Mirror symmetry for quiver stacks and machine learning

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Quiver representation emerges from Lie theory and mathematical physics. Its simplicity and beautiful theory have attracted a lot of mathematicians and physicists. In this talk, I will explain localizations of a quiver algebra, and the relations with SYZ and noncommutative mirror symmetry. I will also explore the applications of quivers to computational models in machine learning.

1. Motivation: sheaves ↔ quiver representations
2. SYZ mirrors and quivers
3. Framed quivers as computers
Motivation: sheaves $\leftrightarrow$ quiver representations

One important source: quiver resolution and quiver gauge theory.

For a local Calabi-Yau singularity $X = \text{Spec } R$, let $Y$ be a crepant resolution ($f^*\omega_X = \omega_Y$). Van den Bergh has formulated quiver algebra $A$, called noncommutative crepant resolution, such that $D^b(\text{coh}(Y)) \cong D^b(\text{mod}(A))$.

$A := \text{End}_R(M)$ where $M$: a reflexive $R$-module ($M^{**} \cong M$).

Douglas-Moore use quiver to encode a system of D-branes wrapping a Calabi-Yau threefold singularity.

Quiver is also useful in studying noncommutative deformations [Donovan-Wemyss].
Ex. conifold singularity \( \{y_0w_1 = y_1w_0\} \).
Crepant resolution:

\[
\mathbb{C}^4 / \mathbb{C}^3 = \mathcal{O}(-1) \oplus \mathcal{O}(-1) \rightarrow \text{conifold}
\]

\[
D^b(\text{coh}(Y)) \cong D^b(\text{mod}(A)).
\]

Ex. Orbifold \( \mathbb{C}^3 / \mathbb{Z}_3 \).
Crepant resolution:
Another important source for \textbf{sheaf} $\leftrightarrow$ \textbf{quiver representation}:

\textbf{ADHM quiver} [Atiyah-Drinfeld-Hitchin-Manin, Donaldson, Nakajima].

Yang-Mills instantons over $S^4$ \quad $(F_A = -* F_A)$

$\leftrightarrow$ stable quiver representations over ADHM quiver

$\leftrightarrow$ framed torsion-free sheaves over $(\mathbb{P}^2, I_{\infty})$.

(multiplication by $O_x - \{0\}$ on $E_x$ is injective)

Generalized to ALE surfaces $\overline{\mathbb{C}^2/\Gamma}$ [Kronheimer-Nakajima].

We will use mirror symmetry to systematically construct the algebro-geometric correspondence: \textbf{sheaf} $\leftrightarrow$ \textbf{quiver representation}.

\textbf{Theorem:} there exists a triangle of functors:

$$\begin{array}{ccc}
\text{Symplectic geometry} & \overset{F_\mathcal{L}}{\longrightarrow} & \text{Tw}(\mathcal{X}) \overset{\mathcal{C}}{\longrightarrow} \text{Rep} \text{theory} \\
\text{Fuk}(M) & \overset{\mathcal{F}_\mathcal{L}}{\longleftarrow} & \overset{\mathcal{F}_\mathcal{U}}{\longleftarrow} \text{dg - mod}(\mathbb{A})
\end{array}$$

- $F_\mathcal{L}$ is SYZ mirror functor.
- $F_\mathcal{L}$ is quiver mirror functor.
- $F_\mathcal{U}$ is constructed from isomorphism $\mathcal{L} \leftrightarrow L$. 
SYZ mirror and quiver

- Mirror symmetry is duality symplectic $(M, \omega) \leftrightarrow$ complex $(X, J)$.
- Found by string theorists in the 90's.
- Powerful prediction of Gromov-Witten invariants proved by [Givental] and [Lian-Liu-Yau].
- Homological mirror symmetry [Kontsevich]: $DFuk(M) \cong DCoh(X)$.
- Mirror symmetry is T-duality [Strominger-Yau-Zaslow].

$$T \rightarrow M \leftrightarrow X \leftarrow \hat{T} = \{ \text{flat } \text{holo} \text{-} \text{conform } \text{on } T \}$$

$$\bigg( H'(T, R) \bigg) \bigg/ H'(T, \mathbb{Z})$$

$$\bigg\{ \text{imaginary} \text{ deforms } \text{of } T \bigg\}.$$
SYZ singular fibers are the sources of quantum corrections, which form wall-crossing and scattering [Kontsevich-Soibelman, Gross-Siebert, Auroux, Gross-Hacking-Keel...]

We glue deformation spaces of SYZ singular fibers to construct the mirror.
[Cho-Hong-L., Hong-L.-Kim, L.-Nan-Tan]

ex. Deformation space of the nodal sphere is

\[ \mathbb{C} \left[ u,v \right] \left[ (uv-1)^{\frac{1}{k}} \right] \]
Note: generally, SYZ singular fiber corresponds to quiver algebra! [Cho-Hong-L.]

Have mirror functor

\[
\text{Fuk}_h(M) \longrightarrow \text{dg-mod}(A)
\]
Gluing quiver algebras together produce a quiver stack. (Algebroid stack was defined by [Kashiwara; O'brian-Toledo-Tong; D'Agnolo-Polesello; Bressler-Gorokhovsky-Nest-Tsygan; Block-Holstein-Wei...])

**Def.** A quiver stack consists of the following:

1. An open cover \( \{ U_i : i \in I \} \) of \( B \).
2. A sheaf of algebras \( \mathcal{A}_i \) over each \( U_i \), coming from localizations of a quiver algebra \( \mathcal{A}_i(U_i) = \mathbb{C}Q^{(i)}/R^{(i)} \).
3. A sheaf of representations \( G_{ij} \) of \( Q_V^{(j)} \) over \( \mathcal{A}_i(V) \) for every \( i, j \) and \( V^{\text{open}} \subseteq U_{ij} \).
4. An invertible element \( c_{ijk}(v) \in \left( e_{G_{ij}(G_{jk}(v))} \cdot \mathcal{A}_i(U_{ijk}) \cdot e_{G_{ik}(v)} \right)^\times \) for every \( i, j, k \) and \( v \in Q_{0}^{(k)} \), that satisfies

\[
G_{ij} \circ G_{jk}(a) = c_{ijk}(h_a) \cdot G_{ik}(a) \cdot c_{ijk}^{-1}(t_a)
\]

such that for any \( i, j, k, l \) and \( v \),

\[
c_{ijk}(G_{kl}(v)) c_{ikl}(v) = G_{ij}(c_{jkl}(v)) c_{ijl}(v).
\]

In this paper, we always set \( G_{ii} = \text{Id}, c_{jjk} \equiv 1 \equiv c_{jkk} \).

\( c_{ijk} \) is called gerbe data (which is necessary for gluing quivers with different number of vertices).

The notion enables us to glue generic SYZ fibers with singular Lagrangians via quasi-isomorphisms in the Fukaya category.
ex. Construction of $K_{\mathbb{P}^2}$ and quiver from mirror curve.

**Seidel Lagrangians** replacing the singular SYZ fibers.

Isomorphisms $(L_0, b_0) \sim (L_i, b_i)$:

$$\alpha_{03} = \mathbb{Q};$$

$$\alpha_{30} = P \cdot b_3^{-1} b_1^{-1}.$$ 

**Theorem. [L.-Nan-Tan]** There exists a quiver stack $\hat{\mathcal{Y}}$ such that $\alpha_{0j}, \alpha_{j0}$ satisfy the isomorphism equations

$$m_1 \hat{\mathcal{G}}, \sigma_j, \sigma_k (\alpha_{jk}) = 0;$$

$$m_2 \hat{\mathcal{G}}, \sigma_j, \sigma_k, \sigma_j (\alpha_{jk}, \alpha_{kj}) = 1_{\mathcal{L}_j}.$$

Such a method helps us to **glue three non-intersecting SYZ fibers together via a middle agent**!
Conclusion:

- Have mirror functor $\mathcal{F}^\mathcal{L}: \text{Fuk}(M) \to \text{Tw}(X)$ by gluing singular SYZ fibers together.

- Have mirror functor $\mathcal{F}^{(L,b)}: \text{Fuk}(M) \to \text{Mod}(Q)$ by a single immersion $L$ and its deformations (from a Lagrangian skeleton of $M$).

- $U := \mathcal{F}^\mathcal{L}(L, b)$ gives $\mathcal{F}^U = \text{Hom}_{\text{Tw}(X)}(U, -): \text{Tw}(X) \to \text{Mod}(Q)$. 

![Diagram](image)

Symplectic geometry

$\text{Fuk}(M)$

$\mathcal{F}^\mathcal{L}$

$\mathcal{F}^L$

Algebraic geometry

$\text{Tw}(\mathcal{X})$

Representative theory

$\text{dg-mod}(\mathcal{A})$
Ongoing work:
Use framed Lagrangian $F \cup S$ to produce the ADHM quiver and sheaves.
Quivers and machine learning

Remarkably, quiver representation is the key object in Deep learning.

Fix $\gamma \in \mathbb{C}Q$ that starts and ends with $i_{in}, i_{out}$.

Have a canonical linear function $L_{\gamma,w} : V_{i_{in}} \rightarrow V_{i_{out}}$ associated to each $w \in R(Q),$ by composing arrow linear maps along $\gamma$.

Representation learning:
Given $K \subset V_{i_{in}}$ and a continuous function $f : K \rightarrow V_{i_{out}}$ (statistically given), minimize the cost function $|L_{\gamma,w} - f|^2 : R(Q) \rightarrow \mathbb{R}$ by taking a stochastic gradient descent in $R(Q)$.

$\Rightarrow$ quiver representation that gives the best linear approximation.
Insight from neural network:

to get **non-linear approximation**, introduce non-linear `activation functions' at vertices.

Some sigmoid functions compared. In the drawing all functions are normalized in such a way that their slope at the origin is 1.

Compose with these activation functions and obtain **network function**

\[ f_{\widetilde{\gamma},w} : V_{i_{\text{in}}} \rightarrow V_{i_{\text{out}}} \]

for every \( w \in \text{Rep}(Q) \).

**Do not occur in usual quiver theory.**
We want to work with moduli space rather than the vector space $R(Q)$ of representations.

$$M(Q) := \frac{[R(Q)]}{G}$$

where

$$G = \prod_{i \in Q_0} \text{GL}(V_i).$$

[Mumford; Kings]

\{\theta - \text{semistable points in } R(Q)\}/G.

GIT or slope stability: choose weights $\theta \in (\mathbb{Z}^{Q_0})^*$. Rep. $V$ is $\theta$-semi-stable if $\theta \left( \overline{\dim V} \right) \leq \theta \left( \overline{\dim V'} \right)$ for $V' \subset V$.

(Can also be understood via symplectic quotient.)

Obstacles of running deep learning over $M(Q)$:

$L_{\tilde{Y}, \tilde{W}}$ is composed with activation functions $\sigma: V_i \to V_i$, which are NOT $G$-equivariant: $\sigma(g \cdot v) \neq g \cdot \sigma(v)$. 
This motivates us to use **framed** quiver representations [Nakajima; Crawley-Boevey; Reineke].

**Framed quiver representation:**
usual quiver representation (linear maps associated to arrows) together with linear maps \( e_i : \mathbb{C}^{ni} \rightarrow V_i \) (called framing).

\[
R^{fr} := \bigoplus_{a \in Q_1} \text{Hom}(V_{t(a)}, V_{h(a)}) \oplus \bigoplus_{i \in Q_0} \text{Hom}(\mathbb{C}^{ni}, V_i) \ni (w, e).
\]

\[\mathcal{M}^{fr} := \{\text{stable points in } R^{fr}\}/G.\]

Think of \( V_i \) as state spaces; \( \mathbb{C}^{ni} \) as spaces for input, output or memory.

**Key:** put the activation functions \( \sigma \) on the framing, rather than on the state spaces \( V \).
Moreover, the moduli space has **universal bundles** $\mathcal{V}_i \rightarrow \mathcal{M}^{\text{fr}}$.
Each arrow $a$ gives a vector-bundle map $\mathcal{V}_{t(a)} \rightarrow \mathcal{V}_{h(a)}$.

Using these, together with metric, can make well-defined learning algorithm over the quiver moduli.
Fix
- a graph $Q$,
- input and output vertices $i_{in}, i_{out}$,
- the representation dimension vector,
- the framing $e_i : F_i \to V_i$,
- Non-linear functions $\sigma_i : F_i \to F_i$,
- the "algorithm" $\tilde{\gamma} \in \mathbb{C}\hat{Q}\{\sigma_1, \ldots, \sigma_N\}$.

Compose arrow maps $V_{t(a)} \to V_{h(a)}$,
framing maps $e_i : F_i \to V_i$,
their metric adjoints $e_i^* : V_i \to F_i$, and
$\sigma_i : F_i \to F_i$,
get a machine function
$f_{\tilde{\gamma}} : V_{i_{in}} \to V_{i_{out}}$ over $\mathcal{M}^{fr}$.

Ex. $A_n$-quiver.

\[ f_{\tilde{\gamma}} = s_{n-1} \circ e_{\text{out}}^* \circ \left( w_{n-1} \circ \left( \cdots \circ e^{(1)} \circ s_1 \circ e^{(1)*} \circ \left( w_1 \circ e_{\text{in}} + b^{(1)} \right) \cdots \right) + b^{(n-1)} \right) \]
To run the algorithm over $\mathcal{M}^{fr}$, need
(1) **vector-bundle metric** $H_i$ on universal bundle $\mathcal{V}_i$;
(2) **Metric** $h_{\mathcal{M}^{fr}}$ on $\mathcal{M}^{fr}$.

**Thm.**
Fix $i \in Q_0$.

- $H_i: R^{fr} \to \text{End}(\mathbb{C}^{d_i})$,

\[(w, e) \mapsto \left( \sum_{h(y) = i} (w_\gamma e_{t(y)})(w_\gamma e_{t(y)})^* \right)^{-1}\]

gives a well-defined metric on $\mathcal{V}_i \to \mathcal{M}$.

- Moreover, assuming $Q$ has no oriented cycle, the Ricci curvature $\sqrt{-1} \sum_i \partial \bar{\partial} \log \det H_i$
of the resulting metric on $\bigotimes_{i \in Q_0} U_i$ defines a Kaehler metric on $\mathcal{M}^{fr}$. 

Moreover, can uniformize with the original Euclidean setup and hyperbolic metric.

Assume $\tilde{n} \geq \tilde{d}$.
Write the framing as $e^{(i)} = (e^{(i)} \ b^{(i)})$.

At points where $e^{(i)}$ is invertible, applying quiver automorphism $\Rightarrow e^{(i)} = \text{Id}$.

This gives a chart:
\[ R_{\tilde{n}-\tilde{d},\tilde{d}}(Q) \leftrightarrow \mathcal{M}^{fr}_{\tilde{n},\tilde{d}}(Q). \]

**Uniformization** gives a unified point of view towards
\[ R_{\tilde{n}-\tilde{d},\tilde{d}}(Q), \mathcal{M}^{fr}_{\tilde{n},\tilde{d}}(Q), \text{ and } \mathcal{M}^{-}_{\tilde{n},\tilde{d}}(Q). \]
Ex. Hyperbolic disc $D \subset \mathbb{CP}^1$.

$D, \mathcal{C}$ and $\mathbb{CP}^1$ can be uniformly understood:

$\mathbb{CP}^1 = \{ \text{lines in } \mathbb{C}^2 \}$.

If we equip $\mathbb{C}^2$ with the quadratic form $y^2 - x^2$, then $D = \{ \text{spacelike lines in } \mathbb{C}^{1,1} \}$.

If we equip $\mathbb{C}^2$ with the quadratic form $H_0 = y^2$, then $\mathcal{C} = \{ \text{spacelike lines in } (\mathbb{C}^2, H_0) \}$.

This is classically well understood for symmetric spaces, and $D \leftrightarrow \mathbb{CP}^1$ is known as **Borel embedding**.

We generalize this to framed quiver moduli.

**Thm.**

There exists spherical, hyperbolic, Euclidean moduli spaces $\mathcal{M}, \mathcal{M}^-, \mathcal{M}^0$ which can be interpolated by a family of metrics.

$$
\left( \rho_i \left( \begin{array}{cc} I_{d_i} & 0 \\ 0 & r \cdot I \end{array} \right) \rho_i^* \right)^{-1}.
$$
Conclusion

- **Quiver near algebras** give a uniform setup for machine learning and quantum computing.
- **Quiver gauge theory** (resolution of local CY singularities) & ADHM provide correspondence between sheaves and quivers.
- The correspondence can be realized by **mirror symmetry**.
- We are using **quivers as fundamental building blocks in both physical and computational models**.

![Diagram]

- Symplectic geometry
  - $\text{Fuk}(M)$
- Algebraic geometry
  - $\text{Tw}(\mathcal{X})$
- Representation theory
  - $\text{dg – mod(\mathcal{A})}$