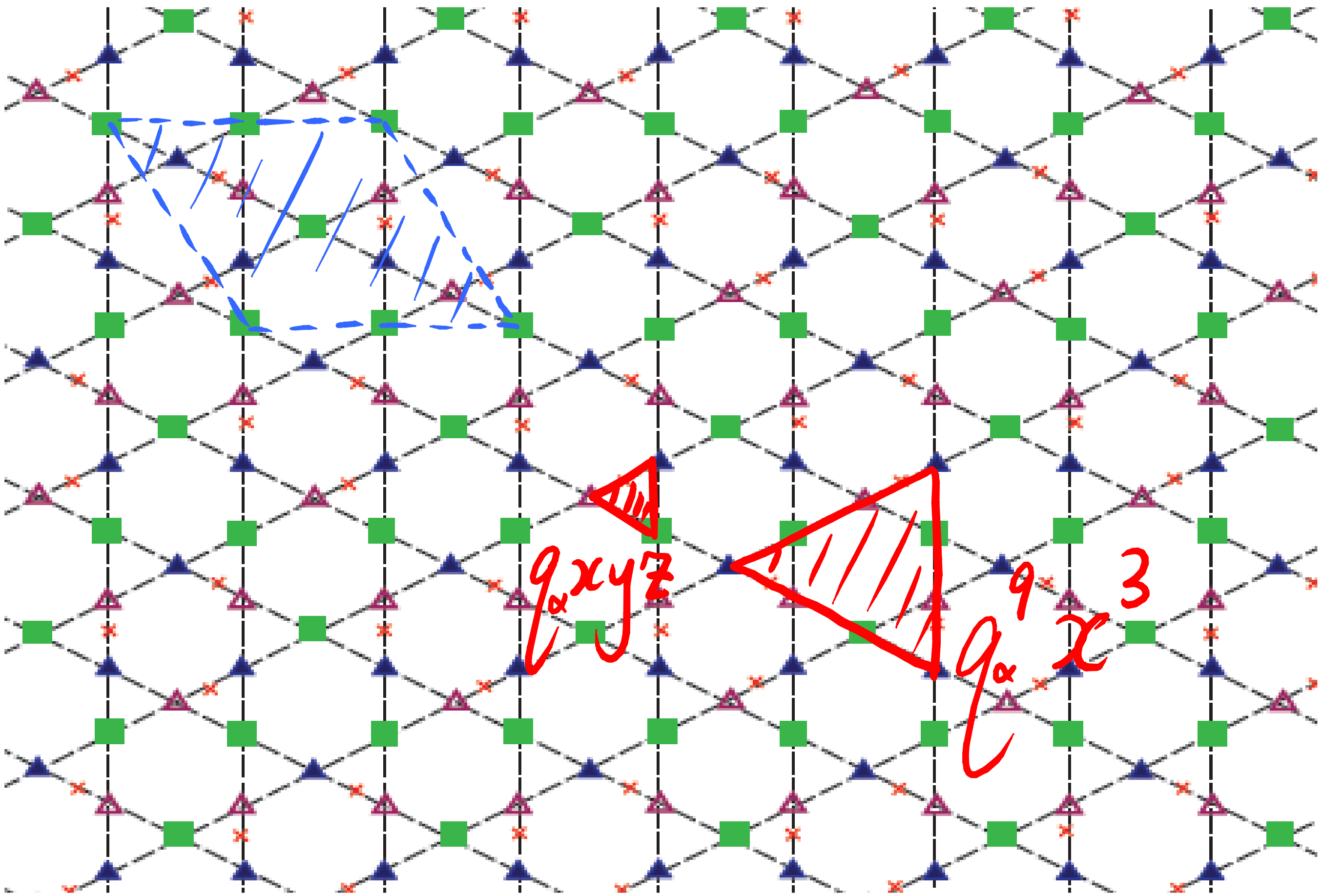
  ▲ = x, △ = y, ■ = z



- Label the vertices by ▲ x, △ y, ■ z.
- Record the labels of vertices of each triangle by a monomial.



- $q_\alpha \triangleq e^{-t/24}$. ($\alpha$ stands for the minimal triangle)
- Record $\exp(-\text{area of each triangle})$ in terms of q_α.



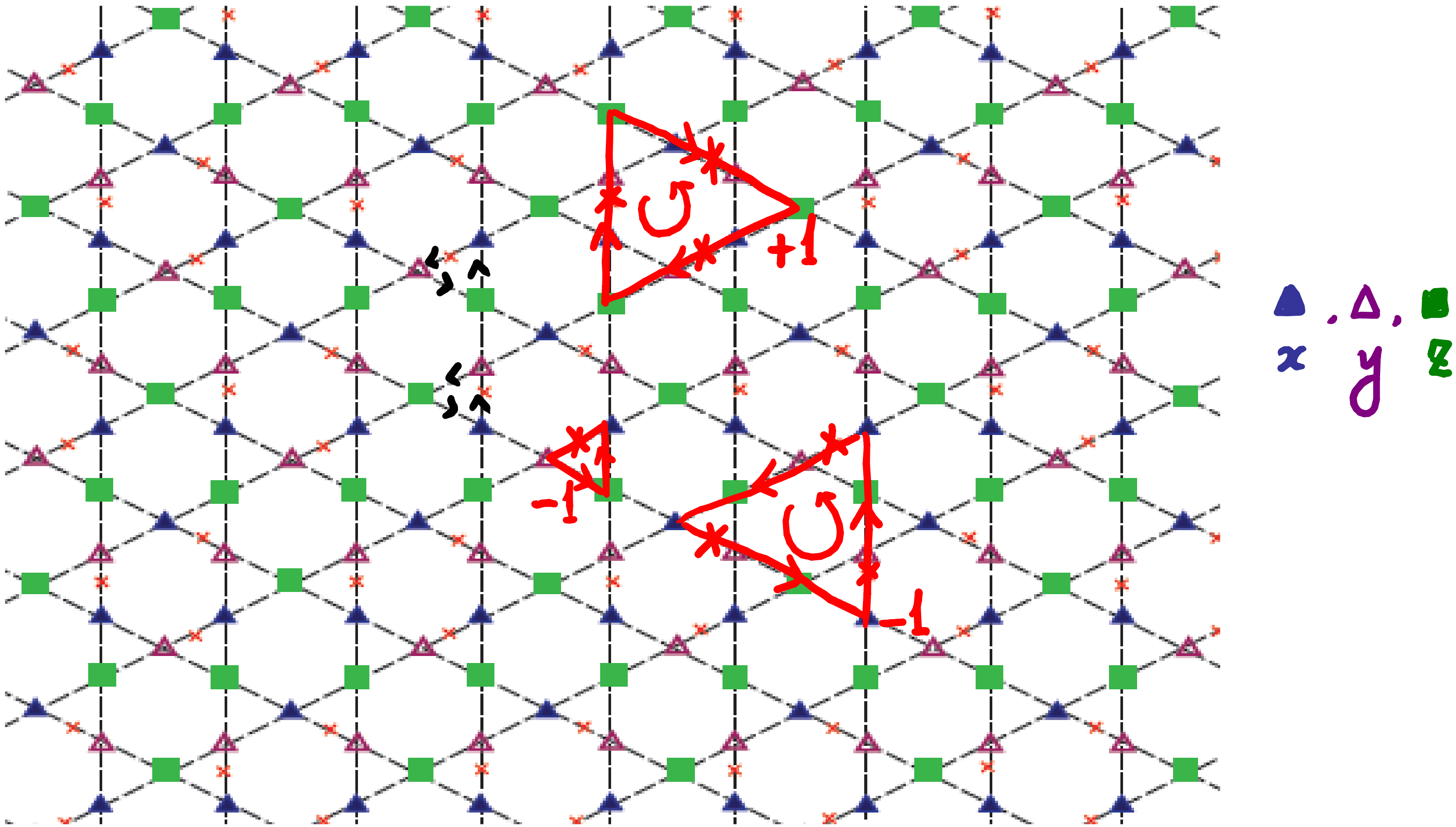
▲, △, ■

x y z

Fix an orientation of L (which is $\mathbb{Z}_3$-invariant).

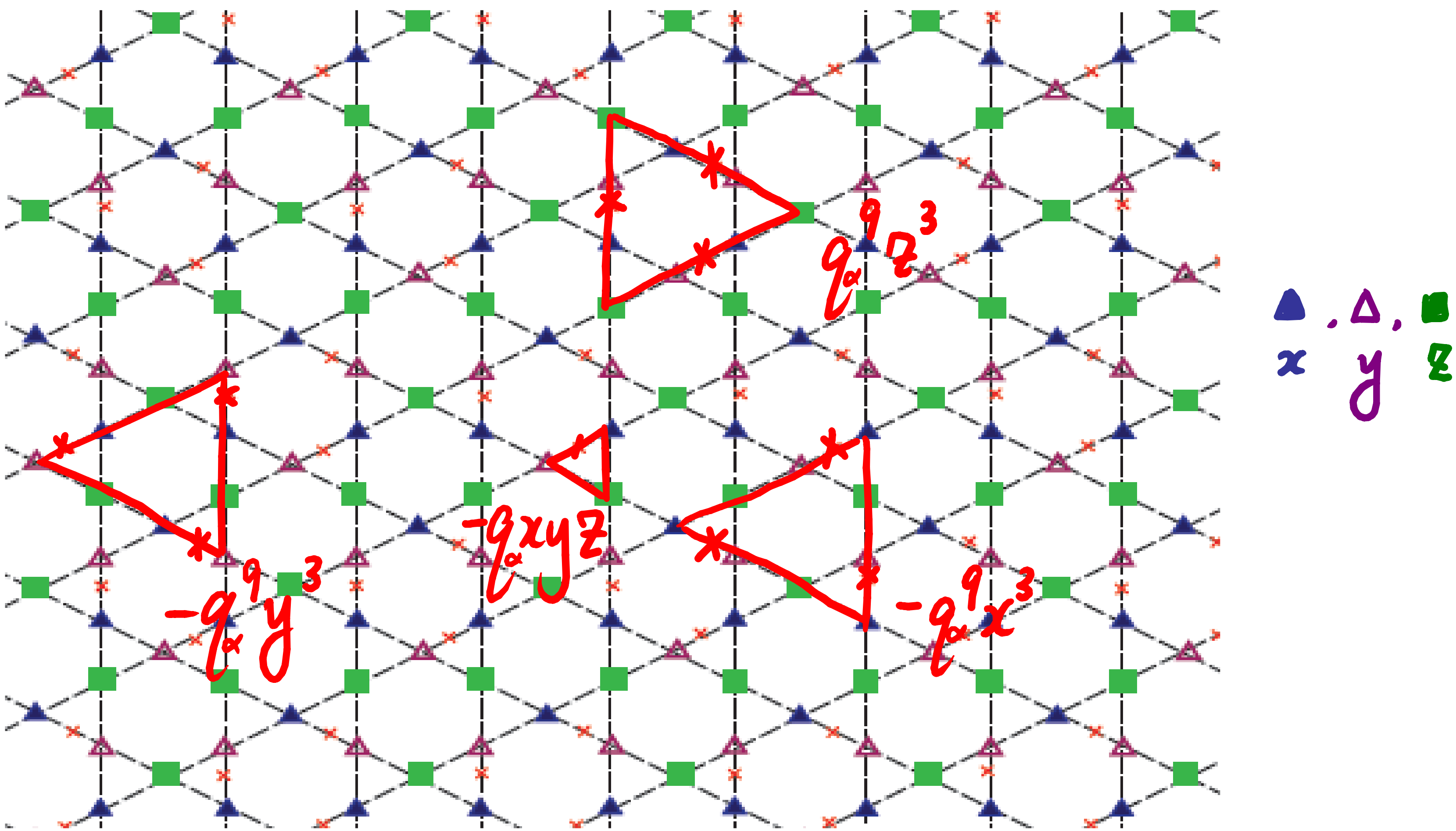
Sign of each triangle: $(-1)^{\#\text{reversed edges}}(-1)^{\#\text{marked points on boundary}}$.

(due to spin structure by Seidel)



$$W \triangleq \sum_{\beta} n_{\beta} q^{\beta} z^{\partial\beta}$$

$$= -q_{\alpha} xyz - q_{\alpha}^{9} x^{3} - q_{\alpha}^{9} y^{3} + q_{\alpha}^{9} z^{3} + \ldots\ldots$$



$$W = -q_\alpha xyz - q_\alpha^9 x^3 - q_\alpha^9 y^3 + q_\alpha^9 z^3 + \ldots\ldots$$

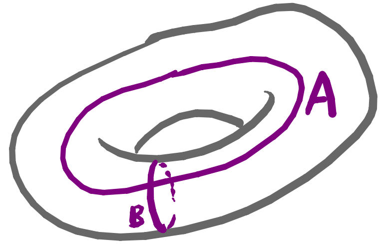
$$= \phi \cdot (-x^3 - y^3 + z^3) - \psi \cdot xyz$$

where $$\phi = \sum_{k=0}^{\infty} (-1)^{3k+1}(2k+1) q_\alpha^{3(12k^2+12k+3)}$$

$$\psi = -q_\alpha + \sum_{k=1}^{\infty} \left((-1)^{3k+1}(6k+1) q_\alpha^{(6k+1)^2} + (-1)^{3k}(6k-1) q_\alpha^{(6k-1)^2} \right).$$

$$\underset{\substack{\text{change of}\\ \text{coordinates in } (x,y,z)}}{\sim} \quad x^3 + y^3 + z^3 \underbrace{-\frac{\psi}{\phi}}_{\text{SYZ map}} xyz.$$

Consider $\{x^3+y^3+z^3+\sigma xyz=0\} \subset \mathbb{P}^2$.



Family of elliptic curves mirror to E.

Flat coordinate is $q(\sigma) = \exp\left(-\dfrac{\pi_B(\sigma)}{\pi_A(\sigma)}\right)$, where

$\underbrace{\pi_A, \pi_B}_{\text{periods}}$ satisfies Picard-Fuchs equation $u'' + \dfrac{3\sigma^2}{\sigma^3+27}u' + \dfrac{\sigma}{\sigma^3+27}u = 0$.

$q(\sigma)$ is called the mirror map.

SYZ map equals to the mirror map

Theorem (Cho-Hong-L.):

$$-\frac{\psi(q_\alpha)}{\phi(q_\alpha)} = \sigma(q = q_\alpha^{8})$$

where $\sigma(q)$ is the inverse mirror map.

$\Rightarrow$ Coefficients of $\sigma(q)$ are integers.

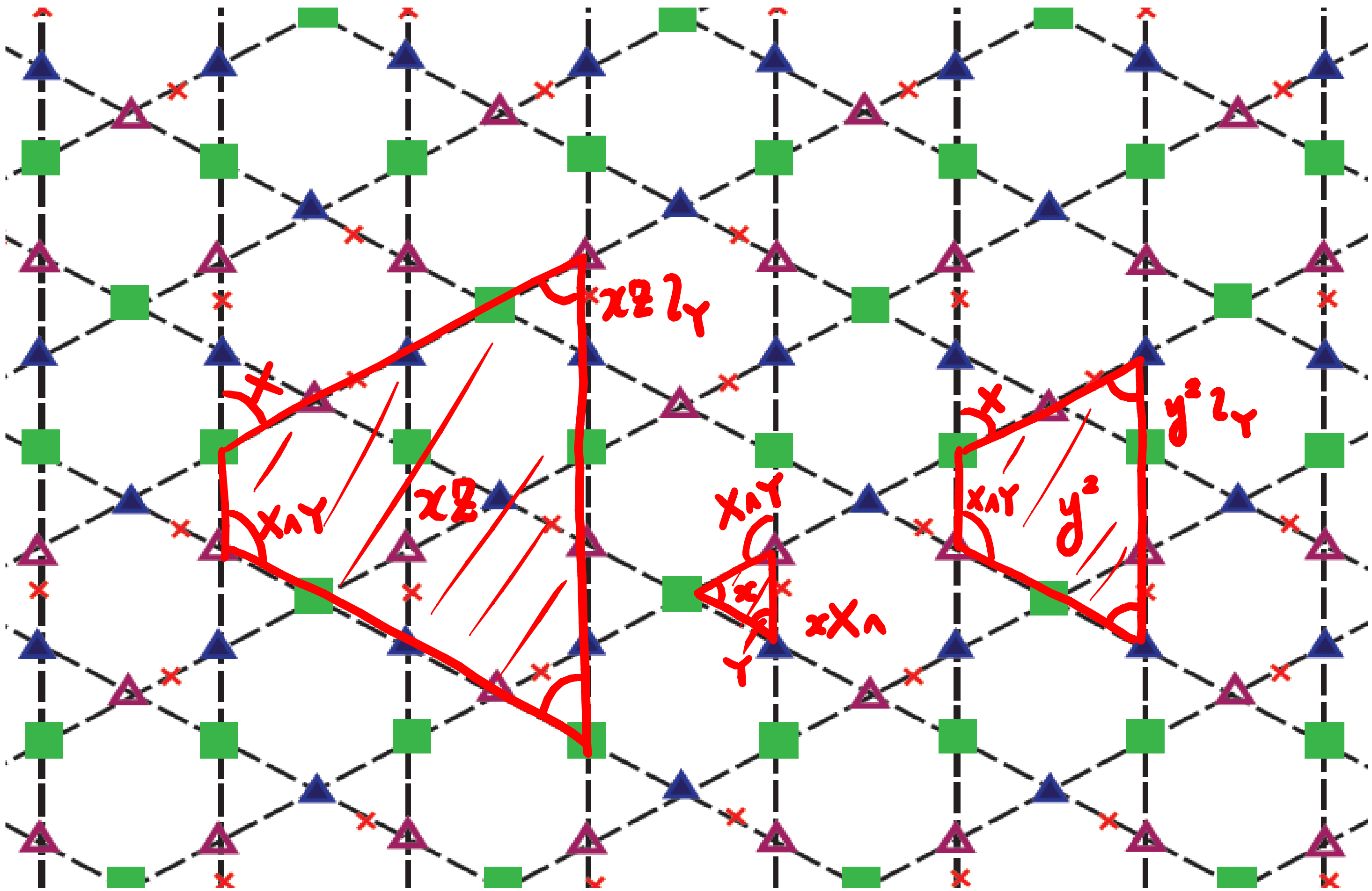
Transform of Seidel Lagrangian L

$$L \xmapsto{\text{Cho-Hong-L.}} \left(\sum_{i=1}^{3} x_i X_i + \sum_{i=1}^{3} \left(x_i^2 + x_{i-1} x_{i+1}\right) \iota_{X_i} \right) \curvearrowright \Lambda^{\bullet} \mathbb{C}^3 .$$

split generates DFuk(X) [AF000]

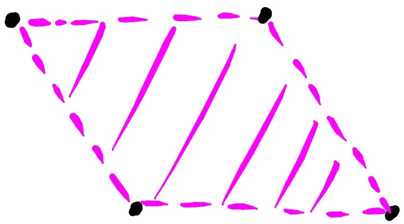
(need non-trivial change of coordinates)

split generates DMF(W) [Dyckerhoff]



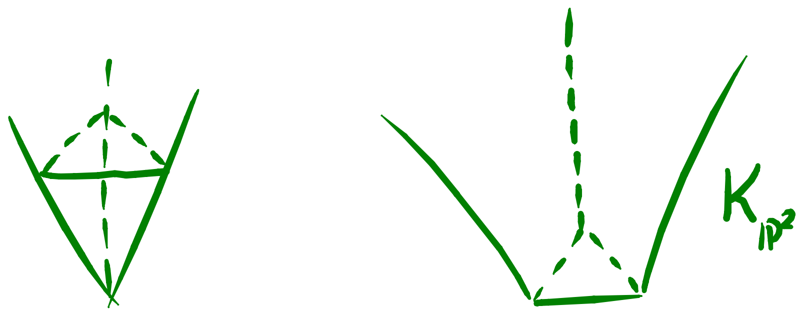
Orbifolding

E



$$\downarrow$$

$$E/\mathbb{Z}_3 = \mathbb{P}^1_{3,3,3}$$



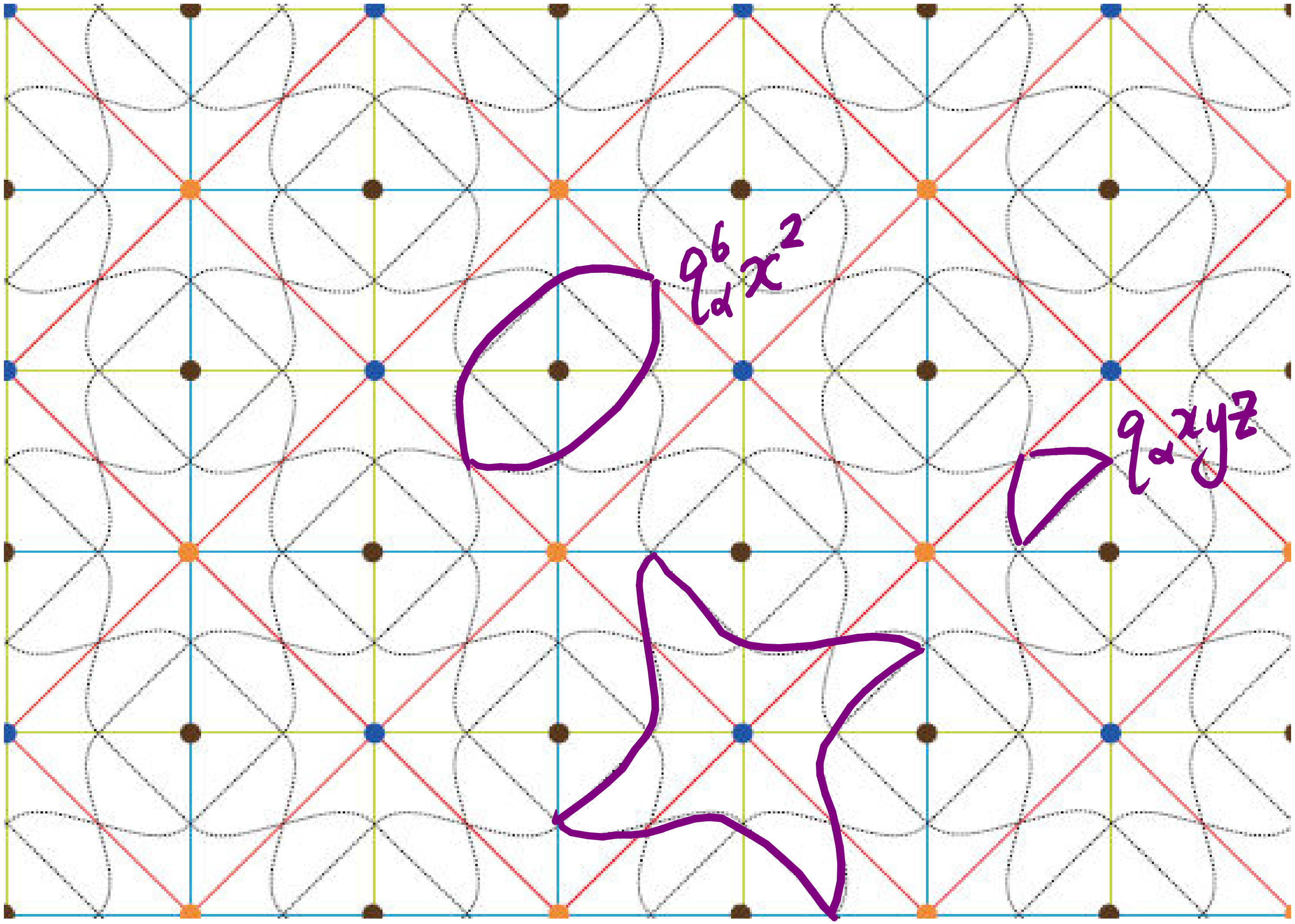
$$(\mathbb{C}^3/\mathbb{Z}_3, W) \simeq \{W=0\} \subset \mathbb{P}^2.$$

$$\Big\| \quad \check{E}$$

$$\uparrow$$

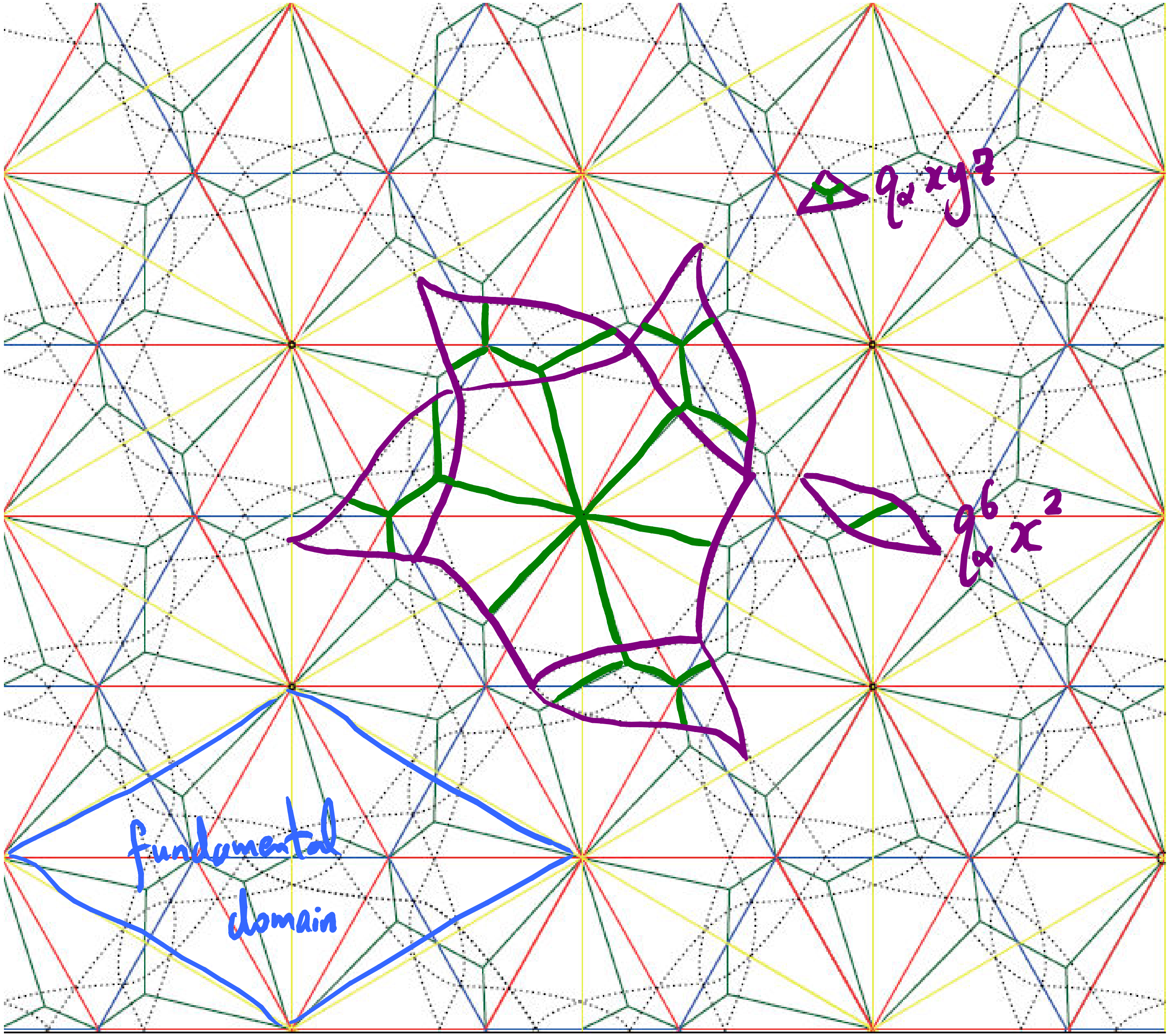
$$\left(\mathbb{C}^3, W = \phi\cdot(x^3+y^3+z^3) - \psi\cdot xyz\right)$$

$$E \longrightarrow E/\mathbb{Z}_4$$



$$W = -q_\alpha xyz + q_\alpha^6 x^2 + \phi\cdot(y^4+z^4) + \psi\cdot y^2z^2.$$

$$E \downarrow E/\mathbb{Z}_6$$



$$W = q_\alpha xyz + q_\alpha^6 x^2 + p_1 z^6 + p_2 y^3 + p_3 y^2 z^2 + p_4 y z^4$$

where $p_1, \dots, p_4$ are explicit series in q_α.

- The construction works for general $\mathbb{P}^1_{(a,b,c)}$ $\forall a,b,c \geq 1$:

$$W = -q_\alpha xyz + (-1)^a q_\alpha^{3a} x^a + (-1)^b q_\beta^{3b} y^b + (-1)^c q_\gamma^{3c} z^c + \dots\dots$$

which is a formal power series.

- [Cho-Hong-Kim-L.]:

Construct an algorithm to compute W $\forall a,b,c$.

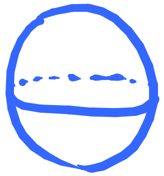
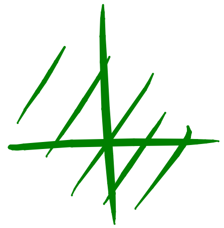
- When $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$,

$\mathbb{P}^1_{(a,b,c)} = \left(\Sigma_{g>1}\right)/G$. $W(x,y,z,q)$ is an infinite series.

Thm: W is convergent in a neighborhood of $(x,y,z,q)=0$.

SYZ map is convergent even for general-type case!

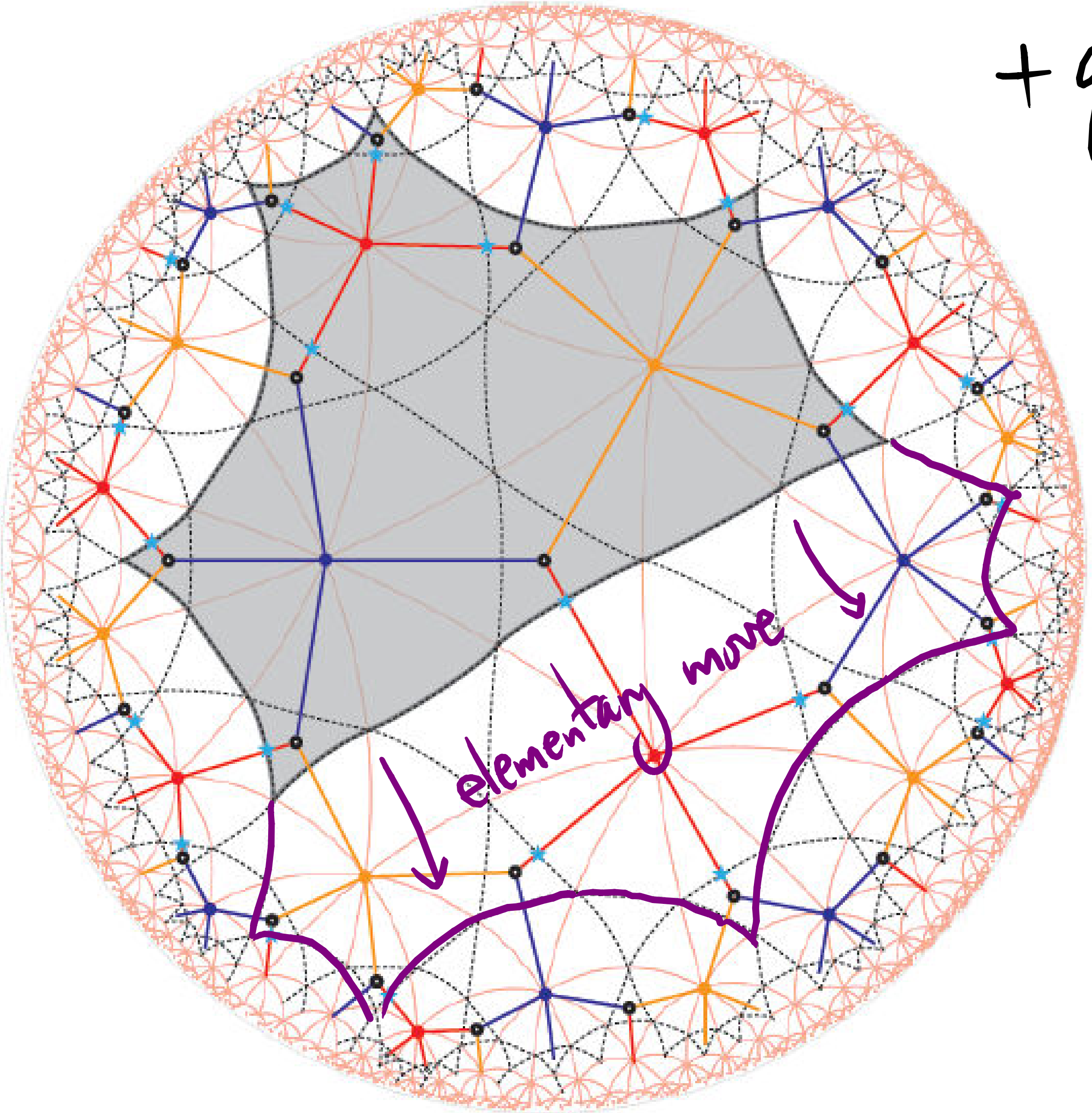
$\mathbb{P}^1_{(a,b,c)}$ is quotient of space-forms

Spherical 			Planar 			hyperbolic 
A	D	E	$\tilde{E}_6$	$\tilde{E}_7$	$\tilde{E}_8$	$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1.$
$a=1$	$(2,2,k)$	$(2,3,k)$ $k=3,4,5$	$(3,3,3)$	$(2,4,4)$	$(2,3,6)$	
W is a polynomial			W is a polynomial with series coefficients			W is a convergent series

We compute W in all cases.

A hyperbolic example

$$W = -q_\alpha xyz + q_\alpha^{12}(x^4 + y^4 + z^4) + q_\alpha^{34} x^2 y z^2 + \ldots\ldots$$



$$\mathbb{D} \downarrow \mathbb{P}^1_{(4,4,4)}$$

Modularity of open Gromov-Witten invariants

[L.-Zhou] (Similar for matrix factorizations mirror to L)

$$\eta = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

$E/\mathbb{Z}_3$: $W = \phi\cdot(x^3+y^3+z^3) - \psi\cdot xyz$ where

$$\phi = \eta(q^3)^3,\ \psi = \eta(q^{1/3})^3 + 3\eta(q^3)^3$$

are $\Gamma(3)$-modular forms.

$$E_2 = 1 - 24\sum_{d=1}^{\infty}\underbrace{\sigma_1(d)}_{\text{sum of divisors of } d} q^d$$

$E/\mathbb{Z}_4$: $W = \phi\cdot(x^4+y^4) + \psi\cdot y^2z^2$ where

$$\phi = -\frac{E_2(q^{1/2})}{24} + \frac{E_2(q)}{8} - \frac{1}{12}E_2(q^2)$$

$$\psi = \frac{1}{4} + \frac{E_2(q)}{4} - \frac{E_2(q^2)}{2}$$ are $\Gamma(4)$-modular forms.

Global mirror symmetry

$$\mathrm{Fuk}(X) \xrightarrow[\text{functor}]{\text{mirror}} MF^{\text{graded}}(W)$$

$$\begin{array}{ccc} \mathcal{M}^{\text{Käh.}}_{\text{global}} & \xrightarrow[\text{`SYZ map'}]{} & \mathcal{M}^{\text{cpx.}}_{\text{global}} \\ \cong & & \cong \\ \mathcal{H}/\Gamma(N) & & \mathcal{H}/\Gamma(N) \end{array}$$