Moduli theory of Lagrangian immersions and mirror symmetry

Siu-Cheong Lau
Boston University

December 2017

Joint work with Cheol-Hyun Cho and Hansol Hong
Section 1

Overview
Moduli theory in the B-side

- Moduli theory for vector bundles has been developed into a deep theory.

Theorem (Donaldson, Uhlenbeck-Yau)

A slope-semistable holomorphic vector bundle admits a Hermitian Yang-Mills metric.

- GIT and stability conditions were essential to the construction.
- Bridgeland developed a general mathematical theory of stability conditions for triangulated categories.
- Toda developed foundational techniques to construct Bridgeland stability conditions for derived categories of coherent sheaves.
- Moduli spaces undergo birational changes (such as flops) in a variation of stability conditions.
- How about moduli of Lagrangians in the mirror A-side?
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Ingredients for moduli theory of Lagrangians

- **Complexification.** The classical moduli spaces are affine manifolds with singularities [Hitchin, McLean]. Complexification is needed in order to compactify. Technically we need to work over the Novikov ring.

- **Quantum correction.** The canonical complex structures need to be corrected using Lagrangian Floer theory [Fukaya-Oh-Ohta-Ono]. The combinatorial structure of quantum corrections for SYZ fibrations was deeply studied by Kontsevich-Soibelman and Gross-Siebert.

- **Landau-Ginzburg model.** The moduli in general are singular varieties. They are described as critical loci of holomorphic functions. It can also be noncommutative in general [Cho-Hong-L.].

- **Singular Lagrangians.** Kontsevich proposed to study them using cosheaves of categories. Nadler is developing a theory of arboreal singularities.
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Why care about Lagrangian immersions

- Lagrangian immersions have a well-defined Floer theory by Akaho-Joyce.
- They are the main sources of wall-crossing phenomenon in the SYZ setting.
- The deformation space of a Lagrangian immersion is ‘bigger’ than its smoothing and covers a local family of Lagrangians (including singular Lagrangians).
- It is hoped that every object in the Fukaya category can be represented by a Lagrangian immersion. If so we do not need to worry about singular Lagrangians.
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SYZ and Family Floer theory

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- It leads to many exciting developments. Gross-Siebert, Leung, Auroux, Chan . . .
- Fukaya proposed to study mirror symmetry by using $\text{CF}(L_b, \cdot)$ for fibers $L_b$ of a Lagrangian torus fibration.
- Tu took this approach to construct mirror spaces away from singular fibers.
- Abouzaid constructed family Floer functors for torus bundles and showed that the functor is fully faithful.
- We construct mirror geometry as the moduli space of stable Lagrangian immersions which are not necessarily tori. This generalizes the SYZ setting.
- Instead of Fukaya trick, we consider pseudo-isomorphisms between Lagrangian immersions, and obtain the gluing maps from cocycle conditions. (In particular we do not need diffeomorphisms.)
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The quantum-corrected moduli

- The moduli space is given as a superpotential $W$ defined on $\{\text{Stable formally deformed immersed Lagrangians in a fixed phase}\} / \{\text{Isomorphisms in the Fukaya category}\}$ (in place of the set of special Lagrangians).

- Formal deformations are given by flat $\mathbb{C}^\times$-connections or smoothings of immersed points. They are required to be weakly unobstructed so that $W$ is well-defined.

- Isomorphisms between objects in the Fukaya category are defined by counting holomorphic strips.

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Example: the two-sphere

- Consider $\mathbb{P}^1$ equipped with the meromorphic top-form $dz/z$. The $S^1$-moment map gives a special Lagrangian fibration.

- Fibers are stable. We can also perturb it by Hamiltonian, which is still stable.

- $\{(\text{fibers},\text{flat } U(1)\text{-connections } \nabla^t)\} = (0, 1) \times S^1$ as sets.

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- Enlarge the local deformation spaces:
  \[ S^1 = \{ \text{flat } U(1)\text{-conn.} \} \subset \Lambda_*^0 \subset \Lambda^* \]
  \[ \Lambda = \{ \sum_{i=0}^{\infty} a_i T^A : A_0 \leq A_1 \leq \ldots \} \]

- ‘Pseudo-deformations’: flat \( \Lambda^* \)-connections. ‘Pseudo’ because they are invalid in the Fukaya category.

- The two intersection points \((\alpha, \beta)\) between a Hamiltonian-perturbed fiber with a neighboring fiber provide a ‘pseudo-isomorphism’.

- \( m_1(\alpha) = 0 \) gives \( t = T^A t' \) where \( A \) is the area of the cylinder bounded by the two fibers. \( t \) is forced to be \( \Lambda^* \)-valued.

- \( \{ (\text{stable Lagrangians in fiber class, flat } \Lambda^*_0\text{-conn.}) \}/\text{Isom.} \) equals to \( \Lambda^*_{0<\text{val}<1} \) as rigid analytic spaces.
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Section 2

Pair-of-pants decompositions
Homological mirror symmetry was proved for punctured Riemann surface by Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Bocklandt, and Heather Lee.

Our focus is on the construction of mirror as the moduli space rather than HMS. The construction comes with a natural functor which derives HMS.

Consider a pair-of-pants decomposition of a punctured Riemann surface. We have a class of Lagrangian immersions as shown in the figure.

$\dim = 1$ contains the essential ingredients. In higher dimensions, Seidel’s Lagrangians are replaced by Sheridan’s Lagrangians.
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dim $= 1$ contains the essential ingredients. In higher dimensions, Seidel’s Lagrangians are replaced by Sheridan’s Lagrangians.
First consider a pair-of-pants, with Seidel’s Lagrangian immersion $L$.

- $\text{CF}(L, L) = \text{Span}\{1, X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}, \text{pt}\}$.
- We proved that $xX + yY + zZ$ is weakly unobstructed.
- The local moduli is $(\mathbb{C}^3, W)$, where $W = xyz$.
- The pair-of-pants can be compactified to $\mathbb{P}^1_{a,b,c}$. We used this to construct the mirror, and derived homological mirror symmetry. We also constructed noncommutative deformations. In an ongoing work with Amorim we prove closed-string mirror symmetry. For elliptic orbifolds the coefficients of $W$ are modular forms [L.-Zhou].
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Consider the four-punctured sphere as shown above.

We need to glue the deformation spaces of the two Seidel Lagrangians $S_1$ and $S_2$.

There are two main processes: smoothing and gauge change.

Let’s pretend to work over $\mathbb{C}$ at this stage. The gluing we need is

$$(\mathbb{C}^3, W) \xrightarrow{\text{smoothing}} (\mathbb{C} \times \mathbb{C}^2) \xrightarrow{\text{gauge change}} (\mathbb{C} \times \mathbb{C}^2) \xrightarrow{\text{smoothing}} \mathbb{C}^3.$$
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Choice of gauge change

- There is a vanishing sphere in a smoothing at an immersed point of $S$. In this case it is simply the union of two points.
- Put a flat $\mathbb{C}^\times$ connection on the smoothing $C$, which is acting by $t \in \mathbb{C}^\times$ when passing through the two points.
- The position of the two points are different for smoothings on the left and on the right.
- When a gauge point $T$ is moved across the immersed point $Y$, the $A_\infty$ algebras are related by $\tilde{y} = ty$.
- There are different ways of moving the gauge points to match them. This results in $\tilde{y} = t^a y$, $\tilde{z} = t^b z$ with $a + b = 2$. 
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When a gauge point $T$ is moved across the immersed point $Y$, the $A_\infty$ algebras are related by $\tilde{y} = ty$.

There are different ways of moving the gauge points to match them. This results in $\tilde{y} = t^ay$, $\tilde{z} = t^bz$ with $a + b = 2$. 
Choice of gauge change

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Smoothing

- We need to glue the deformation spaces of $S$ and $C$.
- They are given by $(x, y, z) \in \mathbb{C}^3$ and $(t, y_0, z_0) \in \mathbb{C}^\times \times \mathbb{C}^2$ respectively.
- Intuitively to match the superpotentials $xyz$ and $ty_0z_0$, we simply put $x = t$, $y = y_0$, $z = z_0$.
- $S$ and $C$ are disjoint to each other!
- We take $S_1$ to be the deformed Seidel Lagrangian which intersects $C$ as in the figure.
- Then we use cocycle conditions to deduce the gluing between $S$ and $C$. 
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Isomorphisms in smoothing

- We consider $a_1 + b_1 \in \text{CF}((C, \nabla_t), (S, xX))$ and $c_2 + d_2 \in \text{CF}((S, xX), (C, \nabla_t))$.

- Consider cocycle conditions on $a_1 + b_1$ and $c_2 + d_2$. Once the cocycle conditions are satisfied, they give isomorphisms between the two objects.
A paradox

- $S$ is isomorphic to $S_1$, and $(S_1, xX)$ is isomorphic to $(C, \nabla_t)$ for $t = x \neq 0$. Thus $(S, xX)$ is isomorphic to $C$.
- But $S$ and $C$ are disjoint, and hence there is no morphism between them!
- We need to take a closer look at areas of holomorphic discs.
- Following Fukaya-Oh-Ohta-Ono, we shall use the Novikov ring $\Lambda_0 = \{\sum_{i=0}^{\infty} a_i T^{A_i} : 0 \leq A_0 \leq A_1 \leq \ldots\}$ to filter deformations into different energy levels.
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Area constraints for isomorphisms

For the cocycle conditions on $(S_1, x_1 X_1) \to (C, \nabla_t)$, indeed we have $t = T^{A_1 + \ldots + A_5 - A_7} x_1$.

Note that $t \in \mathbb{C}^\times$ while $x_1 \in \Lambda_0$. When $A_1 + \ldots + A_5 > A_7$, the condition is never satisfied.

Hence $(S_1, x_1 X_1)$ can only be isomorphic to $(C, \nabla_t)$ if $A_1 + \ldots + A_5 \leq A_7$.

When $A_1 + \ldots + A_5 = A_7$, the change of coordinates $x_1 = t$ does not involve Novikov parameters.

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The two regions $\text{val}(x_1) = 0$ and $\text{val}(x_1) \geq A > 0$ are disjoint. This solves the paradox.
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The gluing $x = t$, $y = y_0$, $z = z_0$ looks pretty trivial. Let’s compactify to get more interesting gluing.

As an example, compactify the four-punctured sphere to a sphere. We still consider the moduli of double-circles.

The gluing is further quantum-corrected by discs emanated from infinite divisors.

The new gluing is $t = x$, $y_0 = y_1 + t_1^{-1}$, $z_0 = z_1 - t_1^{-1}$.

$W = xyz - y_1 + x_1 + z_1$. (The Novikov parameter is suppressed for simplicity.)

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A compactification

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Let’s go back to the four-punctured sphere. We can take another pair-of-pants decomposition.

It corresponds to another choice of a quadratic differential.

This results in a flop of the moduli space $(\mathcal{O}(-1) \oplus \mathcal{O}(-1), W)$.

We can also consider a quadratic differential with double zeros. Then the moduli is the non-commutative resolution of the conifold corresponding to a quiver ([Cho-Hong-L. 15]).

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Section 3

Wall-crossing in SYZ
Immersed sphere

The two-dimensional immersed sphere leads to wall-crossing phenomenons for SYZ Lagrangian fibrations.

**Fukaya** studied this immersion and the relation with the mirror equation. Here we realize it from cocycle conditions.

- It has two degree-one immersed generators $U$ and $V$. It gives the deformations $b = uU + vV$ where $(u, v) \in (\Lambda_0 \times \Lambda_+ ) \cup (\Lambda_+ \times \Lambda_0)$.
- There are constant holomorphic discs with corners $U, V, \ldots, U, V$. To ensure there are only finitely many terms under every energy level, we only allow one of $u, v$ having valuation zero.
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- The constant discs with corners $U, V, \ldots, U, V$ contributing to $\text{pt}$ of $m_b^0$ cancel with that with corners $V, U, \ldots, V, U$. Thus $b = uU + vV$ is weakly unobstructed.
Consider the immersed sphere $S$ and a Chekanov torus $T$. $S$ is made by deforming the immersed fiber.

We glue the deformations $(u, v) \in \Lambda_0 \times \Lambda_+$ with the deformations $(x, y) \in (\mathbb{C}^\times \oplus \Lambda_+)^2$ of $T$. 
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The cocycle conditions give \(u = y\) and \(x = uv - 1\). Note that the second equation is on the region \(x = -1 + \Lambda_+\).

In other words, \(\text{CF}(T, T)\) gives an extension of \(\text{CF}(S, S)\) from \(v \in \Lambda_+\) to \(v \in \Lambda_0\) with \(uv \neq 1\).

Similarly we can glue the immersed sphere \(S\) (with \((u, v) \in \Lambda_+ \times \Lambda_0\)) with a Clifford torus \(T'\). It is \(v = y'\) and \(x' = uv - 1\).

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Grassmannians

- With **Hansol Hong and Yoosik Kim**, we are applying this to construct the compactified mirror of $\text{Gr}(2, n)$.
- Flag manifolds have Gelfand-Cetlin systems serving as Lagrangian torus fibrations.
- Immersed spheres are important in that case because they appear as critical fibers of the superpotential.
- For instance, by **Nohara-Ueda** for $\text{Gr}(2, 4)$ there is a critical point $(0, 0)$ which corresponds to a certain fiber $S^3 \times S^1$ of the Gelfand-Cetlin system.
- The trouble is $S^3$ is rigid!
Consider the symplectic reduction picture on $S^2$. We push in one singular point and consider the corresponding moduli.

- There is one monotone Lagrangian torus above and below the wall respectively.
- The deformation spaces $(\mathbb{C}^\times)^2$ of these two tori provide the cluster charts of the mirror. However there is still one point missing, which is crucial since a critical point of $W$ lies there.
- We glue the deformation spaces of an immersed sphere (times $T^2$) with that of the two monotone tori like in the last slide.
- This recovers the Rietsch Lie theoretical mirror for $Gr(2, 4)$. 