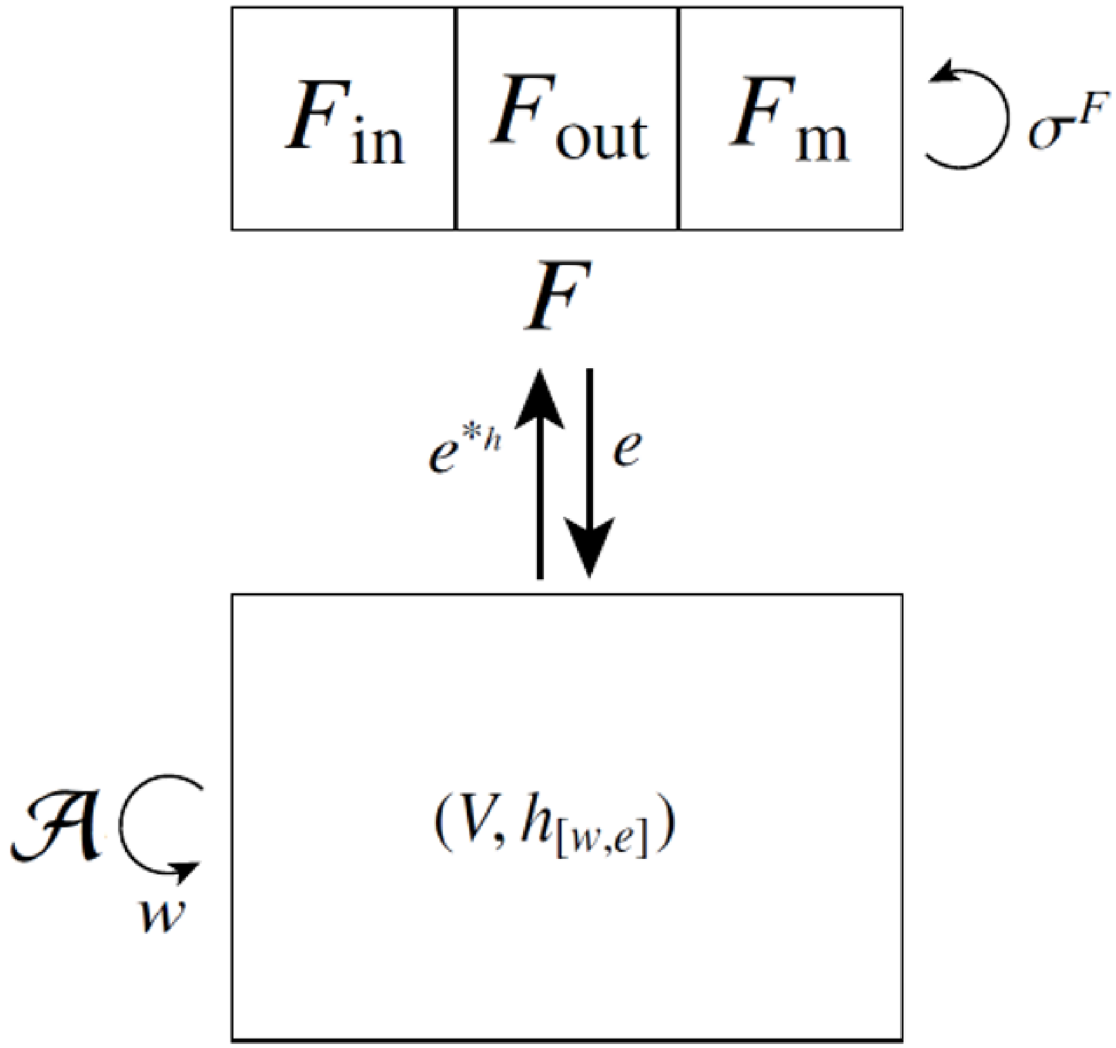
Mirror symmetry for quiver stacks and machine learning

Wednesday, September 14, 2022 8:12 AM

Siu Cheong Lau

Quiver representation emerges from Lie theory and mathematical physics. Its simplicity and beautiful theory have attracted a lot of mathematicians and physicists. In this talk, I will explain localizations of a quiver algebra, and the relations with SYZ and noncommutative mirror symmetry. I will also explore the applications of quivers to computational models in machine learning.

1. Motivation: sheaves ↔ quiver representations
2. SYZ mirrors and quivers
3. Framed quivers as computers



Motivation: sheaves ↔ quiver representations

One important source: **quiver resolution and quiver gauge theory**.

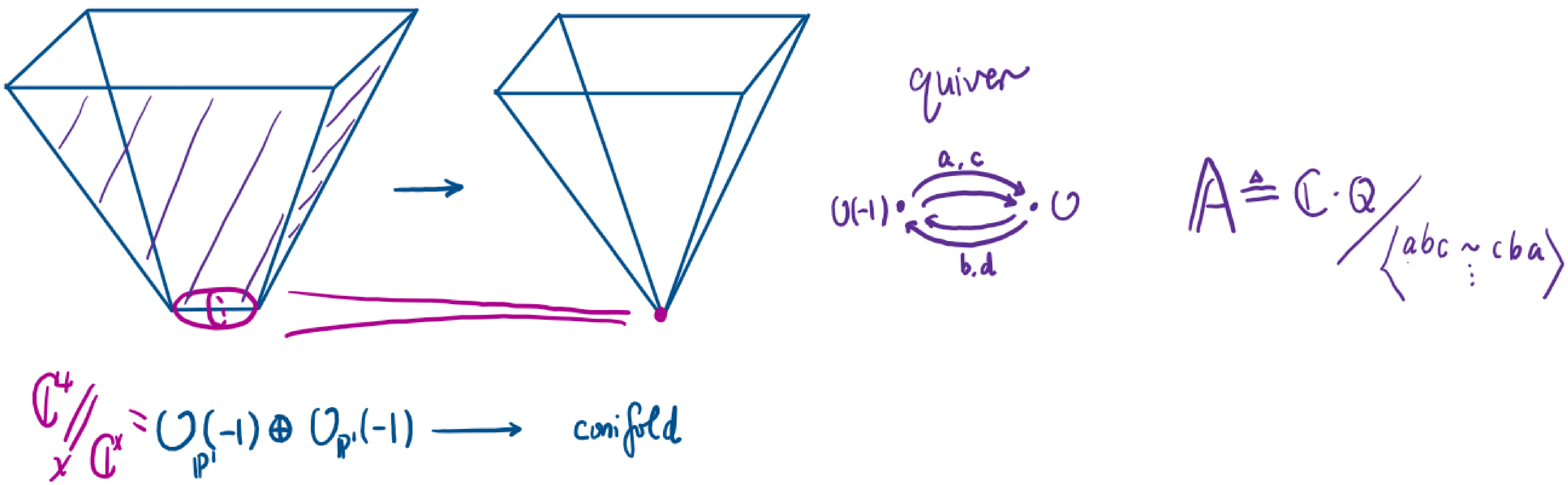
For a local Calabi-Yau singularity $X = \text{Spec } R$,
let Y be a crepant resolution ($f^*\omega_X = \omega_Y$).
Van den Bergh has formulated quiver algebra $\boldsymbol{A}$, called **noncommutative crepant resolution**, such that $D^b(coh(Y)) \cong D^b(mod(\boldsymbol{A}))$.

$\boldsymbol{A} \coloneqq \mathbf{End}_{\boldsymbol{R}}(\boldsymbol{M})$ where
M: a reflexive R-module ($M^{**} \cong M$).

Douglas-Moore use quiver to encode a system of D-branes wrapping a Calabi-Yau threefold singularity.

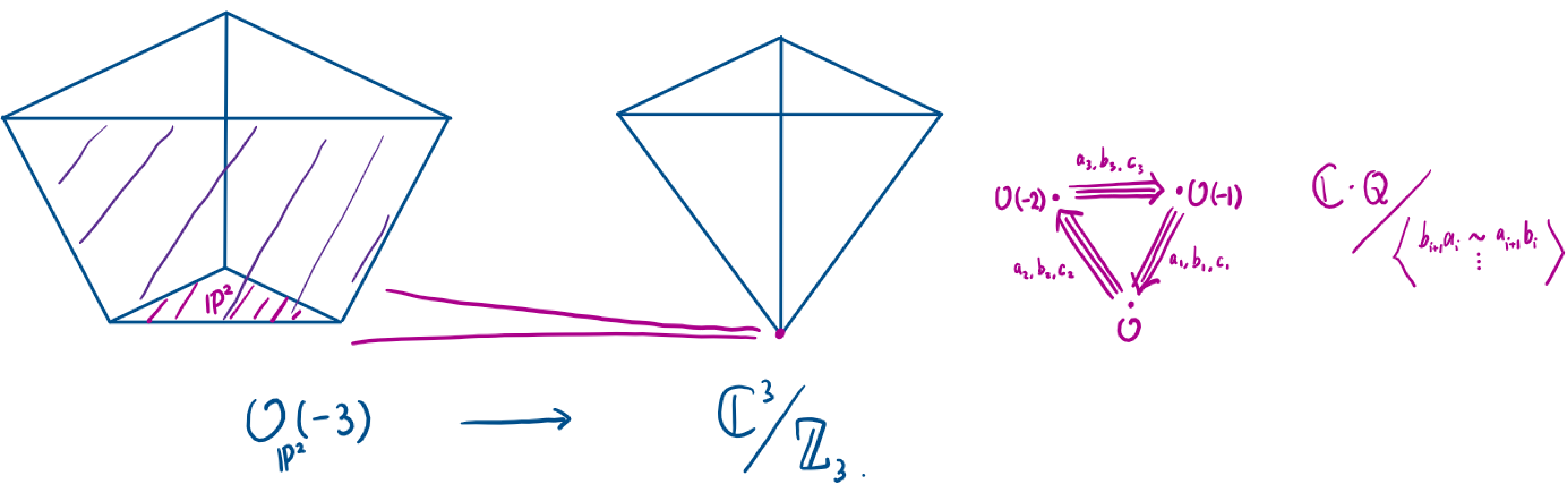
Quiver is also useful in studying noncommutative deformations [**Donovan-Wemyss**].

Ex. conifold singularity $\{y_0 w_1 = y_1 w_0\}$.
Crepant resolution:



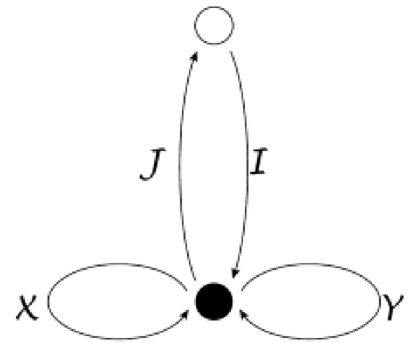
$$D^b(coh(Y)) \cong D^b(mod(\boldsymbol{A})).$$

Ex. Orbifold $\mathbb{C}^3/\mathbb{Z}_3$.
Crepant resolution:



Another important source for **sheaf ↔ quiver representation**:
ADHM quiver [Atiyah-Drinfeld-Hitchin-Manin, Donaldson, Nakajima].

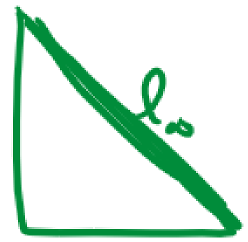
Yang-Mills instantons over $\mathbb{S}^4$ $\quad (F_A = -* F_A)$



$xy - yx + ij = 0$

↔ stable quiver representations over ADHM quiver

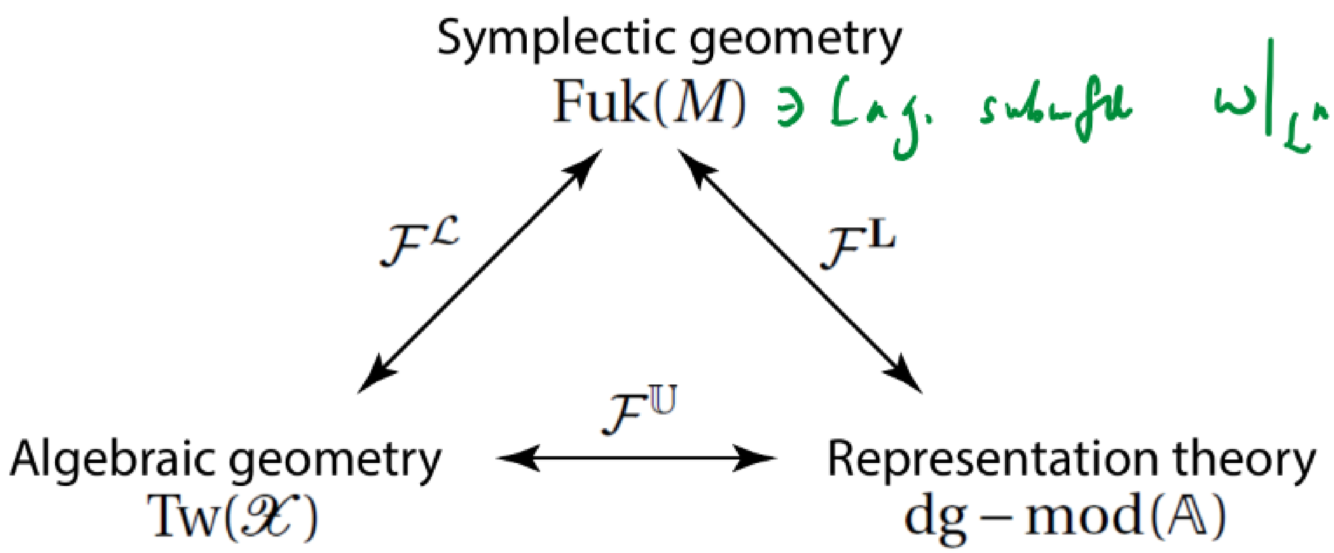
↔ framed torsion-free sheaves over $\left(\mathbb{P}^2, l_\infty\right)$.
(multiplication by $O_x - \{0\}$ on E_x is injective)



Generalized to ALE surfaces $\widehat{\mathbb{C}^2/\Gamma}$ [**Kronheimer-Nakajima**].

We will use mirror symmetry to systematically construct the algebro-geometric correspondence: **sheaf ↔ quiver representation**.

Theorem: there exists a triangle of functors:



$\ni$ Lag. submfld $\omega|_{L^n \subset M^{2n}} = 0$.

- $\mathcal{F}^{\mathcal{L}}$ is SYZ mirror functor.
- $\mathcal{F}^{\mathbf{L}}$ is quiver mirror functor.
- $\mathcal{F}^{U}$ is constructed from isomorphism $\mathcal{L} \leftrightarrow \boldsymbol{L}$.

SYZ mirror and quiver

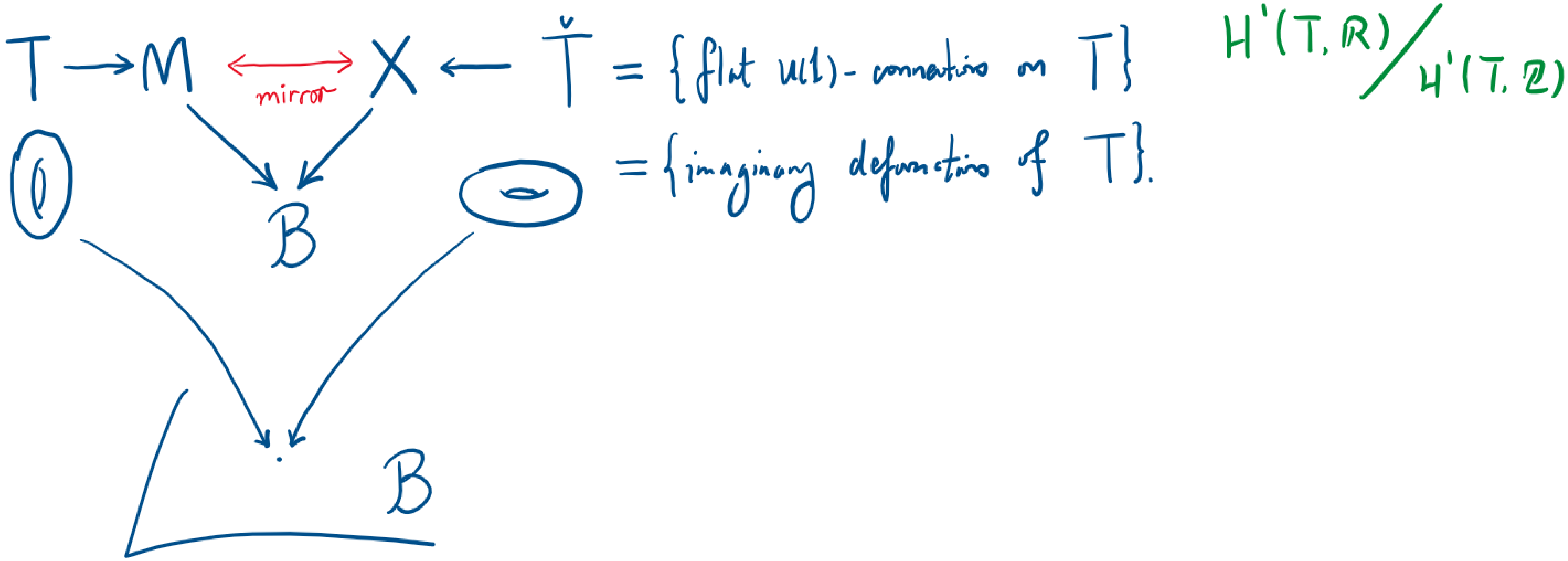
- Mirror symmetry is duality
  symplectic $(M,\omega) \leftrightarrow$ complex (X,J).

- Found by string theorists in the 90's.

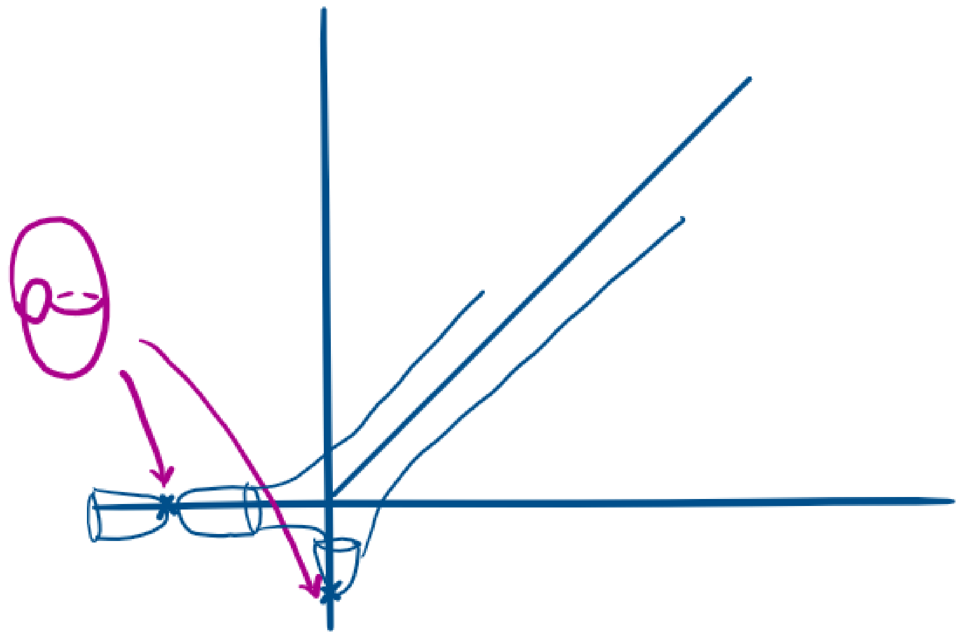
- Powerful prediction of Gromov-Witten invariants
  proved by **[Givental]** and **[Lian-Liu-Yau]**.

- Homological mirror symmetry **[Kontsevich]**:
  $D\mathrm{Fuk}(M) \cong D\mathrm{Coh}(X)$.

- Mirror symmetry is T-duality **[Strominger-Yau-Zaslow]**.



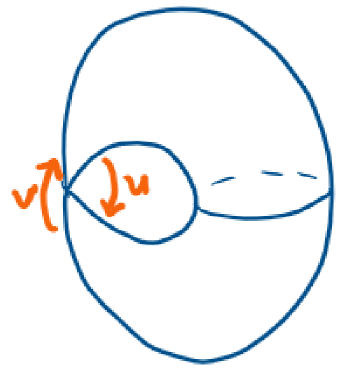
SYZ singular fibers are the sources of quantum corrections, which form wall-crossing and scattering [**Kontsevich-Soibelman, Gross-Siebert, Auroux, Gross-Hacking-Keel…**]



We glue **deformation spaces of SYZ singular fibers** to construct the mirror.
[Cho-Hong-L., Hong-L.-Kim, L.-Nan-Tan]

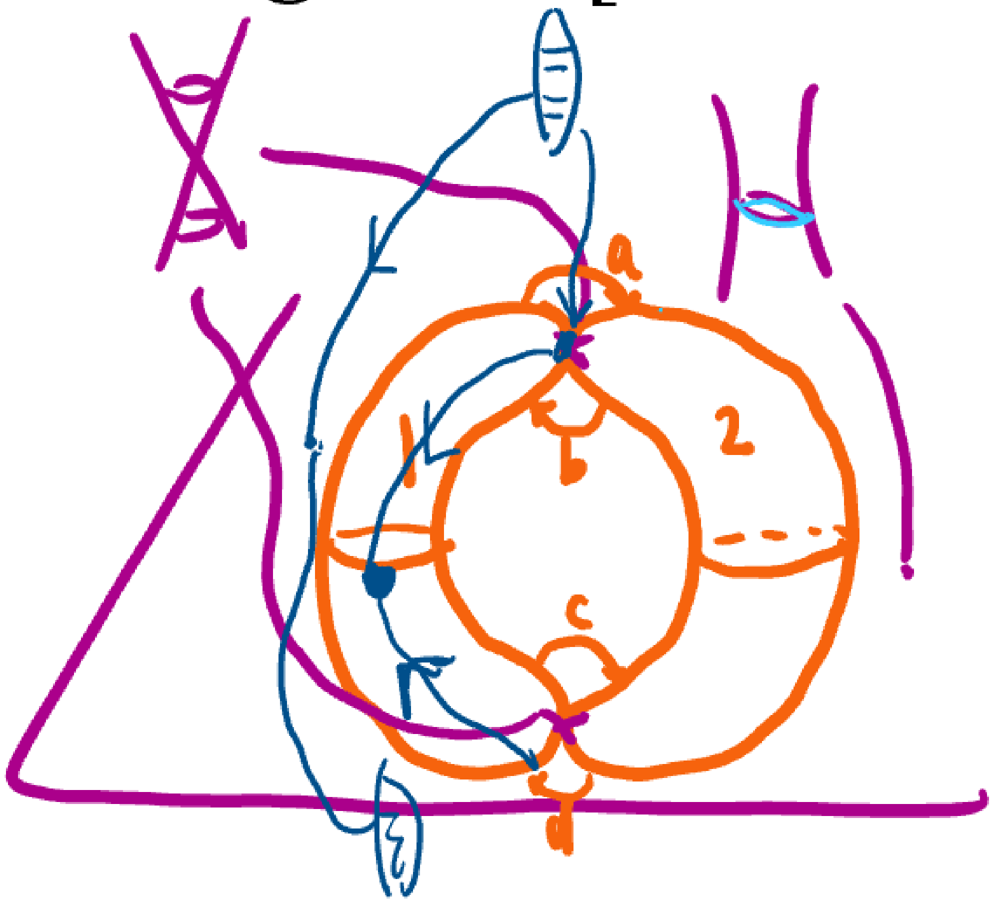
ex. Deformation space of the nodal sphere is

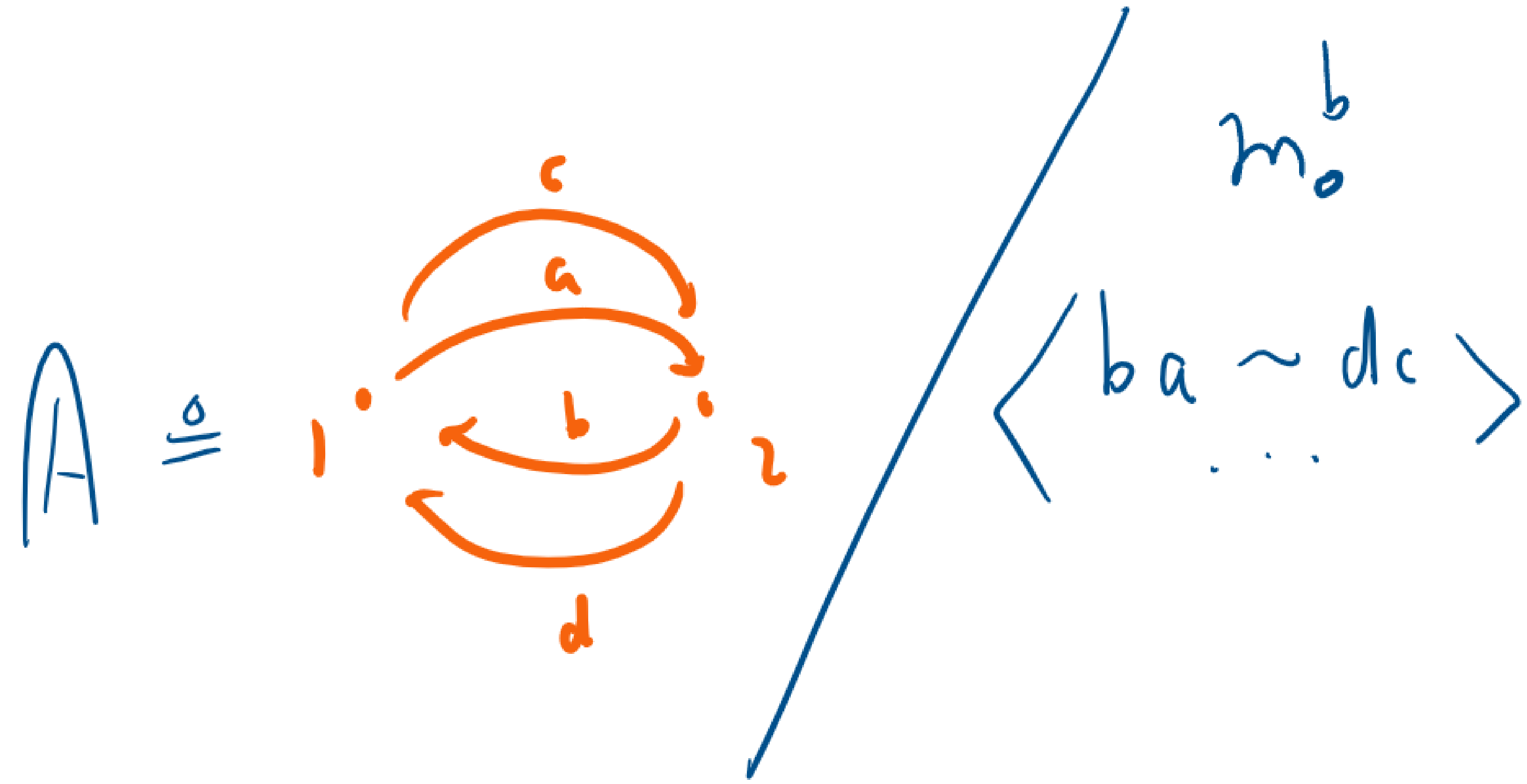
$$\mathbb{C}[u,v][(uv-1)^{-1}]$$



Note: generally, SYZ singular fiber corresponds to quiver algebra! [**Cho-Hong-L.**]

ex.





Have mirror functor

$$\mathrm{Fuk}(M) \longrightarrow \text{dg-mod}(A)$$

Gluing quiver algebras together produce a quiver stack.
(Algebroid stack was defined by **[Kashiwara; O'brian-Toledo-Tong; D'Agnolo-Polesello; Bressler-Gorokhovsky-Nest-Tsygan; Block-Holstein-Wei...]**)

Def. A quiver stack consists of the following:

(1) *An open cover* $\{U_i : i \in I\}$ *of* B.
(2) *A sheaf of algebras* $\mathscr{A}_i$ *over each* U_i*, coming from localizations of a quiver algebra* $\mathscr{A}_i(U_i) = \mathbb{C}Q^{(i)}/R^{(i)}$.
(3) *A sheaf of representations* G_{ij} *of* $Q_V^{(j)}$ *over* $\mathscr{A}_i(V)$ *for every* i, j *and* $V \overset{\text{open}}{\subset} U_{ij}$.
(4) *An invertible element* $c_{ijk}(v) \in \left(e_{G_{ij}(G_{jk}(v))} \cdot \mathscr{A}_i(U_{ijk}) \cdot e_{G_{ik}(v)}\right)^{\times}$ *for every* i, j, k *and* $v \in Q_0^{(k)}$*, that satisfies*

$$G_{ij} \circ G_{jk}(a) = c_{ijk}(h_a) \cdot G_{ik}(a) \cdot c_{ijk}^{-1}(t_a) \tag{2.13}$$

such that for any i, j, k, l *and* v,

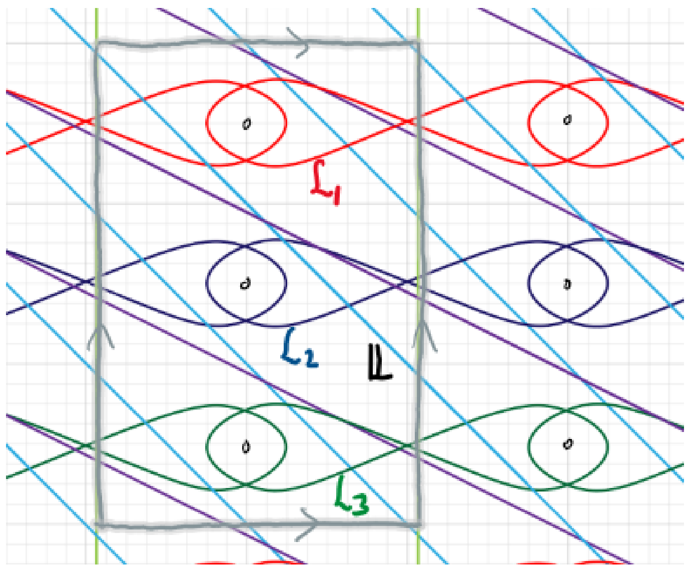
$$c_{ijk}(G_{kl}(v))c_{ikl}(v) = G_{ij}(c_{jkl}(v))c_{ijl}(v). \tag{2.14}$$

In this paper, we always set $G_{ii} = \mathrm{Id}, c_{jjk} \equiv 1 \equiv c_{jkk}$.

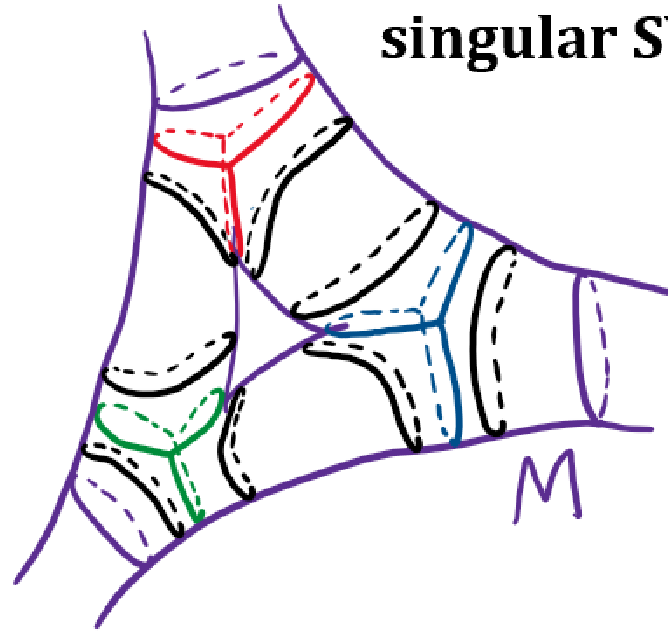
c_{ijk} is called gerbe data (which is necessary for gluing quivers with different number of vertices).

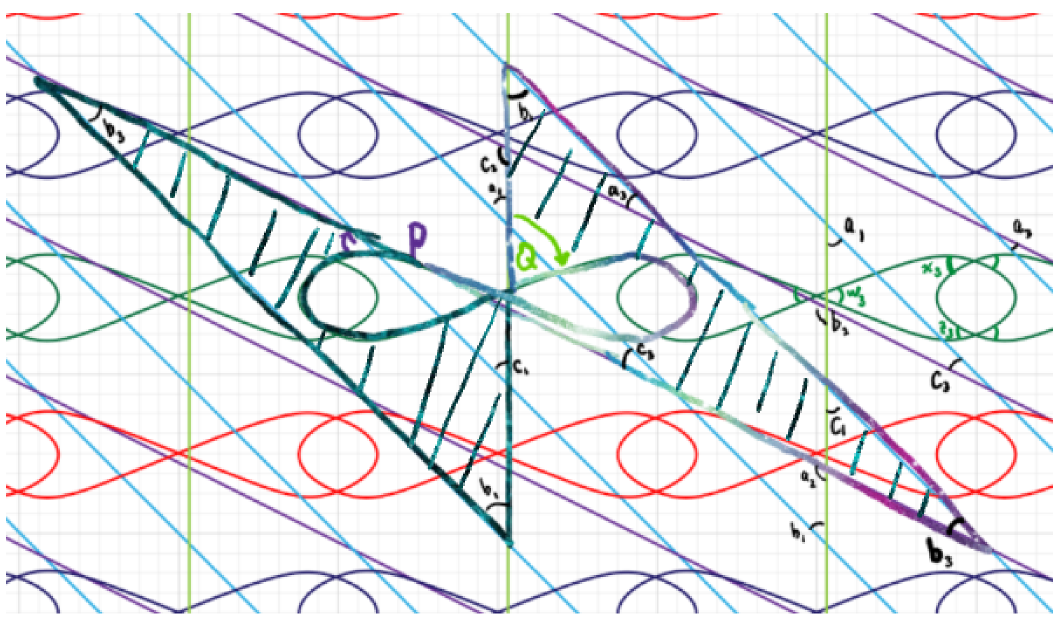
The notion enables us to **glue generic SYZ fibers with singular Lagrangians** via quasi-isomorphisms in the Fukaya category.

ex. Construction of $K_{\mathbb{P}^2}$ and quiver from mirror curve.



Seidel Lagrangians **replacing the singular SYZ fibers.**

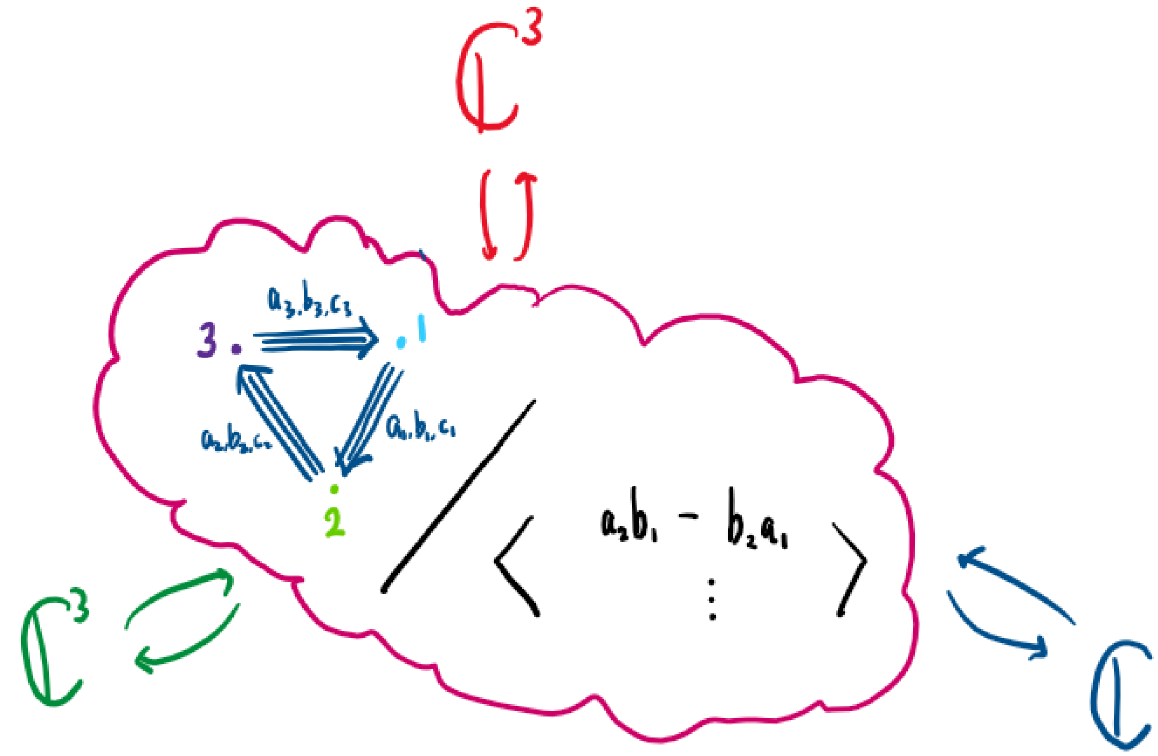




Isomorphisms $(L_0, b_0) \simeq (L_i, b_i)$:
$i = 1,2,3$

$\alpha_{03} = Q$;

$\alpha_{30} = P \cdot b_3^{-1} b_1^{-1}$.



Theorem. [**L.-Nan-Tan**] There exists a quiver stack $\hat{\mathcal{Y}}$ such that α_{0j}, α_{j0} satisfy the isomorphism equations

$m_1^{\hat{\mathcal{Y}}, b_j, b_k}(\alpha_{jk}) = 0;$

$m_2^{\hat{\mathcal{Y}}, b_j, b_k, b_j}(\alpha_{jk}, \alpha_{kj}) = \mathbf{1}_{L_j}.$

$\longrightarrow K_{\mathbb{P}^2}$.

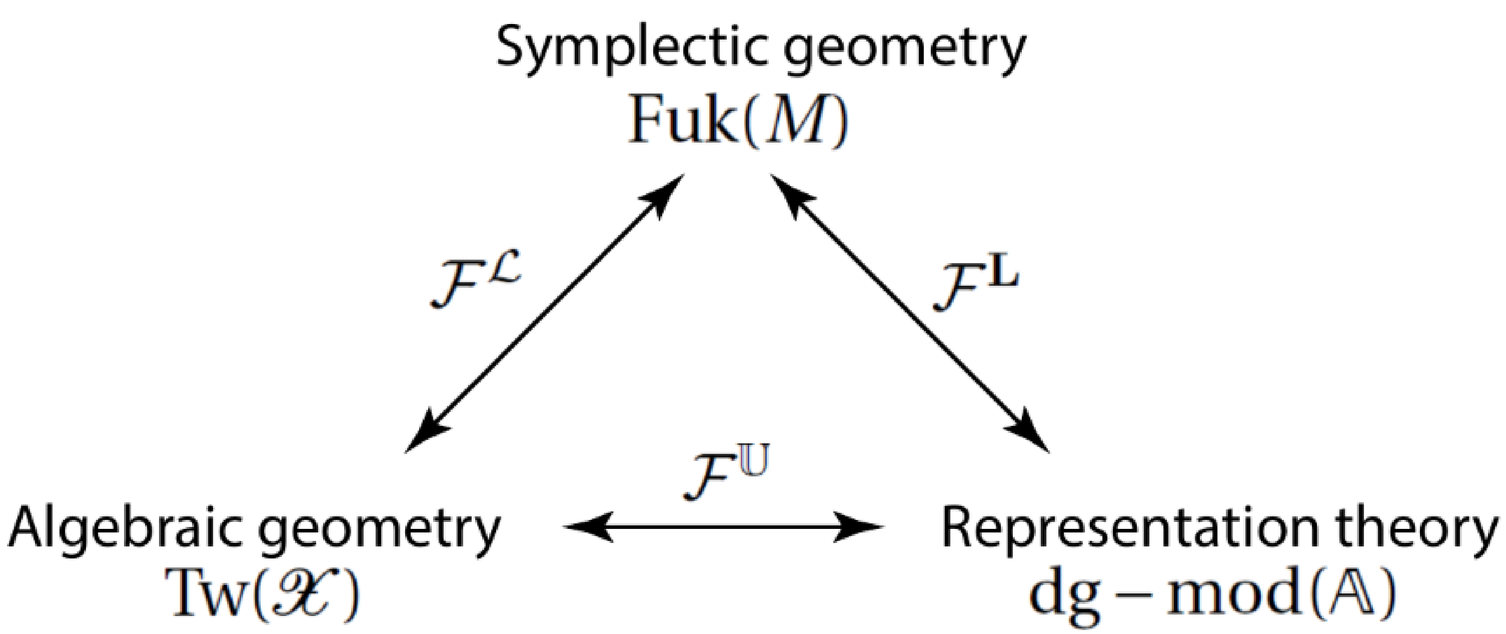
Such a method helps us to **glue three non-intersecting SYZ fibers together via a middle agent**!

Conclusion:

- Have mirror functor $\mathcal{F}^{\mathcal{L}}: Fuk(M) \to Tw(X)$ by gluing singular SYZ fibers together.

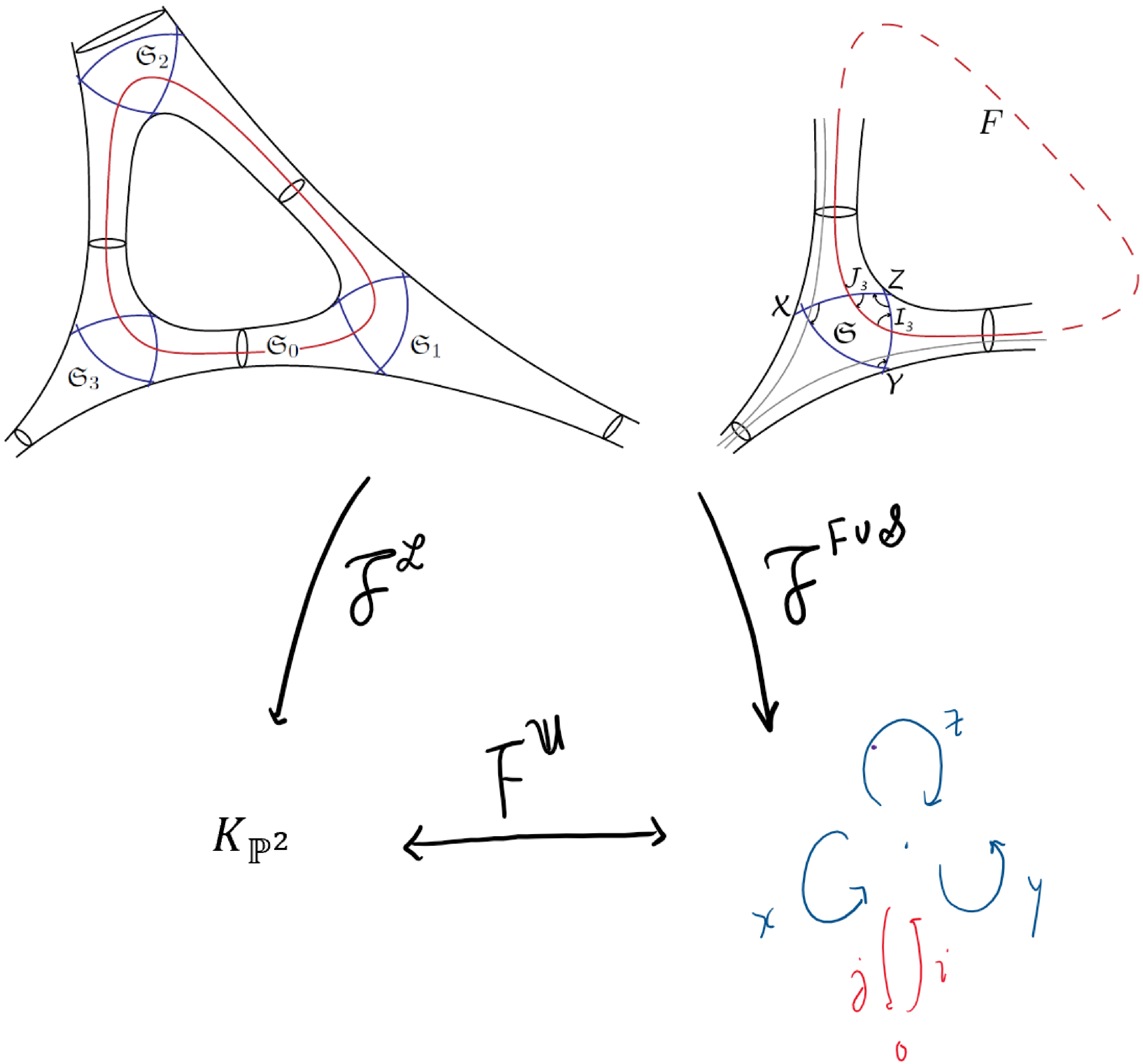
- Have mirror functor $\mathcal{F}^{(\boldsymbol{L},\boldsymbol{b})}: Fuk(M) \to Mod(Q)$ by a single immersion $\boldsymbol{L}$ and its deformations (from a Lagrangian skeleton of M).

- $U := \mathcal{F}^{\mathcal{L}}(\boldsymbol{L},\boldsymbol{b})$ gives $\mathcal{F}^{U} = \mathrm{Hom}_{\mathrm{Tw(X)}}(U,-): Tw(X) \to Mod(Q)$.



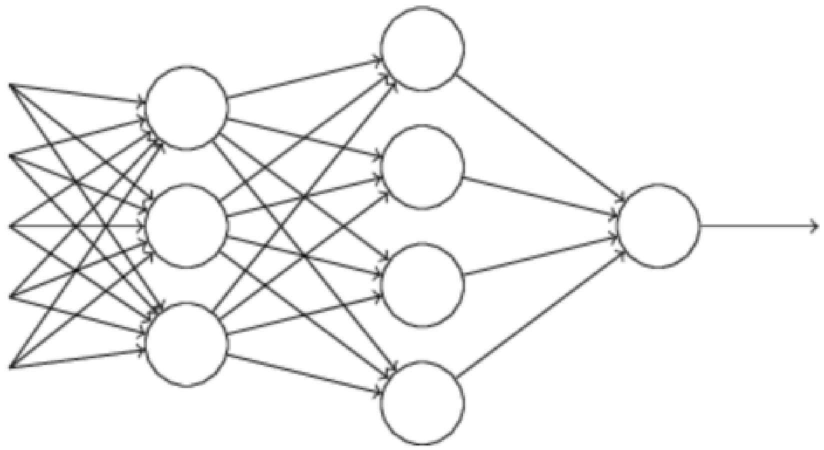
Ongoing work:

Use framed Lagrangian $F \cup \mathcal{S}$ to produce the ADHM quiver and sheaves.



Quivers and machine learning

Remarkably, quiver representation is the key object in **Deep learning**.



Fix $\gamma \in \mathbb{C}Q$ that starts and ends with $i_{\text{in}}, i_{\text{out}}$.

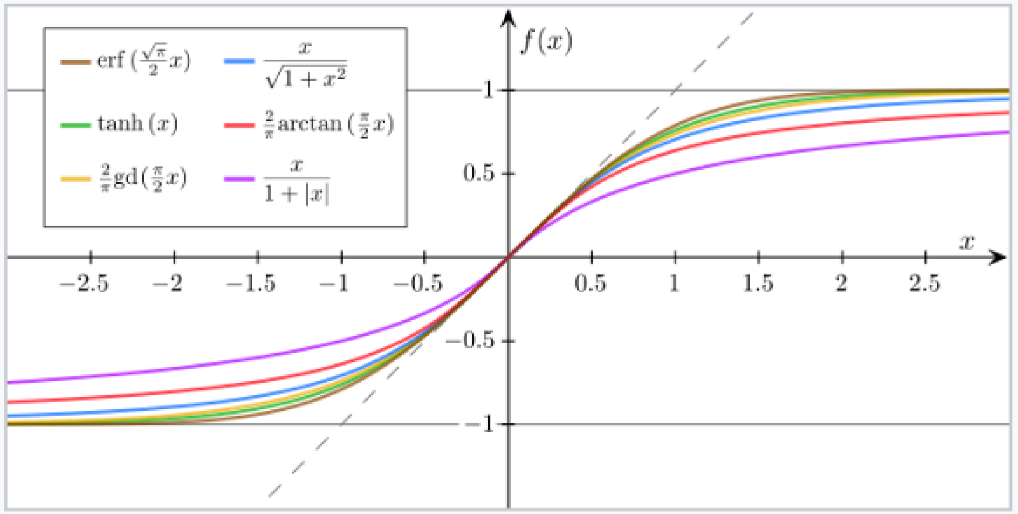
Have a canonical linear function
$\boldsymbol{L_{\gamma,w}: V_{i_{\text{in}}} \to V_{i_{\text{out}}}}$
associated to each $w \in R(Q)$,
by composing arrow linear maps along γ.

Representation learning:
Given $K \subset V_{i_{\text{in}}}$ and a continuous function
$f: K \to V_{i_{\text{out}}}$ (statistically given),
minimize the cost function
$\left|L_{\gamma,w} - f\right|^2: R(Q) \to \mathbb{R}$
by taking a stochastic gradient descent in $R(Q)$.

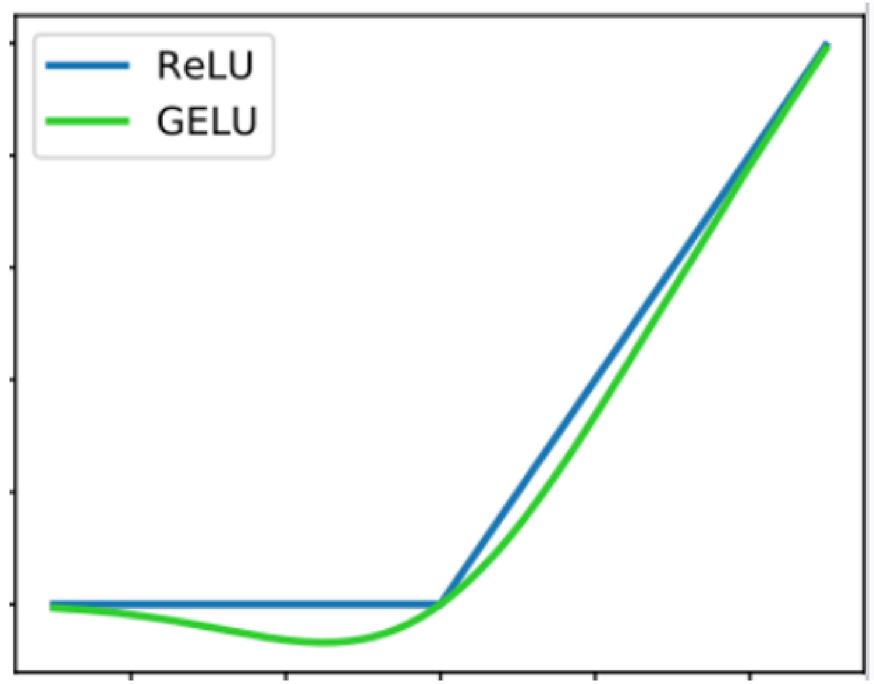
$\Rightarrow$ quiver representation that gives the best linear approximation.

Insight from neural network:

to get non-linear approximation, introduce non-linear `activation functions' at vertices.



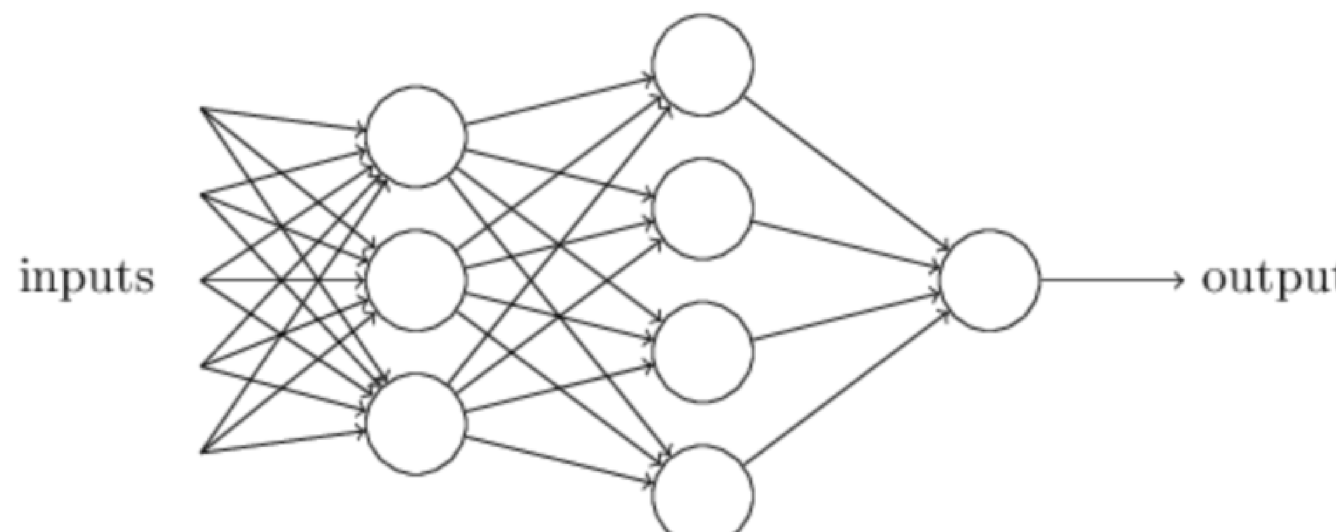
Some sigmoid functions compared. In the drawing all functions are normalized in such a way that their slope at the origin is 1.



Compose with these activation functions and obtain **network function**

$f_{\tilde{\gamma},w}: V_{i_{\text{in}}} \to V_{i_{\text{out}}}$

for every $w \in \mathrm{Rep}(Q)$.



Do not occur in usual quiver theory.

We want to work with moduli space
rather than the vector space $R(Q)$ of representations.

$\boldsymbol{M(Q) := [R(Q)/G]}$

where

$$G = \prod_{i \in Q_0} \mathrm{GL}(V_i) .$$

[Mumford; Kings]

$\{\boldsymbol{\theta}$ **– semistable points in** $\boldsymbol{R(Q)}\}//\ \boldsymbol{G}$.

GIT or slope stability: choose weights $\theta \in \left(\mathbb{Z}^{Q_0}\right)^*$.

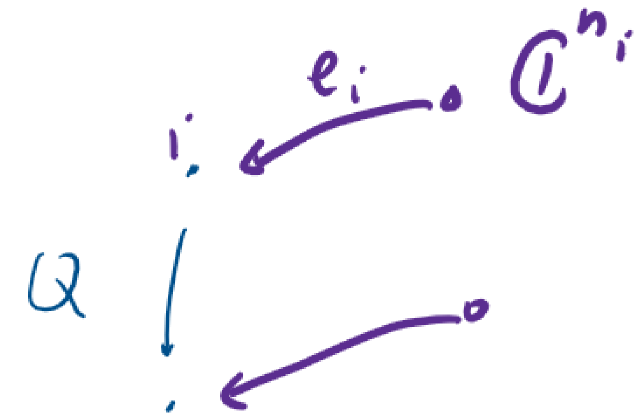
Rep. V is θ-semi-stable if $\theta\left(\overrightarrow{\dim V'}\right) \leq \theta\left(\overrightarrow{\dim V}\right)$ for $V' \subset V$.

(Can also be understood via symplectic quotient.)

Obstacles of running deep learning over $M(Q)$:

$L_{\tilde{\gamma},w}$ is composed with activation functions $\sigma: V_i \to V_i$, which are NOT G-equivariant: $\sigma(g \cdot v) \neq g \cdot \sigma(v)$.

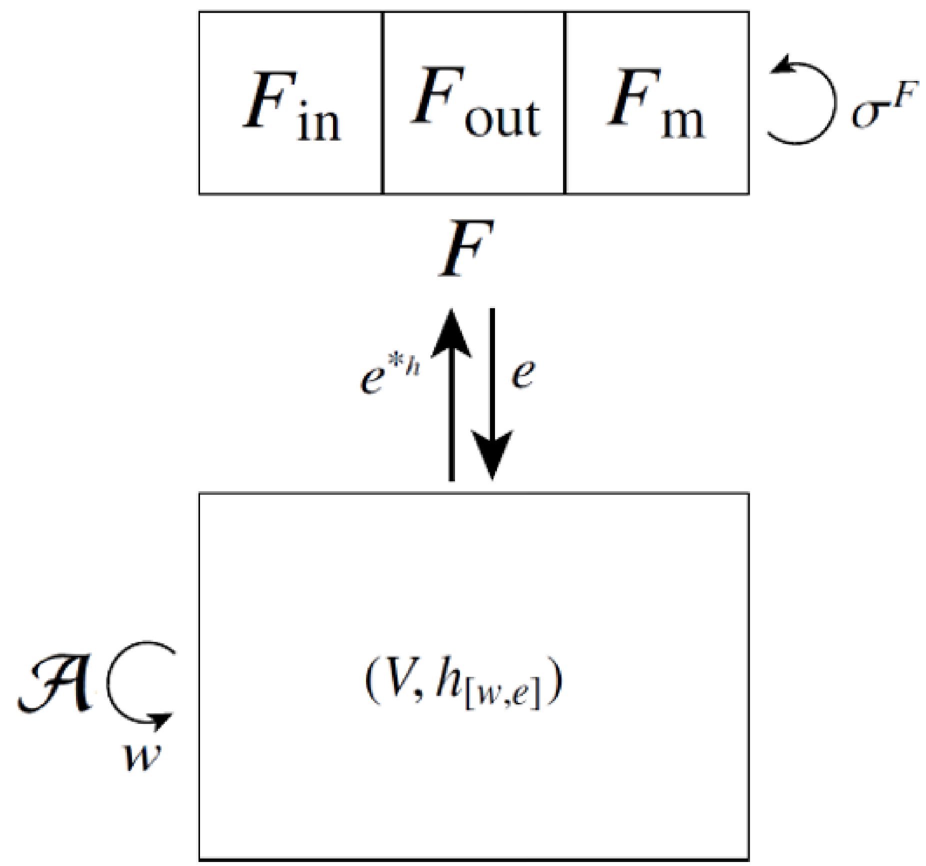
This motivates us to use **framed** quiver representations [**Nakajima; Crawley-Boevey; Reineke**].



Framed quiver representation:
usual quiver representation
(linear maps associated to arrows)
together with linear maps $\boldsymbol{e_i}: \mathbb{C}^{n_i} \to \boldsymbol{V_i}$ **(called framing)**.

$$R^{\mathrm{fr}} := \bigoplus_{a \in Q_1} \mathrm{Hom}\left(V_{t(a)}, V_{h(a)}\right) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(\mathbb{C}^{n_i}, V_i) \ni (w, e).$$

$$\boldsymbol{\mathcal{M}^{\mathrm{fr}} := \{\text{stable points in } R^{\mathrm{fr}}\}/G}.$$

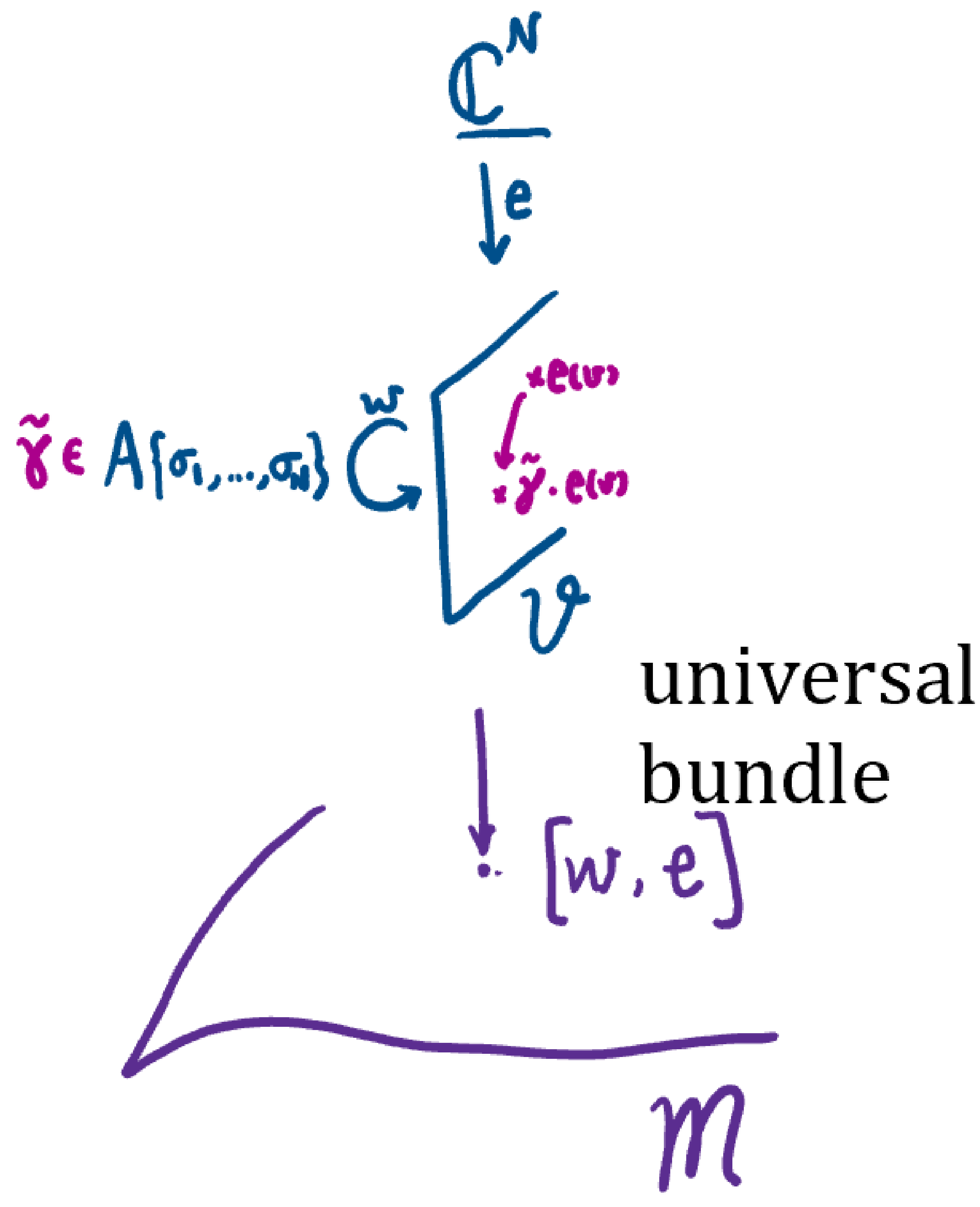


Think of V_i as state spaces;
$\mathbb{C}^{n_i}$ as spaces for input, output or memory.

Key: put the activation functions σ on the framing, rather than on the state spaces V.

Moreover, the moduli space has
universal bundles $\mathcal{V}_i \to \mathcal{M}^{\text{fr}}$.
Each arrow a gives a vector-bundle map
$\mathcal{V}_{t(a)} \to \mathcal{V}_{h(a)}$.

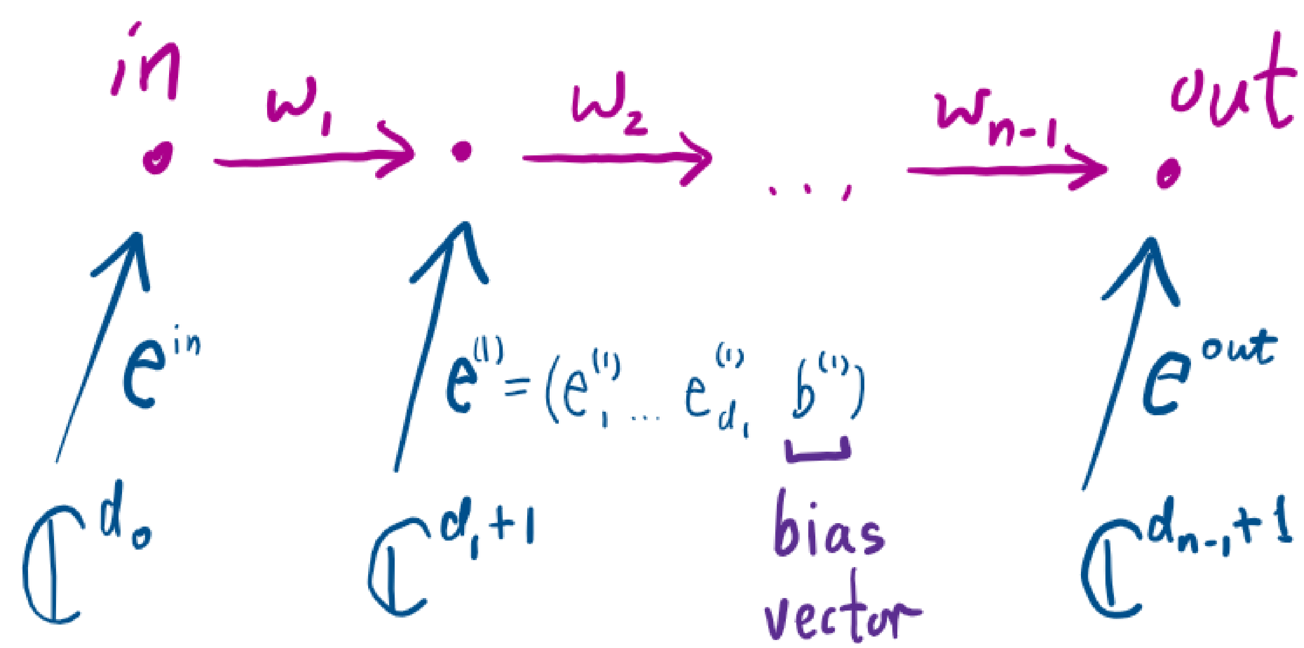
Using these, together with metric,
can make well-defined learning algorithm over the quiver moduli.



Fix

- a graph Q,
- input and output vertices $i_{\text{in}}, i_{\text{out}}$,
- the representation dimension vector,
- the framing $e_i: F_i \to V_i$,
- Non-linear functions $\sigma_i: F_i \to F_i$,
- the "algorithm" $\tilde{\gamma} \in \mathbb{C}\hat{Q}\{\sigma_1, \dots, \sigma_N\}$,

Compose
arrow maps $\mathcal{V}_{t(a)} \to \mathcal{V}_{h(a)}$,
framing maps $e_i: \boldsymbol{F_i} \to \mathcal{V}_i$,
their metric adjoints $e_i^*: \mathcal{V}_i \to \boldsymbol{F_i}$, and
$\sigma_i: F_i \to F_i$,
get a machine function
$f_{\tilde{\gamma}}: V_{i_{\text{in}}} \to V_{i_{\text{out}}}$ **over $\boldsymbol{\mathcal{M}}^{\text{fr}}$**.



Ex. A_n-quiver.

$$f_{\tilde{\gamma}} = s_{n-1} \circ e^{\text{out}^*} \circ \left(w_{n-1} \circ \left(\dots \circ e^{(1)} \circ s_1 \circ e^{(1)^*} \circ \left(w_1 \circ e^{\text{in}} + b^{(1)} \right) \dots \right) + b^{(n-1)} \right).$$

To run the algorithm over $\mathcal{M}^{\mathrm{fr}}$, need
(1) **vector-bundle metric** H_i on universal bundle $\mathcal{V}_i$;
(2) **Metric** $h_{\mathcal{M}^{\mathrm{fr}}}$ on $\mathcal{M}^{\mathrm{fr}}$.

Thm.
Fix $i \in Q_0$.

- $H_i : R^{\mathrm{fr}} \to \mathrm{End}\left(\mathbb{C}^{d_i}\right)$,

$$(w, e) \mapsto \left(\sum_{h(\gamma)=i} \left(w_\gamma e_{t(\gamma)}\right)\left(w_\gamma e_{t(\gamma)}\right)^* \right)^{-1}$$

gives a well-defined metric on $\mathcal{V}_i \to \mathcal{M}$.

- Moreover, assuming Q has no oriented cycle, the Ricci curvature $\sqrt{-1}\sum_i \partial\bar{\partial} \log \det H_i$ of the resulting metric on $\bigotimes_{i \in Q_0} U_i$ defines a Kaehler metric on $\mathcal{M}^{\mathrm{fr}}$.

Moreover, can uniformize with the original Euclidean setup and hyperbolic metric.

Assume $\vec{n} \geq \vec{d}$.
Write the framing as $e^{(i)} = \begin{pmatrix} \epsilon^{(i)} & b^{(i)} \end{pmatrix}$.

At points where $\epsilon^{(i)}$ is invertible,
applying quiver automorphism $\Rightarrow \epsilon^{(i)} = \mathrm{Id}$.

This gives a chart:
$\mathrm{R}_{\vec{n}-\vec{d},\vec{d}}(Q) \hookrightarrow \mathcal{M}^{\mathrm{fr}}_{\vec{n},\vec{d}}(Q)$.

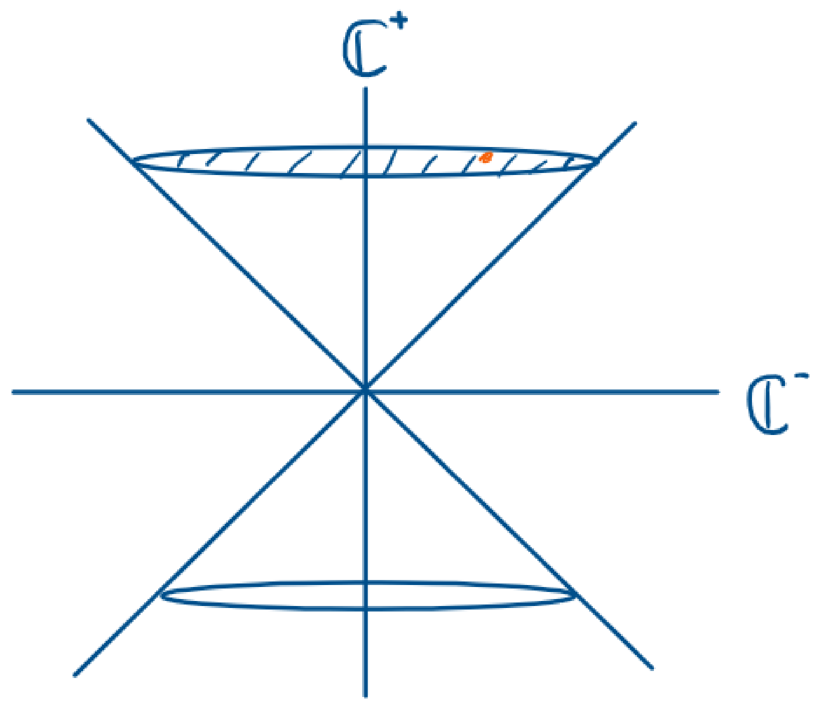
Uniformization gives a unified point of view towards
$\mathrm{R}_{\vec{n}-\vec{d},\vec{d}}(Q)$, $\mathcal{M}^{\mathrm{fr}}_{\vec{n},\vec{d}}(Q)$, and $\mathcal{M}^{-}_{\vec{n},\vec{d}}(Q)$.

Ex. Hyperbolic disc $D \subset \mathbb{CP}^1$.
$D, \mathbb{C}$ and $\mathbb{CP}^1$ can be uniformly understood:

$\mathbb{CP}^1 = \{\text{lines in } \mathbb{C}^2\}$.

If we equip $\mathbb{C}^2$ with the quadratic form $y^2 - x^2$, then
$D = \{\text{spacelike lines in } \mathbb{C}^{1,1}\}$.

If we equip $\mathbb{C}^2$ with the quadratic form $H_0 = y^2$, then
$\mathbb{C} = \left\{\text{spacelike lines in } \left(\mathbb{C}^2, H_0\right)\right\}$.

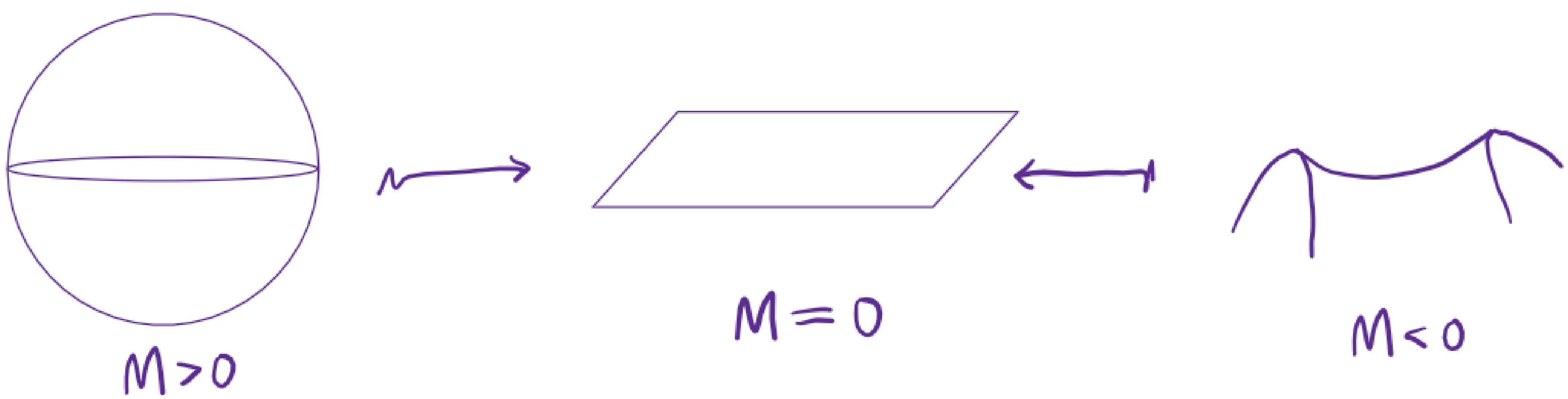


This is classically well understood for symmetric spaces, and
$D \hookrightarrow \mathbb{CP}^1$ is known as **Borel embedding**.

We generalize this to framed quiver moduli.

Thm.
There exists spherical, hyperbolic, Euclidean moduli spaces
$\mathcal{M}, \mathcal{M}^-, \mathcal{M}^0$ which can be interpolated by a family of metrics.

$$\left(\rho_i \begin{pmatrix} I_{d_i} & 0 \\ 0 & r \cdot I \end{pmatrix} \rho_i^*\right)^{-1}.$$



Conclusion

- **Quiver near algebras** give a uniform setup for machine learning and quantum computing.
- **Quiver gauge theory** (resolution of local CY singularities) & **ADHM** provide correspondence between sheaves and quivers.
- The correspondence can be realized by **mirror symmetry**.
- We are using ***quivers as fundamental building blocks in both physical and computational models***.