Equivariant disc potentials for SYZ torus fibrations

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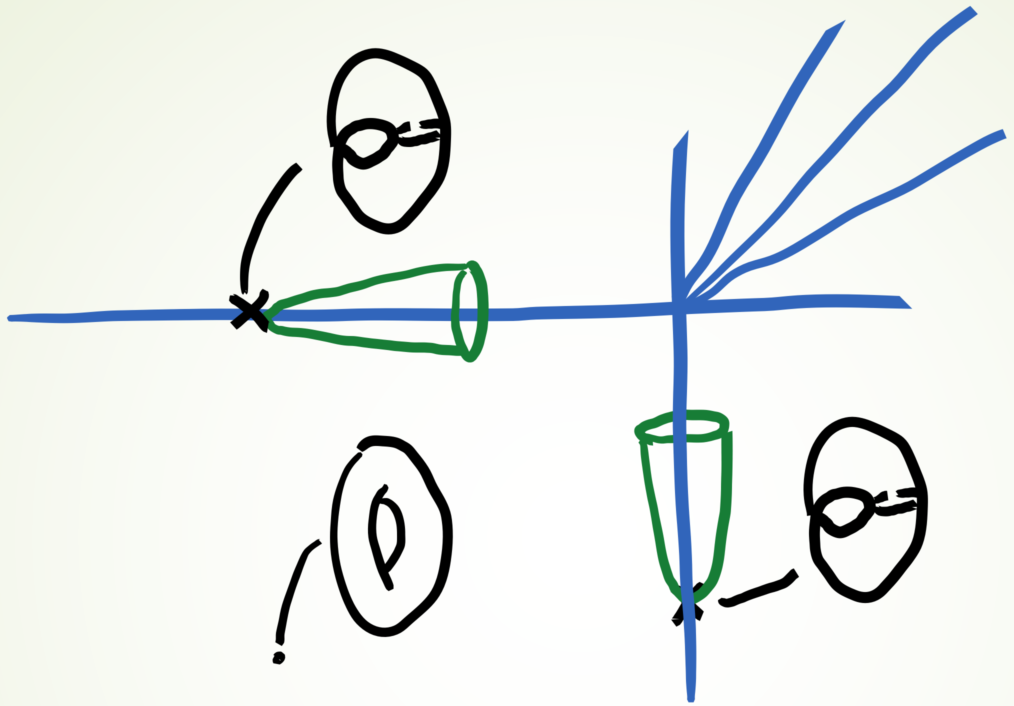
- Isomorphisms and gluing of formal deformation spaces of Lagrangians
- Equivariant construction
- Application to toric Calabi-Yau manifolds

Strominger-Yau-Zaslow: Mirror symmetry is T-duality

- From **homological mirror symmetry conjecture [Kontsevich]**, $\mathcal{O}_p$ in $\check{M}$ is mirror to a Lagrangian brane L in M.
- $Ext^*(\mathcal{O}_p, \mathcal{O}_p) = \Lambda^* \mathbb{C}^n$. So expect $L \cong T^n$.
- $\mathcal{O}_p$ moves in the whole space.
  Thus L should be a leaf of a foliation, or simply a (special) **Lagrangian fibration**.
- Need to complexify. Take (L, ∇).
  ∇ is a flat $U(1)$ connection in $Hom(\pi_1(L), U(1)) = (T^n)^*$.
- $\{\text{flat } U(1) \text{ connections on } L\} \cong T^*$ gives a torus in $\check{M}$.
  Thus $\check{M}$ should admit a **dual torus fibration**.

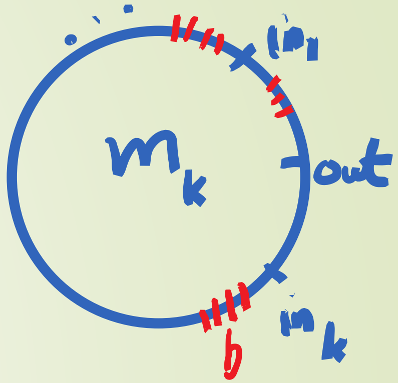
Quantum corrections from **singular fibers**



- Algebraic structure: wall-crossing formula, scattering diagram, cluster algebra. [**Kontsevich-Soibelman, Gross-Siebert, Fock-Goncharov, Gross-Hacking-Keel…**]
- Symplectic interpretation: **Fukaya trick**. Non-trivial A_∞ homotopy between fibers on different sides of a wall, due to **Maslov-zero stable discs** over the wall.
  Family Floer theory [**Fukaya, Tu, Abouzaid**].
- Singular fibers occur. Construct and glue in their deformation spaces. [**Hong-L.-Kim**]
  Glue up a mirror functor. [**Cho-Hong-L.**]

Symplectic geometry of quantum corrections

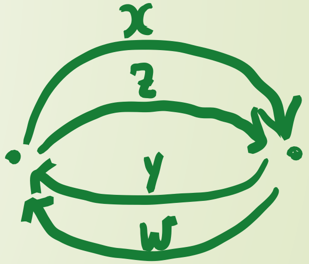
- Floer cohomology $\boldsymbol{HF^*(L,L)}$. [**Fukaya-Oh-Ohta-Ono**]



- Take boundary deformations by degree one chains b (or flat connections). Obstruction term:

$$m_0^b = W_L(b)\cdot[L] + \sum_Y h_Y(b)\cdot Y \in C^*(L).$$

- **Landau-Ginzburg model** $\left(\check{M} \coloneqq \{b:\ h_Y(b)=0\ \ \forall\, Y\}, W(b)\right)$.
  Such b are called *weak bounding cochains*. W is called the disc potential. [**Cho-Oh, Auroux, Fukaya-Oh-Ohta-Ono, Chan-L.-Leung-Tseng, Cho-Hong-L.…**].

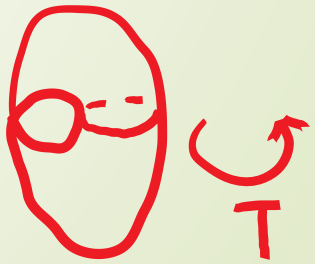


- May have no commutative solutions to $h_Y = 0$.
  Enlarge the base ring [**Cho-Hong-L.**] by quiver algebras.
  Ex. Noncommutative resolution of conifold.

- Need **Novikov field**

$$\Lambda = \left\{\sum_{i=0}^{\infty} a_i T^{A_i} \colon A_i \to +\infty\right\}.$$

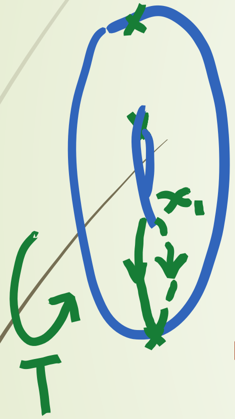
  Mirror $\check{M}$ defined over Λ.



- Today: T-equivariant version of mirror construction.
  In particular: **T-equivariant disc potential W^T**.

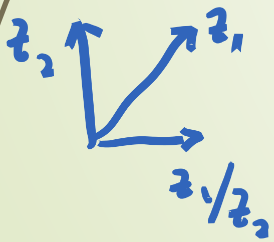
Main results in the equivariant setting

- Let G acts on (X, ω, J), and L is a Lagrangian (with minimal Maslov index 0) preserved by G. Then construct A_∞ structure on $C_f^*(L) \otimes H^*(BG)$, and show that
$$m_k^G(\boldsymbol{\lambda} \cdot x_1, \dots, x_k) = \boldsymbol{\lambda} \cdot m_k^G(x_1, \dots, x_k)\ \forall \lambda \in H^*(BG).$$
- Let $G = T^p$ and suppose T^p acts freely on L^n. Then (for suitable $b = \sum x_i X_i$ degree one generators)
$$m_0^{T^p, b} = m_0^b + \sum_{i=1}^{p} (x_i + h_i)\, \lambda_i$$
where λ_i are the equivariant parameters, $h_i \in \Lambda_+$ are contributed by non-constant Maslov-zero holomorphic discs.



- ⇒ get explicit equation for (local) mirror for $X//G$:
$$x_i + h_i = 0\ \forall\, i. \quad \text{(Weakly unobstructedness)}$$

ex. [**Hori-Vafa; Teleman**] Mirror of $\mathbb{C}^2$: $W = z_2 + z_1/z_2$. Take diagonal $U(1)$ action. $W^{U(1)} = z_2 + e^{x_1}/z_2 + x_1\lambda$. Mirror of the quotient $\mathbb{P}^1$ is given by $z_1 = e^{x_1} = 1$. Homotopy between $\mathrm{CF}^G(L)$ and $CF(L/G)$. [**Woodward-Xu, Daemi-Fukaya**]

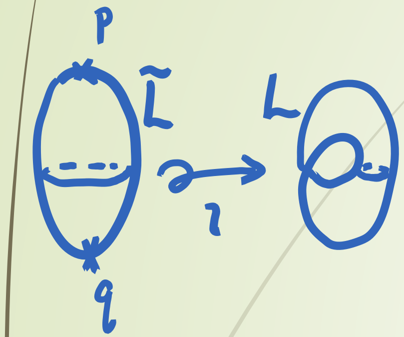


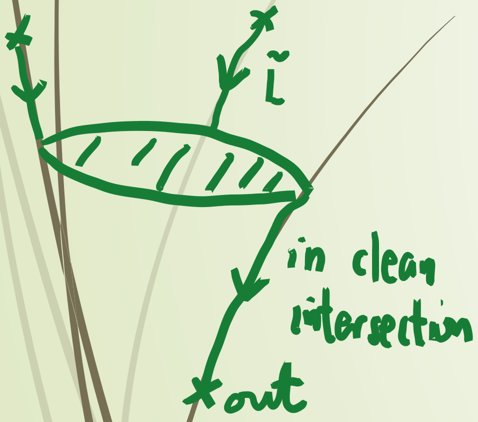
- Suppose L_1, L_2 are preserved by G, and have isomorphism between $(L_1, b_1), (L_2, b_2)$. Then still have **the isomorphism in the equivariant setting**.
- ⇒ equivariant disc potential $W_{L_1}^G$ deduces $W_{L_2}^G$.

Gluing of immersed Lagrangians

with **Cheol-Hyun Cho and Hansol Hong**;
with **Hansol Hong and Yoosik Kim**.

Immersed Lagrangian Floer theory



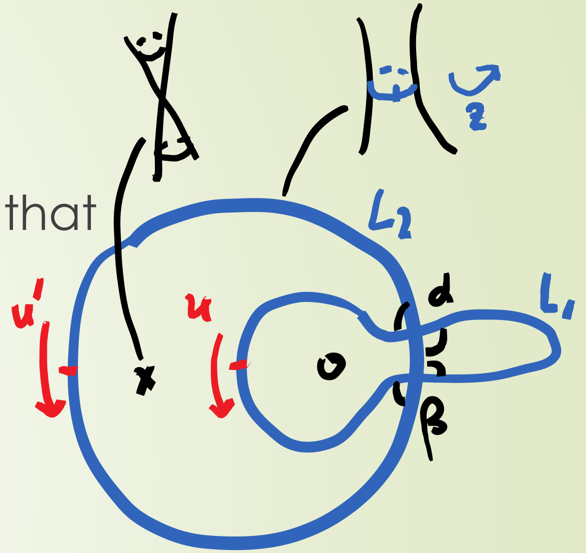


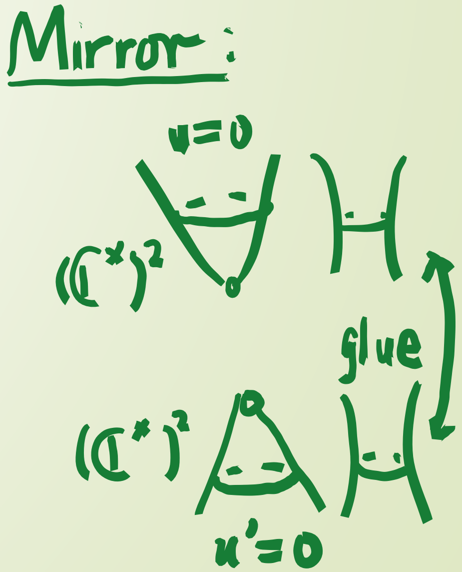
- **[Fukaya-Oh-Ohta-Ono, Seidel, Akaho-Joyce]**
- We use Morse model.
- $\iota: \tilde{L} \to L \subset X$ Lagrangian immersion with self clean intersections.
- $\tilde{L} \times_L \tilde{L} = \tilde{L} \coprod \{(p,q): \iota(p) = \iota(q)\} \coprod \{(q,p): \iota(p) = \iota(q)\}$.
- Holomorphic polygons of (X, L) with boundary lifting to $\tilde{L}$.
  Evaluation at marked point (including corners) targets in a component of $\tilde{L} \times_L \tilde{L}$.

- Fix a Morse function f on $\tilde{L} \times_L \tilde{L}$. Morse chains: $C_f^*(\tilde{L} \times_L \tilde{L})$.
- m_k count pearl trees [**Biran-Cornea; Oh**] (union of flow lines and holomorphic polygons). Use $\boldsymbol{C_f^*(\tilde{L} \times_L \tilde{L}) \to C_{sing}^*(\tilde{L} \times_L \tilde{L})}$ [**FOOO: homological perturbation**].
  Need to add degenerate chains systematically.
- *The maximum point $[\tilde{L}]$ may not be a unit.*
  Enlarge $C_f^*(\tilde{L} \times_L \tilde{L})$ by including **1 (the strict unit)** and F (the homotopy), with
  $$m_1(F) = \mathbf{1} - [\tilde{L}] + h,$$
  $h \in \Lambda_+$ is contributed by Maslov-zero polygons. **[FOOO, Charest-Woodward]**
- Take boundary deformations (L, b) by degree-one chains (or flat connections).

Gluing: isomorphisms of Lagrangians

- Suppose have weakly unobstructed (L_1, b_1) and (L_2, b_2). Find isomorphisms between their formal deformations.
- **Isomorphism**: $\alpha \in CF((L_1, b_1), (L_2, b_2)), \ \beta \in CF\big((L_2, b_2), (L_1, b_1)\big)$ such that
$$m_1(\alpha) = 0, m_1(\beta) = 0; m_2(\alpha, \beta) = \mathbf{1}_{L_2}; m_2(\beta, \alpha) = \mathbf{1}_{L_1}.$$
- Solve the equations and get the gluing formula between b_i.
Theorem [Cho-Hong-L.]: fixing these isomorphisms, there exists a canonical functor from $Fuk(X)$ to the glued category.
- **Lemma**: The disc potential automatically matches under isomorphism:
$$W_{L_1}(b_1) = W_{L_2}(b_2)$$
if (L_1, b_1) is isomorphic to (L_2, b_2).
- A uniform approach to construct mirrors.
- **[Seidel]** and **[Pascaleff-Tonkonog]**: gluing for Chekanov and Clifford torus in the exact/monotone setting. $u' = u(1+z)$.
- $u = u' = 0$ is missing!



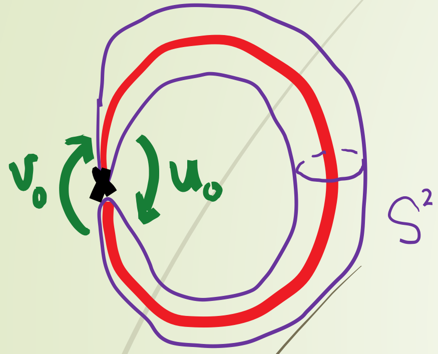


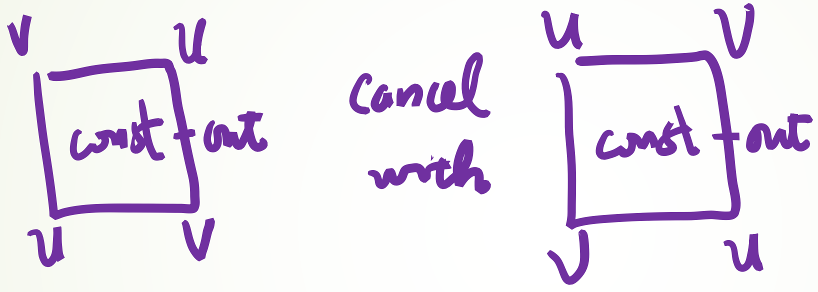
`Dual' of pinched torus

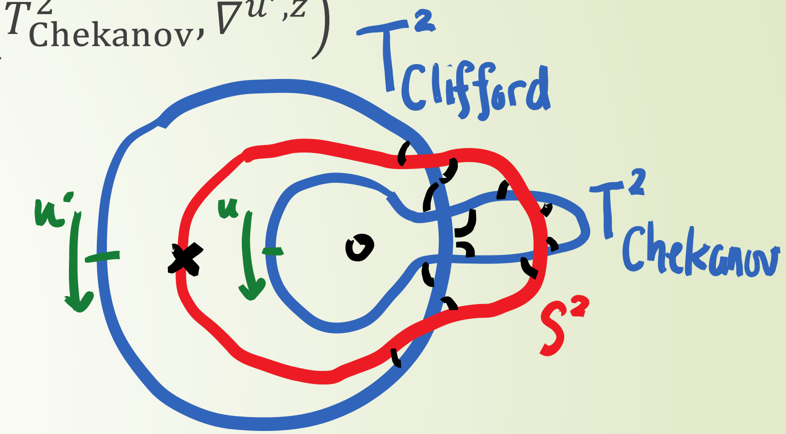
Theorem [Hong-Kim-L.]: The formal deform. $u_0 U + v_0 V$ of immersed S^2 is unobstructed.

$$\left(S^2, u_0 U + v_0 V\right) \cong \left(T^2_{\text{Clifford}}, \nabla^{u,z}\right) \cong \left(T^2_{\text{Chekanov}}, \nabla^{u',z}\right)$$

for $u_0 = u; v_0 = u'; u_0 v_0 = 1 + z.$







- The deformation space $\boldsymbol{\Lambda_0^2} - \{\mathbf{val}(\boldsymbol{u_0 v_0}) = \mathbf{0}\}$ is the (thickened) `dual' of immersed S^2. It fills in the missing $u = u' = 0$.
- **[Dimitroglou Rizell-Ekholm-Tonkonog]** developed a different approach to immersed Floer theory using Legendrian topology.
- Application: **SYZ mirrors of $\boldsymbol{Gr(2,n)}$.**
  - Constructed finitely many suitable immersed Lagrangians
  - showed they are weakly unobstructed
  - computed their disc potentials, using the toric disc potential was computed by [**Nishinou-Nohara-Ueda**]
  - verified that the glued LG mirror agrees with Lie theoretical mirror of Rietsch

Equivariant construction

with **Yoosik Kim and Xiao Zheng**

Motivation

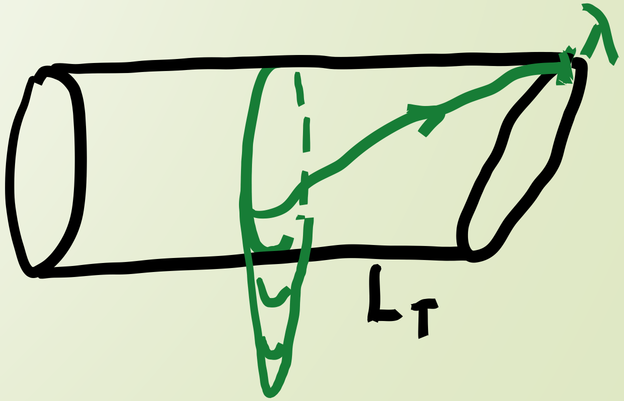
- Let G acts on (X, ω, J).
- **[Hori-Vafa; Teleman conjecture]:** G-equivariant mirror is a holomorphic fibration. The mirror of symplectic quotient $X//G$ is a fiber.
- For toric Calabi-Yau manifold, the Aganagic-Vafa branes have very interesting $\mathbb{S}^1$-equivariant open Gromov-Witten invariants. **[Katz-Liu, Graber-Zaslow, Fang-Liu-Zong, Fang-Liu-Tseng…]**
- Have SYZ construction for toric CY manifolds [**Chan-L.-Leung**] and have computed their open GW invariants [**Chan-Cho-L.-Tseng**].
- Want to understand the relation between the two.
- Ex. compact toric manifold. Equivariant quantum cohomology is mirror to the Jacobian ring of

$$W = W^{\text{non-equiv.}}(z_1, \dots, z_n) + \sum_{i=1}^{n} \lambda_i \log z_i \,.$$

- $W^{\text{non-equiv.}}(z_1, \dots, z_n)$ equals to the disc potential **[Cho-Oh, Fukaya-Oh-Ohta-Ono]**. How about the log terms?

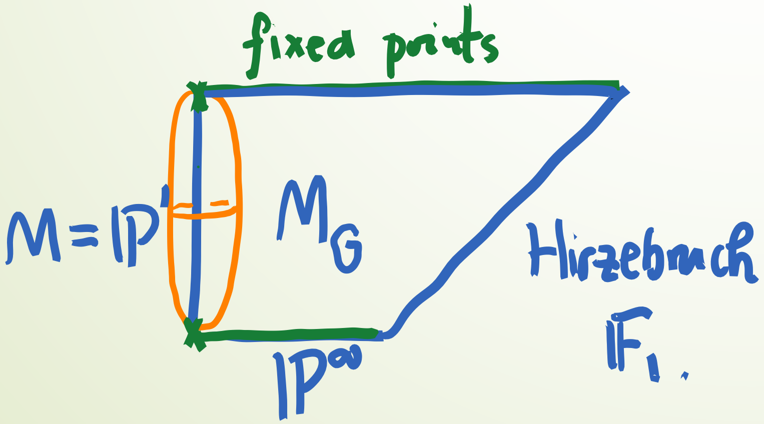
Approaches to equivariant Floer theory

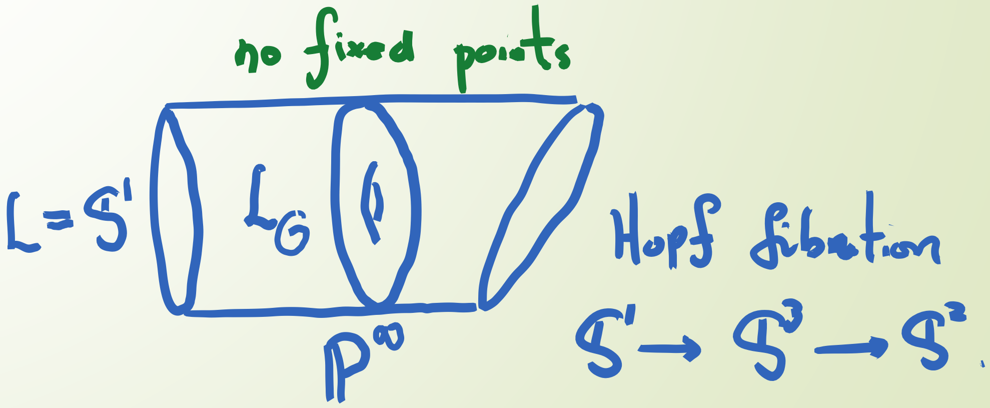
- [**Seidel-Smith**] studied $\mathbb{Z}_2$-equivariant Floer theory in the exact setting. They found a method of studying $HF_{\mathbb{Z}_2}(L_1, L_2)$ by family Morse theory.
- [**Hendricks-Lipshitz-Sarkar**] used homotopy method to formulate.
- [**Daemi-Fukaya**] is developing G-equivariant Floer cohomology for divisor complements using differential forms.
- Our approach is closer to Seidel-Smith. We use family Morse model on $L_G \to BG$, combined with fiberwise holomorphic discs.
  Explicit generators, good to compute equivariant disc potential.
- Enlarge the chain space by including $\boldsymbol{\lambda} := \mathbf{1}_L \otimes \lambda$ and$F_L \otimes \lambda$ for all $\lambda \in H^*(BG)$.
  $$m_2(\boldsymbol{\lambda}, x \otimes \lambda') = x \otimes (\lambda \cup \lambda');$$
  $$m_k(\boldsymbol{\lambda}, X_1, \dots, X_{k-1}) = 0 \text{ for } k > 2.$$
- Get an A_∞ algebra over $H^*(BG)$.
  Ex. $H^*(BT) = \mathbb{C}[\lambda_1, \dots, \lambda_k]$.



Borel construction for equivariant cohomology

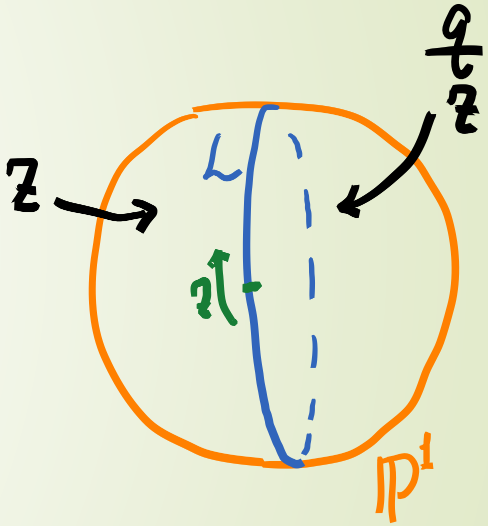
- If G acts freely on M, $H_G(M) = H(M/G)$.
- In general, enlarge the space: $M \times EG$ so that G acts freely.
- EG: a contractible space that admits a free G-action.
- $BG := EG/G$ is known as the classifying space.
  $M_G := (M \times EG)/\mathrm{G}$ replaces M/G.
- $H_G(M) := H(M_G)$.
- $M \to M_G \to BG$ fiber bundle.
- Ex. $G = U(1)$. $EG = \mathbb{S}^{2\infty+1}$. $BG = \mathbb{CP}^{\infty}$. $H(BG) = \mathbb{C}[\lambda]$.





Equivariant superpotential for compact toric varieties

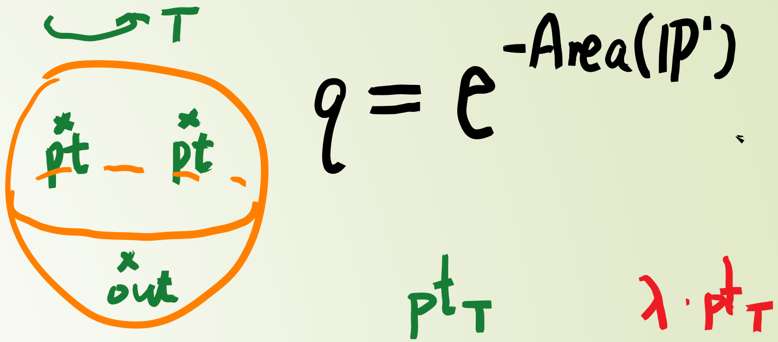
- **[Givental]** and **[Hori-Vafa]** found the Landau-Ginzburg mirrors for compact toric varieties.
- Ex. $\mathbb{P}^1$. $W = z + \frac{q}{z}$. $W_T = z + \frac{q}{z} + \lambda \log z$.
- **[Cho-Oh]** W is a weighted sum of holomorphic discs.
- z: holonomy of flat U(1) connections on L.
- The term $\lambda \log z$ is not even a series at $z = 0$! Hard to interpret as counting!
- Instead, write $z = \exp x$. $W_T = \exp x + q \exp(-x) + \lambda x$. x parametrizes boundary deformations.
- What disc does λx corresponds to?

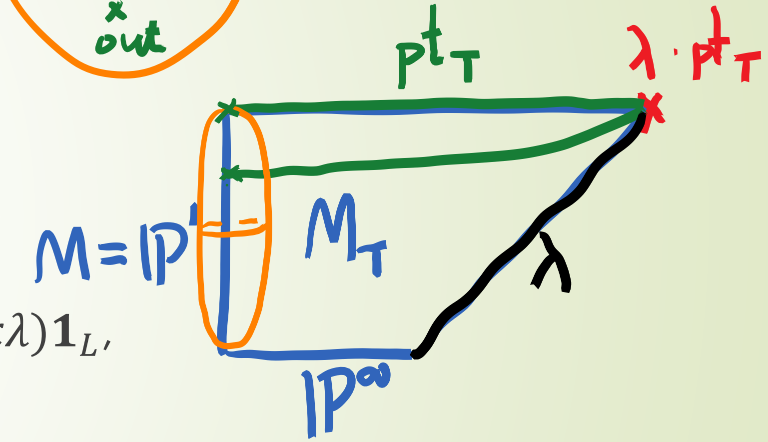


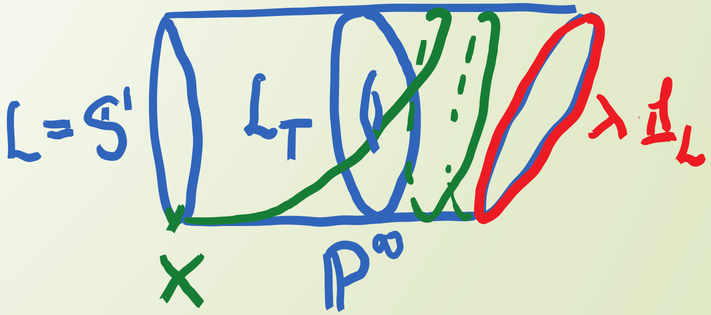
$$q = e^{-\text{Area}(\mathbb{P}^1)}$$

Hint from equivariant QH.

- Example: $\mathrm{H}^*(\mathbb{P}^1) = \langle \mathbf{1}_{\mathbb{P}^1}, [\mathrm{pt}] \rangle$.
- $\mathrm{H}_T^*(\mathbb{P}^1) = \langle \mathbf{1}_T, [\mathrm{pt}]_T \rangle \otimes \mathbb{C}[\lambda]$.
- $[\mathrm{pt}] \cup [\mathrm{pt}] = 0$.
- $[\mathrm{pt}]_T *_q [\mathrm{pt}]_T = q \cdot \mathbf{1}_T + \lambda \cdot [\mathrm{pt}]_T$.
- Want: understand $\left(C_T(L), m_k^T\right)$ similarly.
- Taking x-derivative of $m_0^T = (\exp x + q \exp(-x) + x\lambda)\mathbf{1}_L$, expect $m_1^T(X) = \left((\text{usual}) + \lambda\right)\mathbf{1}_L$.
- X is understood as a hypertorus of L. $\lambda \cdot \mathbf{1}_L$ is understood as the preimage of hyperplane in $\mathbb{P}^\infty$.
- T acts on L freely. $L_T \to BT$ is non-trivial. When we try to take a "section" of X, it is not well-defined and "flow" to the whole $\lambda \cdot \mathbf{1}_L$! "Thus" $m_1^T(X)$ has the term $\lambda \cdot \mathbf{1}_L$.

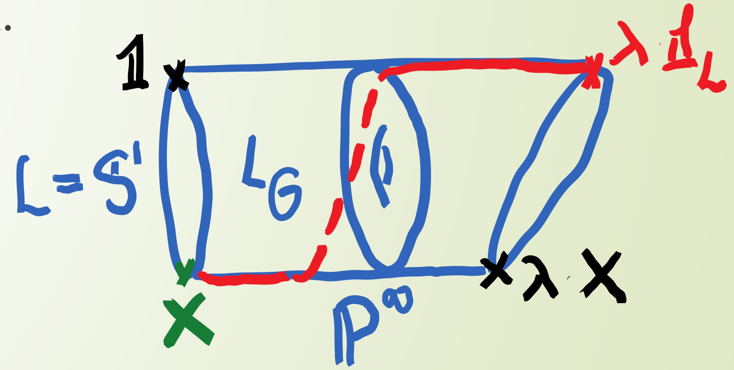






Family Morse for $L_T \to BT$

- **[Hutchings]: family Morse theory** for $F \to M \to B$.
- Take a Morse function on B. The pull-back to M is not Morse.
- Take a function on M, thought as family of functions on fibers. Cannot be Morse on every fiber!
- Assume Morse on fibers over critical points of B. Use these critical points to make a chain complex.
- Apply to $L \to L_G \to BG$. Ex. G= $U(1)$ acts freely on $L = \mathbb{S}^1$. We take a perfect Morse function on $BG = \mathbb{CP}^\infty$.
- Critical points: $\mathbf{1}_L \otimes \lambda^k, X \otimes \lambda^k$ for $k \geq 0$. Equivariant parameters are simply critical points.
- Non-trivial bundle change leads to unique flow line from $X \otimes \mathbf{1}_{BG}$ to $\mathbf{1}_L \otimes \lambda$.
- This accounts for the term $\lambda \cdot \mathbf{1}_L$ in $m_1^T(X)$.

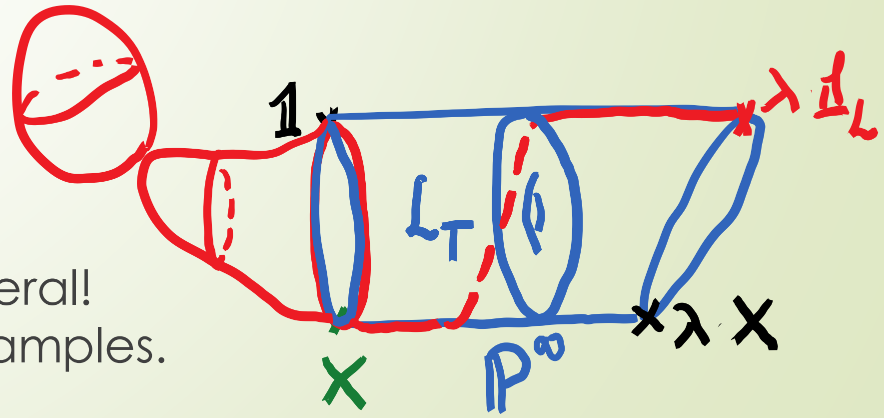


Equivariant disc potential for a compact toric manifold

Theorem [Kim-L.-Zheng]: For a compact semi-Fano toric manifold, L_T is weakly unobstructed, and

$$W_T = \sum_{i=1}^{m} \big(1 + \delta_i(q)\big)\, q^{\beta_i} \exp(\vec{x}, \partial\beta_i) + \sum_{j=1}^{n} \lambda_j x_j \,.$$

- **[Chan-L.-Leung-Tseng 15]**:
  $\big(1 + \delta_i(q)\big)$ are **generating functions of stable bubbled discs**, and they equal to the **inverse mirror map**.
- The second term is the equivariant contribution. Taking $\lambda \to \mathbf{0}$, it recovers the non-equivariant W.
- The first term counts Maslov-two discs.
  The second term counts Maslov-zero discs.
  They are constant discs in this case.
- We have **non-constant Maslov-zero discs** in general!
  Toric Calabi-Yau manifolds provide excellent examples.

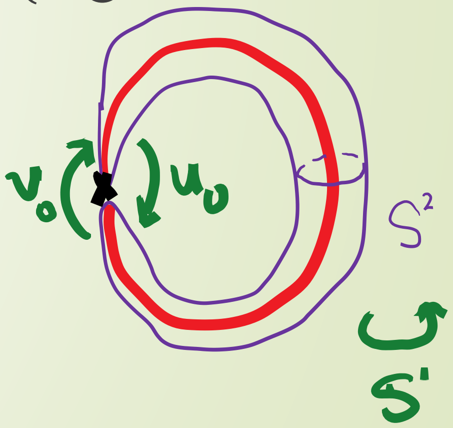


Gluing in the equivariant setting

Theorem [Kim-L.-Zheng]:
Assume L has minimal Maslov index zero, graded with respect to a meromorphic volume form. Suppose T acts on L freely.
If (L, b) is weakly unobstructed, then $(L_T, b \otimes [BG])$ is also `weakly unobstructed':

$$m_0^T = W \cdot \mathbf{1}_L + \sum_i (x_i + h_i) \cdot \boldsymbol{\lambda}_{\boldsymbol{i}}\,.$$

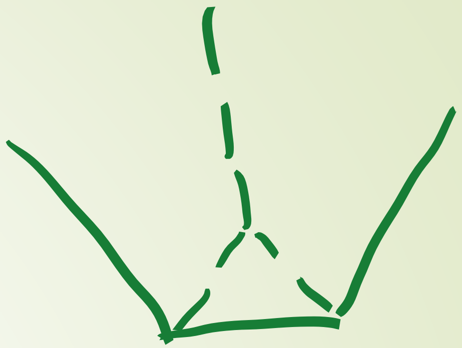
Moreover, if (α, β) is an isomorphism between $(L_1, b_1), (L_2, b_2)$, then $(\alpha \otimes$



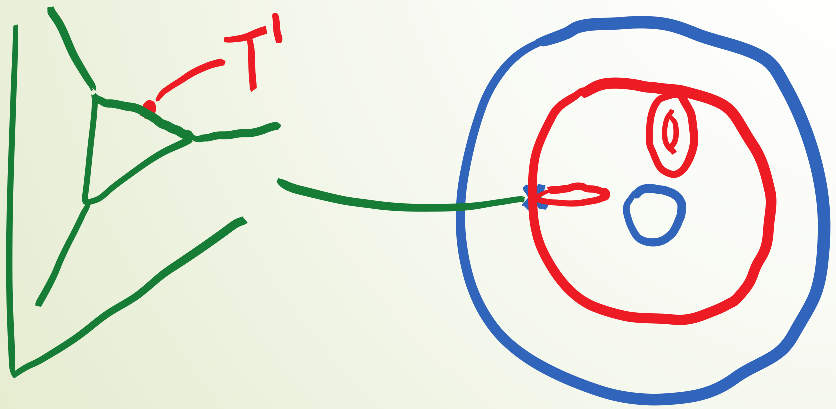
Application to toric CY manifolds

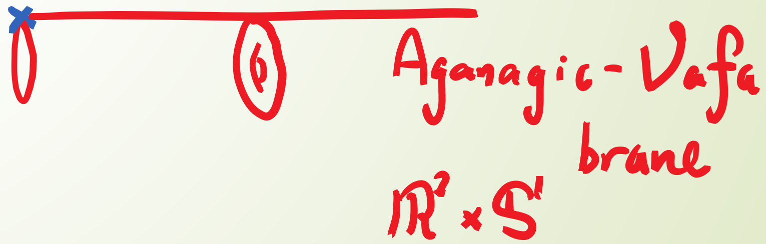
With **Kwokwai Chan, Cheol-Hyun Cho, Naichung Leung and Hsian-Hua Tseng**
Current work with **Hansol Hong, Yoosik Kim and Xiao Zheng**

Toric Calabi-Yau manifolds



- Local building blocks of compact CY.
- Ex. Total spaces of canonical line bundles of toric Fano manifolds. $M = K_{\mathbb{P}^2}$.
- Have **Lagrangian torus fibration** for SYZ. **[Gross; Goldstein]**
  Symplectic T^{n-1} reduction: curves in the reduced plane give Lagrangians.
  Take concentric circles.
- **Aganagic-Vafa branes** correspond to rays. **[Katz-Liu; Fang-Liu-Zong…]**





SYZ mirrors of toric CY manifolds

Theorem [**Chan-L.-Leung 12**]:

The SYZ mirror of $K_{\mathbb{P}^2}$ equals to the local Calabi-Yau threefold

$$uv = \left(1 - 2q + 5q^2 - 32q^3 + \cdots\right) + z_1 + z_2 + qz_1^{-1}z_2^{-1}.$$

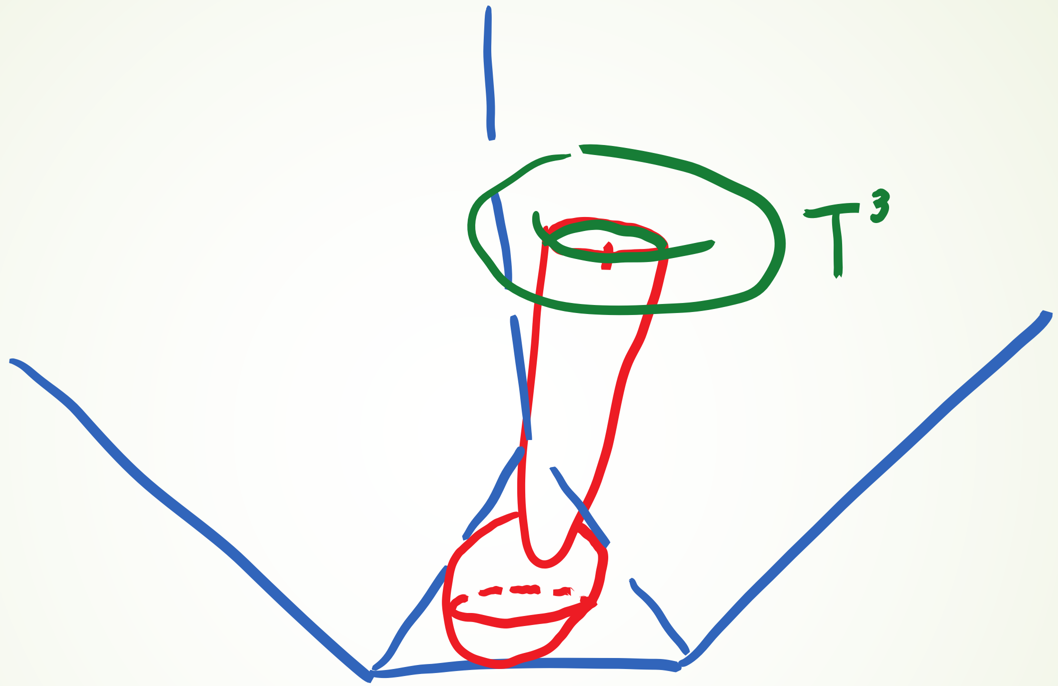
- $\left(1 - 2q + 5q^2 - 32q^3 + \cdots\right)$ are recording stable discs bounded by SYZ fibers.
- q are the Kaehler parameters. z_i are variables on T^* parametrizing flat connections.
- Agrees with the physics result of **[Hori-Iqbel-Vafa]** via the mirror map.

Theorem [**Chan-Cho-L.-Tseng 16**]:

The generating functions $\left(1 - 2q + 5q^2 - 32q^3 + \cdots\right)$ equal to the inverse mirror map.

- RHS is the **normalized slab function of [Gross-Siebert]**.
- [**Abouzaid-Auroux-Katzarkov 16**] carried out the SYZ construction which produces toric Calabi-Yau manifolds as mirrors.

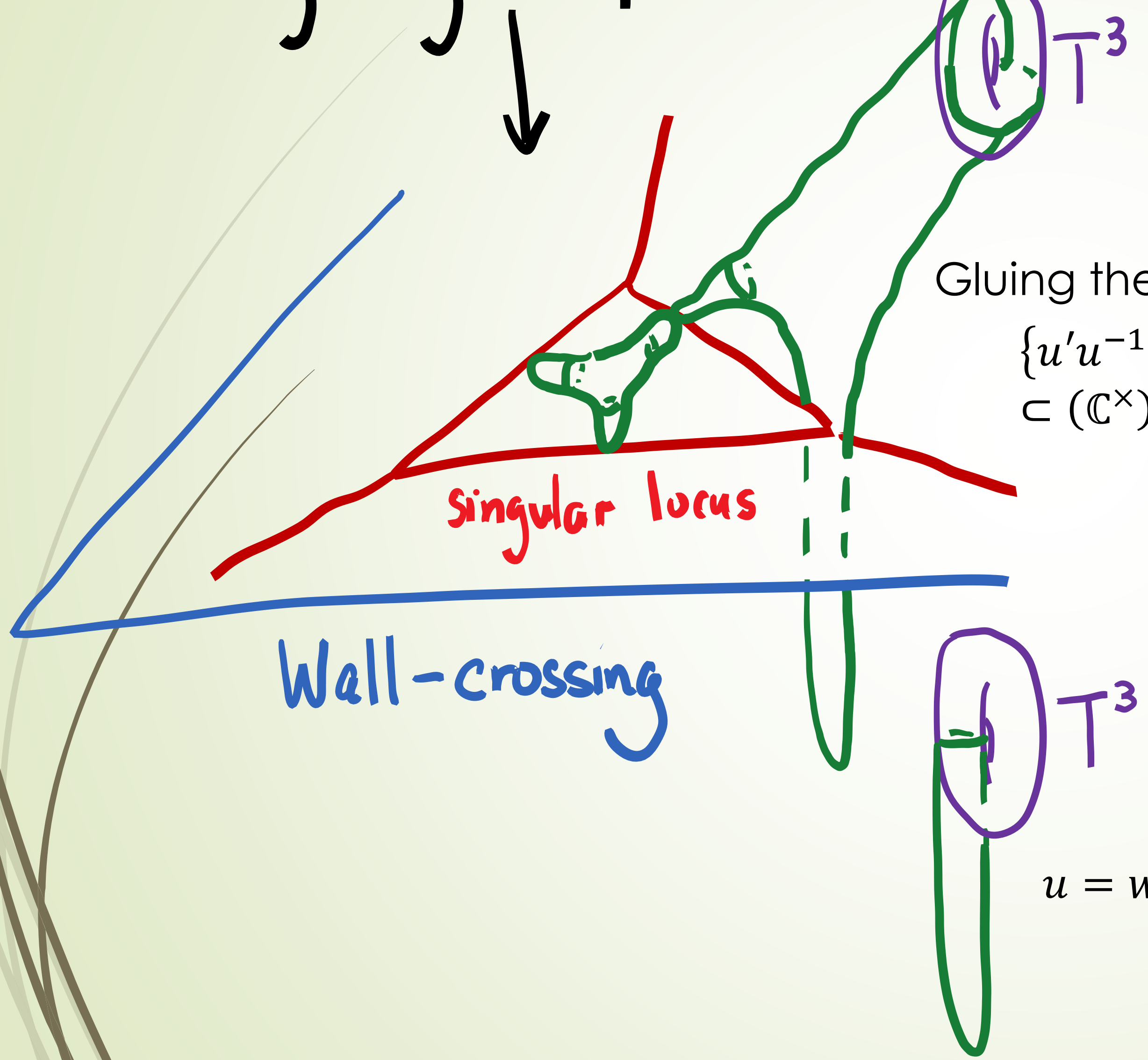
Open GW invariants of SYZ torus fibers



- $CF^*(L, L)$ has quantum corrections by stable discs.
- Non-triviality of disc countings $(1 - 2q + 5q^2 - 32q^3 + \cdots)$: **sphere bubbling**.
- The disc moduli is **obstructed** and hard to compute directly.
- We show that it is isomorphic to the corresponding moduli of rational curves in the compactification. Thus we can compute them systematically and show that they equal to the inverse mirror map.

Lagrangian fibration

$$u' = w\left(\left(1 - 2q + 5q^2 - 32q^3 + \cdots\right) + z_1 + z_2 + qz_1^{-1}z_2^{-1}\right)$$

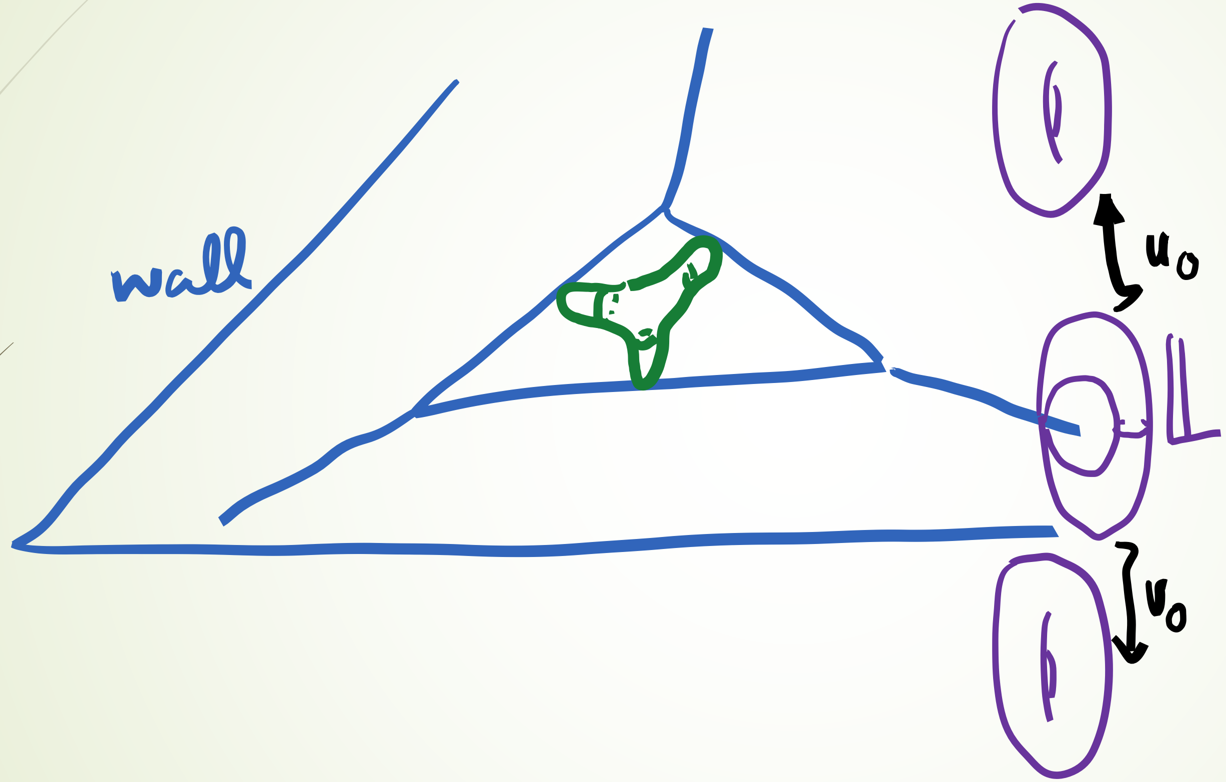


Gluing the two dual tori $\left(\mathbb{C}^\times_{u,z_1,z_2}\right)^3$ and $\left(\mathbb{C}^\times_{u',z_1,z_2}\right)^3$ gives

$$\{u'u^{-1} = \left(1 - 2q + 5q^2 - 32q^3 + \cdots\right) + z_1 + z_2 + qz_1^{-1}z_2^{-1}\} \subset (\mathbb{C}^\times)^4.$$

$u = w$

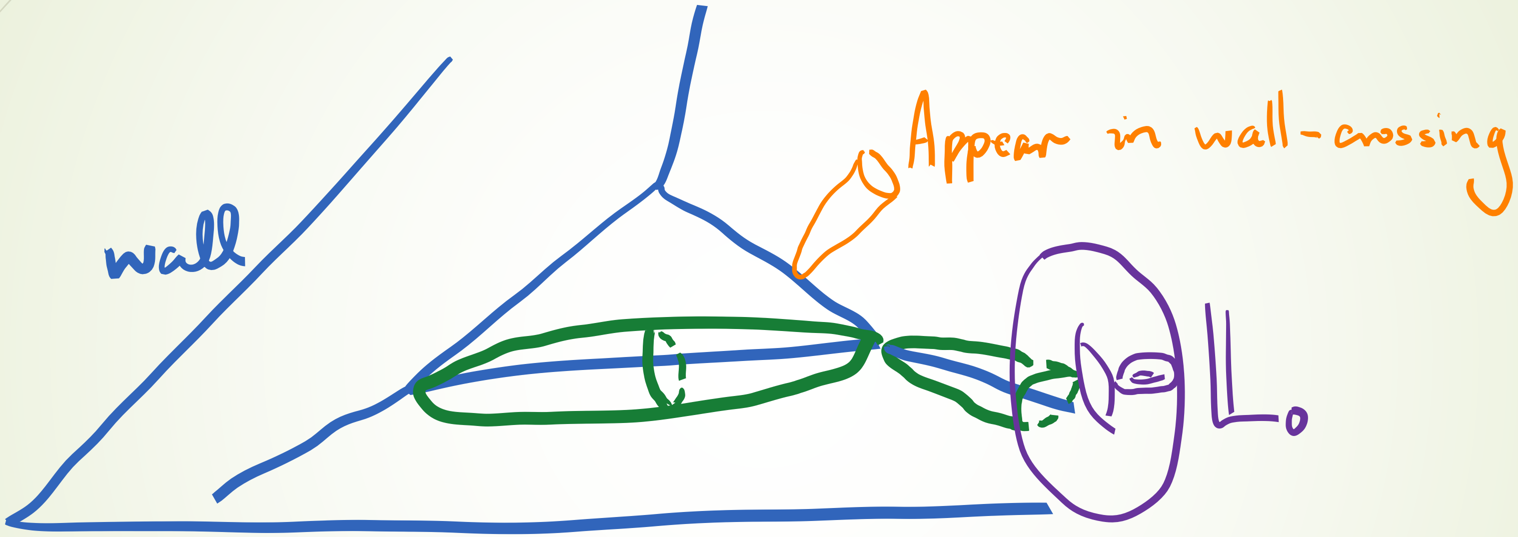
Gluing in dual of singular fibers



L_0 corresponds to $u_0 = v_0 = 0$.

$u_0 . v_0$ can be understood as formal smoothings of the nodal point.

Another type of open GW invariants



- The immersed Lagrangian bounds a different type of discs.
- Main interest of physicists [**Aganagic-Klemm-Vafa…**].
- Not directly seen in the usual Floer theory.
- Use $\mathbb{S}^1$-equivariant $m_0^{\mathbb{S}^1,b}$ of $\boldsymbol{L_0}$ to capture these discs.

Equivariant disc potential for $\bar{S}^2 \times T^{n-2}$

Theorem [Hong-Kim-L.-Zheng current]:
The immersed Lagrangian $\boldsymbol{L_0}$ in a toric CY manifold is unobstructed.

The $\mathbb{S}^1$-equivariant disc potential of the immersed Lagrangian $\boldsymbol{L_0} \subset K_{\mathbb{P}^2}$ equals to

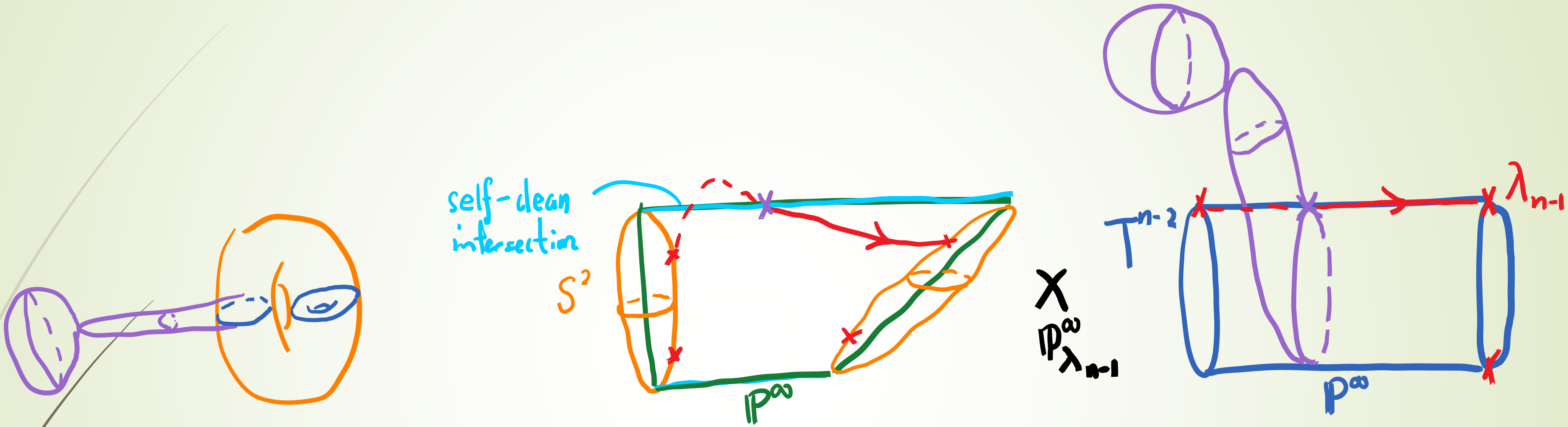
$$m_0^{\mathbb{S}^1,(\boldsymbol{L_0},uU+vV,\nabla^{(z)})} = \boldsymbol{\lambda} \cdot \log f(uv,z)$$

where f is obtained by solving $-w$ from the mirror equation

$$uv = \left(1 - 2q + 5q^2 - 32q^3 + \cdots\right) + z + w + q\, zw.$$

- Can formulate and compute for all toric Calabi-Yau manifolds.
- Recall that $m_0^{\mathbb{S}^1,(T,xX,\nabla^{(z,u)})} = \boldsymbol{\lambda} \cdot x$. $-w = e^x$ in the above notation.
- Set $u = v = 0$. $\log f = \sum k_{d,p}\, q^d z^p$ is $z\partial_z$ of the physicists potential for Aganagic-Vafa brane.
- [**Fang-Liu**], [**Fang-Liu-Tseng**], [**Fang-Liu-Zong**] formulated the invariants for Aganagic-Vafa brane by localization.
  They proved the genus-zero formula, and proved the BKMP remodeling conjecture in all genus.
- Our method computes stable discs with corners at immersed points, giving a connection with SYZ and Lagrangian Floer theory, and works in any dimension.

What are the counting




```
Series[Log[1 + (((D + z) + Sqrt[1 + 2 D + D² - 4 z^(-1) q + 2 z + 2 D z + z^2] - 1) / 2)] /.
  D → -2 q + 5 q^2 - 32 q^3 + 286 q^4 - 3038 q^5 + 35 870 q^6 - 454 880 q^7, {q, 0, 4}, {z, 0, 4}]
```

$$\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+O[z]^5\right)+\left(-\frac{1}{z}-z+2z^2-3z^3+4z^4+O[z]^5\right)q+$$
$$\left(-\frac{3}{2z^2}+\frac{2}{z}+5z-\frac{27z^2}{2}+27z^3-47z^4+O[z]^5\right)q^2+$$
$$\left(-\frac{10}{3z^3}+\frac{8}{z^2}-\frac{12}{z}-40z+122z^2-\frac{838z^3}{3}+560z^4+O[z]^5\right)q^3+$$
$$\left(-\frac{35}{4z^4}+\frac{30}{z^3}-\frac{65}{z^2}+\frac{104}{z}+399z-\frac{2597z^2}{2}+3210z^3-\frac{28027z^4}{4}+O[z]^5\right)q^4+O[q]^5$$

Thank you!