Moduli theory of Lagrangian immersions and mirror symmetry

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Joint work with Cheol-Hyun Cho and Hansol Hong

Section 1

Overview

Moduli theory in the B-side

- Moduli theory for vector bundles has been developed into a deep theory.

Theorem (**Donaldson, Uhlenbeck-Yau**)

A slope-semistable holomorphic vector bundle admits a Hermitian Yang-Mills metric.

- GIT and stability conditions were essential to the construction.
- **Bridgeland** developed a general mathematical theory of stability conditions for triangulated categories.
- **Toda** developed foundational techniques to construct Bridgeland stability conditions for derived categories of coherent sheaves.
- Moduli spaces undergo birational changes (such as flops) in a variation of stability conditions.
- **How about moduli of Lagrangians in the mirror A-side?**

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Ingredients for moduli theory of Lagrangians

- *Complexification.* The classical moduli spaces are affine manifolds with singularities [**Hitchin,McLean**]. Complexification is needed in order to compactify. Technically we need to work over the Novikov ring.
- *Quantum correction.* The canonical complex structures need to be corrected using Lagrangian Floer theory [**Fukaya-Oh-Ohta-Ono**]. The combinatorial structure of quantum corrections for SYZ fibrations was deeply studied by **Kontsevich-Soibelman** and **Gross-Siebert**.
- *Landau-Ginzburg model.* The moduli in general are singular varieties. They are described as critical loci of holomorphic functions. It can also be noncommutative in general [**Cho-Hong-L.**].
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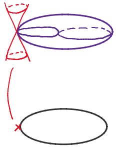

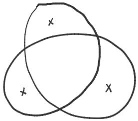

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- They are the main sources of wall-crossing phenomenon in the SYZ setting.
- The deformation space of a Lagrangian immersion is 'bigger' than its smoothing and covers a local family of Lagrangians (including singular Lagrangians).
- It is hoped that every object in the Fukaya category can be represented by a Lagrangian immersion. If so we do not need to worry about singular Lagrangians.

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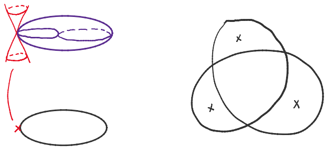

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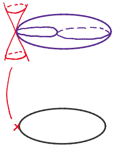

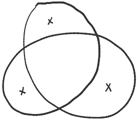

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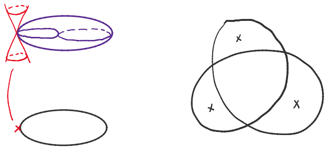

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- **Strominger-Yau-Zaslow** proposed that mirror symmetry can be understood as dual special Lagrangian torus fibrations.
- It leads to many exciting developments. **Gross-Siebert, Leung, Auroux, Chan . . .**
- **Fukaya** proposed to study mirror symmetry by using $\mathrm{CF}(L_b, \cdot)$ for fibers L_b of a Lagrangian torus fibration.
- **Tu** took this approach to construct mirror spaces away from singular fibers.
- **Abouzaid** constructed family Floer functors for torus bundles and showed that the functor is fully faithful.
- We *construct mirror geometry as the moduli space of stable Lagrangian immersions which are not necessarily tori. This generalizes the SYZ setting.*
- Instead of Fukaya trick, *we consider pseudo-isomorphisms between Lagrangian immersions, and obtain the gluing maps from cocycle conditions.* (In particular we do not need diffeomorphisms.)

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The quantum-corrected moduli

- The moduli space is given as a superpotential W defined on {Stable *formally deformed* immersed Lagrangians in a fixed phase} / {Isomorphisms in the Fukaya category} (in place of the set of special Lagrangians).
- Formal deformations are given by flat $\mathbf{C}^{\times}$-connections or smoothings of immersed points. They are required to be *weakly unobstructed* so that W is well-defined.
- Isomorphisms between objects in the Fukaya category are defined by counting holomorphic strips.
- We consider *'pseudo-isomorphisms'*. They provide gluings between the local 'pseudo deformation spaces'.
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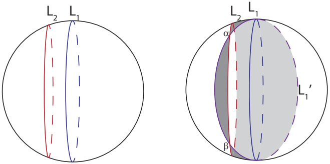
Example: the two-sphere



- Consider $\mathbb{P}^1$ equipped with the meromorphic top-form dz/z. The $\mathbf{S}^1$-moment map gives a special Lagrangian fibration.
- Fibers are stable. We can also perturb it by Hamiltonian, which is still stable.
- $\{(\text{fibers,flat } U(1)\text{-connections } \nabla^t)\} = (0,1) \times \mathbf{S}^1$ as sets.
- Don't have relations between different fibers yet!

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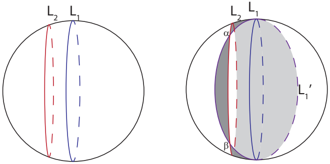

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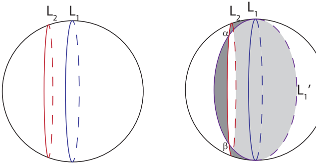

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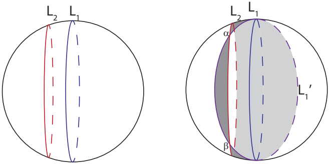

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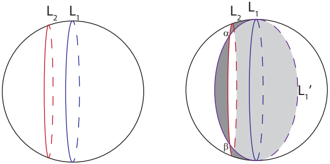

- Enlarge the local deformation spaces: $\mathbf{S}^1 = \{ \text{ flat } U(1)\text{-conn.}\} \subset \Lambda_0^* \subset \Lambda^*$ where $\Lambda = \{\sum_{i=0}^{\infty} a_i T^{A_i} : A_0 \leq A_1 \leq \ldots\}$.
- 'Pseudo-deformations': flat Λ^*-connections. 'Pseudo' because they are invalid in the Fukaya category.
- The two intersection points (α, β) between a Hamiltonian-perturbed fiber with a neighboring fiber provide a 'pseudo-isomorphism'.
- $m_1(\alpha) = 0$ gives $t = \mathbf{T}^A t'$ where A is the area of the cylinder bounded by the two fibers. t is forced to be Λ^*-valued.
- $\{(\text{stable Lagrangians in fiber class, flat } \Lambda_0^*\text{-conn. })\}/\text{Isom.}$ equals to $\Lambda^*_{0<\text{val}<1}$ as rigid analytic spaces.

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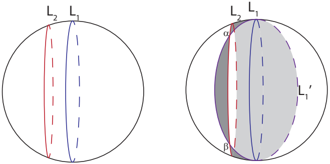

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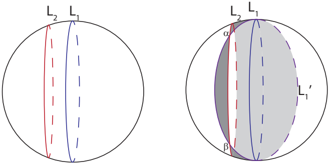

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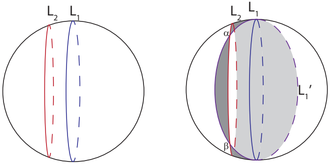

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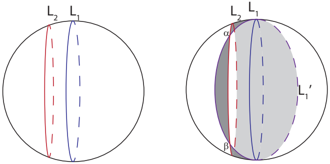

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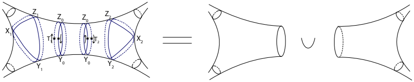
Section 2

Pair-of-pants decompositions

Punctured Riemann surfaces



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- Our focus is on the construction of mirror as the moduli space rather than HMS. The construction comes with a natural functor which derives HMS.
- Consider a pair-of-pants decomposition of a punctured Riemann surface. We have a class of Lagrangian immersions as shown in the figure.
- dim = 1 contains the essential ingredients. In higher dimensions, Seidel's Lagrangians are replaced by Sheridan's Lagrangians.

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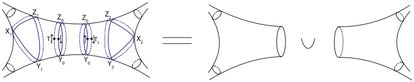

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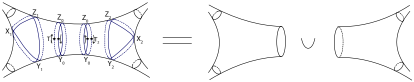

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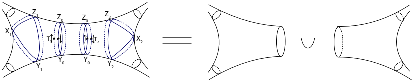

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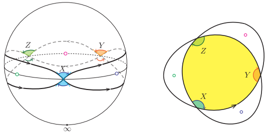
Review on local moduli construction



- First consider a pair-of-pants, with Seidel's Lagrangian immersion L.
- $\mathrm{CF}(L, L) = \mathrm{Span}\{\mathbf{1}, X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}, \mathrm{pt}\}$.
- We proved that $xX + yY + zZ$ is weakly unobstructed.
- The local moduli is $(\mathbf{C}^3, W)$, where $W = xyz$.
- The pair-of-pants can be compactified to $\mathbb{P}^1_{a,b,c}$. We used this to construct the mirror, and derived homological mirror symmetry. We also constructed noncommutative deformations. In an ongoing work with **Amorim** we prove closed-string mirror symmetry. For elliptic orbifolds the coefficients of W are modular forms [**L.-Zhou**].
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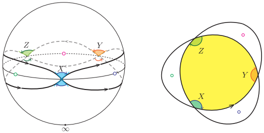

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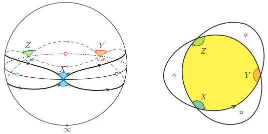

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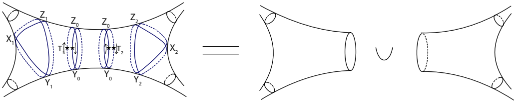
Gluing



- Consider the four-punctured sphere as shown above.
- We need to glue the deformation spaces of the two Seidel Lagrangians S_1 and S_2.
- There are two main processes: smoothing and gauge change.
- Let's pretend to work over $\mathbf{C}$ at this stage. The gluing we need is

$$(\mathbf{C}^3, W) \overset{\text{smoothing}}{\longleftrightarrow} (\mathbf{C}^\times \times \mathbf{C}^2) \overset{\text{gauge change}}{\longleftrightarrow} (\mathbf{C}^\times \times \mathbf{C}^2) \overset{\text{smoothing}}{\longleftrightarrow} \mathbf{C}^3.$$

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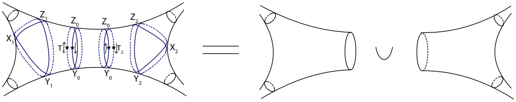

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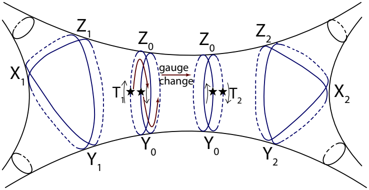
Choice of gauge change



- There is a vanishing sphere in a smoothing at an immersed point of S. In this case it is simply the union of two points.
- Put a flat $\mathbf{C}^\times$ connection on the smoothing C, which is acting by $t \in \mathbf{C}^\times$ when passing through the two points.
- The position of the two points are different for smoothings on the left and on the right.
- When a gauge point T is moved across the immersed point Y, the A_∞ algebras are related by $\tilde{y} = ty$.
- There are different ways of moving the gauge points to match them. This results in $\tilde{y} = t^a y$, $\tilde{z} = t^b z$ with $a + b = 2$.

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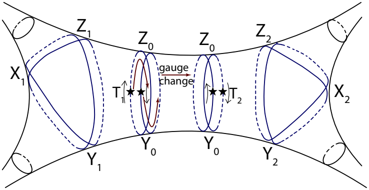

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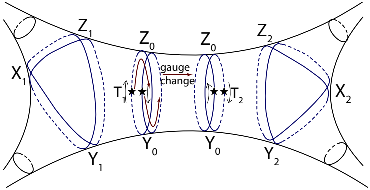

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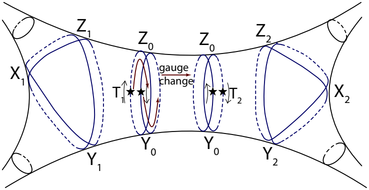

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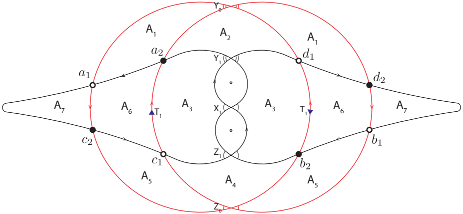
Smoothing

- We need to glue the deformation spaces of S and C.
- They are given by $(x, y, z) \in \mathbf{C}^3$ and $(t, y_0, z_0) \in \mathbf{C}^\times \times \mathbf{C}^2$ respectively.
- Intuitively to match the superpotentials xyz and ty_0z_0, we simply put $x = t, y = y_0, z = z_0$.
- S and C are disjoint to each other!
- We take S_1 to be the deformed Seidel Lagrangian which intersects C as in the figure.
- Then we use cocycle conditions to deduce the gluing between S and C.



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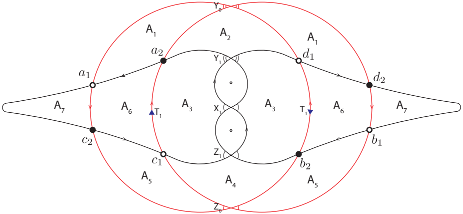
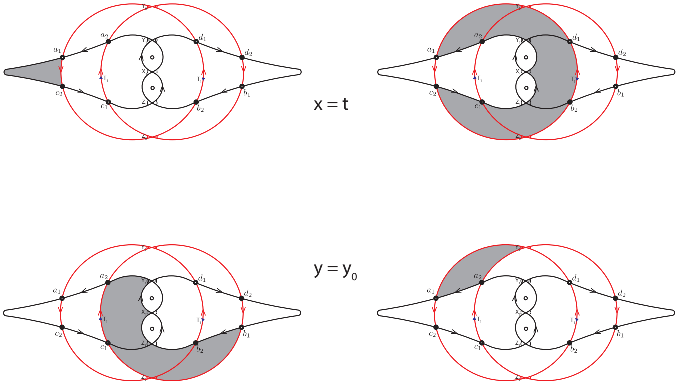
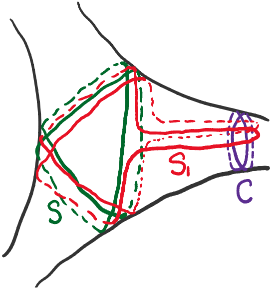


Isomorphisms in smoothing

- We consider $a_1 + b_1 \in \mathrm{CF}((C, \nabla_t), (S, xX))$ and $c_2 + d_2 \in \mathrm{CF}((S, xX), (C, \nabla_t))$.
- Consider cocycle conditions on $a_1 + b_1$ and $c_2 + d_2$. Once the cocycle conditions are satisfied, they give isomorphisms between the two objects.



A paradox



- S is isomorphic to S_1, and (S_1, xX) is isomorphic to (C, ∇_t) for $t = x \neq 0$. Thus (S, xX) is isomorphic to C.
- But S and C are disjoint, and hence there is no morphism between them!
- We need to take a closer look at areas of holomorphic discs.
- Following **Fukaya-Oh-Ohta-Ono**, we shall use the Novikov ring $\Lambda_0 = \{\sum_{i=0}^{\infty} a_i T^{A_i} : 0 \leq A_0 \leq A_1 \leq \ldots\}$ to filter deformations into different energy levels.

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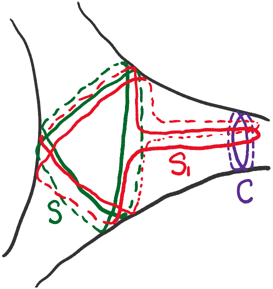

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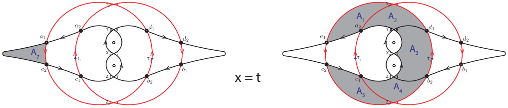
Area constraints for isomorphisms



- For the cocycle conditions on $(S_1, x_1 X_1) \to (C, \nabla_t)$, indeed we have $t = T^{A_1 + \ldots + A_5 - A_7} x_1$.
- Note that $t \in \mathbf{C}^\times$ while $x_1 \in \Lambda_0$. When $A_1 + \ldots + A_5 > A_7$, the condition is never satisfied.
- Hence $(S_1, x_1 X_1)$ can only be isomorphic to (C, ∇_t) if $A_1 + \ldots + A_5 \leq A_7$.
- When $A_1 + \ldots + A_5 = A_7$, the change of coordinates $x_1 = t$ does not involve Novikov parameters.
- For the cocycle conditions on $(S, xX) \to (S_1, x_1 X_1)$, we have $x_1 = T^A x$ for certain $A > 0$. Thus $\mathrm{val}(x_1) \geq A > 0$ in order to have them to be isomorphic.
- The two regions $\mathrm{val}(x_1) = 0$ and $\mathrm{val}(x_1) \geq A > 0$ are disjoint. This solves the paradox.

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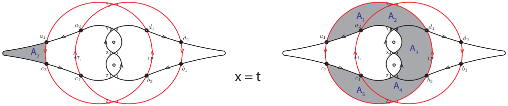

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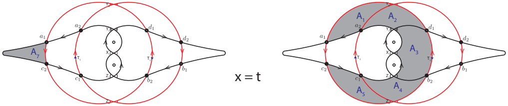

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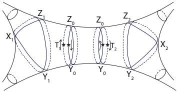
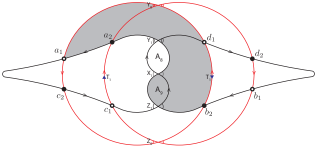
A compactification





- The gluing $x = t$, $y = y_0$, $z = z_0$ looks pretty trivial. Let's compactify to get more interesting gluing.
- As an example, compactify the four-punctured sphere to a sphere. we still consider the moduli of double-circles.
- The gluing is further quantum-corrected by discs emanated from infinite divisors.
- The new gluing is $t = x$, $y_0 = y_1 + t_1^{-1}$, $z_0 = z_1 - t_1^{-1}$. $W = xyz - y_1 + x_1 + z_1$. (The Novikov parameter is suppressed for simplicity.)
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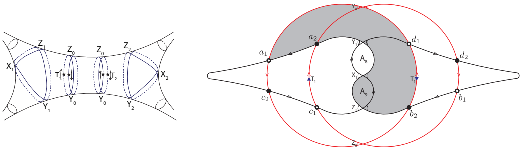

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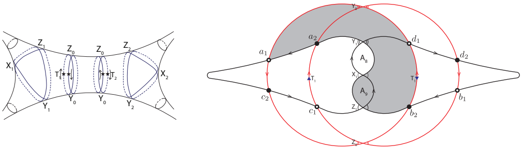

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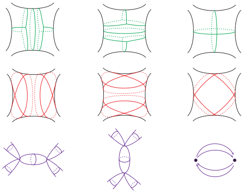
Changing stability and flop



- Let's go back to the four-punctured sphere. We can take another pair-of-pants decomposition.
- It corresponds to another choice of a quadratic differential.
- This results in a flop of the moduli space $(\mathcal{O}(-1) \oplus \mathcal{O}(-1), W)$.
- We can also consider a quadratic differential with double zeros. Then the moduli is the non-commutative resolution of the conifold corresponding to a quiver ([**Cho-Hong-L. 15**]).
- In [**Fan-Hong-L.-Yau**] we studied a 3d version of this ($T^*\mathbf{S}^3$).

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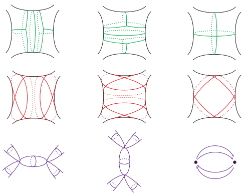

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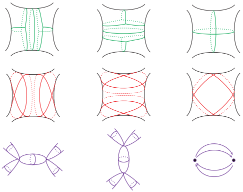

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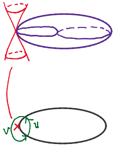
Section 3

Wall-crossing in SYZ

Immersed sphere



- The two-dimensional immersed sphere leads to wall-crossing phenomenons for SYZ Lagrangian fibrations.
- **Fukaya** studied this immersion and the relation with the mirror equation. Here we realize it from cocycle conditions.
- It has two degree-one immersed generators U and V. It gives the deformations $b = uU + vV$ where $(u, v) \in (\Lambda_0 \times \Lambda_+) \cup (\Lambda_+ \times \Lambda_0)$.
- There are constant holomorphic discs with corners $U, V, \ldots, U, V$. To ensure there are only finitely many terms under every energy level, we only allow one of u, v having valuation zero.

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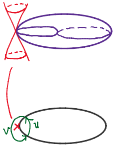

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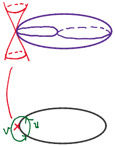
Immersed sphere



- The two-dimensional immersed sphere leads to wall-crossing phenomenons for SYZ Lagrangian fibrations.
- **Fukaya** studied this immersion and the relation with the mirror equation. Here we realize it from cocycle conditions.
- It has two degree-one immersed generators U and V. It gives the deformations $b = uU + vV$ where $(u, v) \in (\Lambda_0 \times \Lambda_+) \cup (\Lambda_+ \times \Lambda_0)$.
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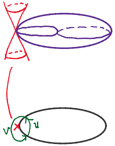

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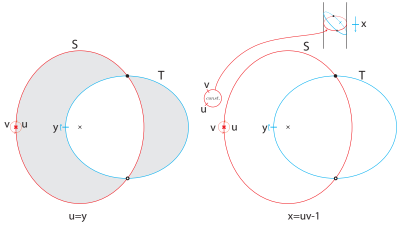
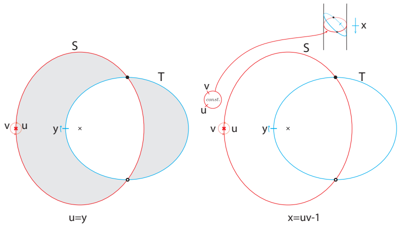
- There are constant holomorphic discs with corners $U, V, \ldots, U, V$. To ensure there are only finitely many terms under every energy level, we only allow one of u, v having valuation zero.
- The constant discs with corners $U, V, \ldots, U, V$ contributing to pt of m_0^b cancel with that with corners $V, U, \ldots, V, U$. Thus $b = uU + vV$ is weakly unobstructed.

Gluing



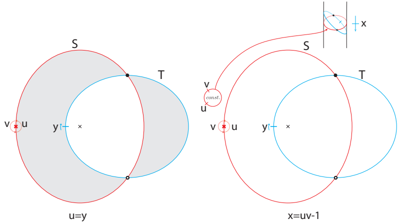
- Consider the immersed sphere S and a Chekanov torus T. S is made by deforming the immersed fiber.
- We glue the deformations $(u, v) \in \Lambda_0 \times \Lambda_+$ with the deformations $(x, y) \in (\mathbf{C}^\times \oplus \Lambda_+)^2$ of T.

Gluing



- We glue the deformations $(u, v) \in \Lambda_0 \times \Lambda_+$ with the deformations $(x, y) \in (\mathbf{C}^\times \oplus \Lambda_+)^2$ of T.
- The cocycle conditions give $u = y$ and $x = uv - 1$. Note that the second equation is on the region $x = -1 + \Lambda_+$.
- In other words, $\mathrm{CF}(T, T)$ gives an extension of $\mathrm{CF}(S, S)$ from $v \in \Lambda_+$ to $v \in \Lambda_0$ with $uv \neq 1$.
- Similarly we can glue the immersed sphere S (with $(u, v) \in \Lambda_+ \times \Lambda_0$) with a Clifford torus T'. It is $v = y'$ and $x' = uv - 1$.
- The resulting moduli is $\{(u, v) \in \Lambda_0 \times \Lambda_0 : uv \neq 1\}$.

Gluing



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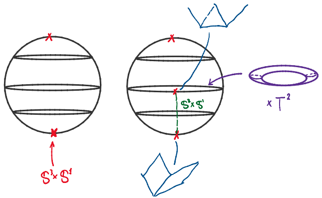
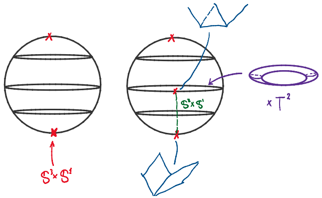
Grassmannians



- With **Hansol Hong and Yoosik Kim**, we are applying this to construct the compactified mirror of $\mathrm{Gr}(2, n)$.
- Flag manifolds have Gelfand-Cetlin systems serving as Lagrangian torus fibrations.
- Immersed spheres are important in that case because they appear as critical fibers of the superpotential.
- For instance, by **Nohara-Ueda** for $\mathrm{Gr}(2, 4)$ there is a critical point $(0, 0)$ which corresponds to a certain fiber $\mathbf{S}^3 \times \mathbf{S}^1$ of the Gelfand-Cetlin system.
- The trouble is $\mathbf{S}^3$ is rigid!

Grassmannians



- Consider the symplectic reduction picture on $\mathbf{S}^2$. We push in one singular point and consider the corresponding moduli.
- There is one monotone Lagrangian torus above and below the wall respectively.
- The deformation spaces $(\mathbf{C}^\times)^2$ of these two tori provide the cluster charts of the mirror. However there is still one point missing, which is crucial since a critical point of W lies there.
- We glue the deformation spaces of an immersed sphere (times T^2) with that of the two monotone tori like in the last slide.
- This recovers the **Rietsch** Lie theoretical mirror for $\mathrm{Gr}(2,4)$.