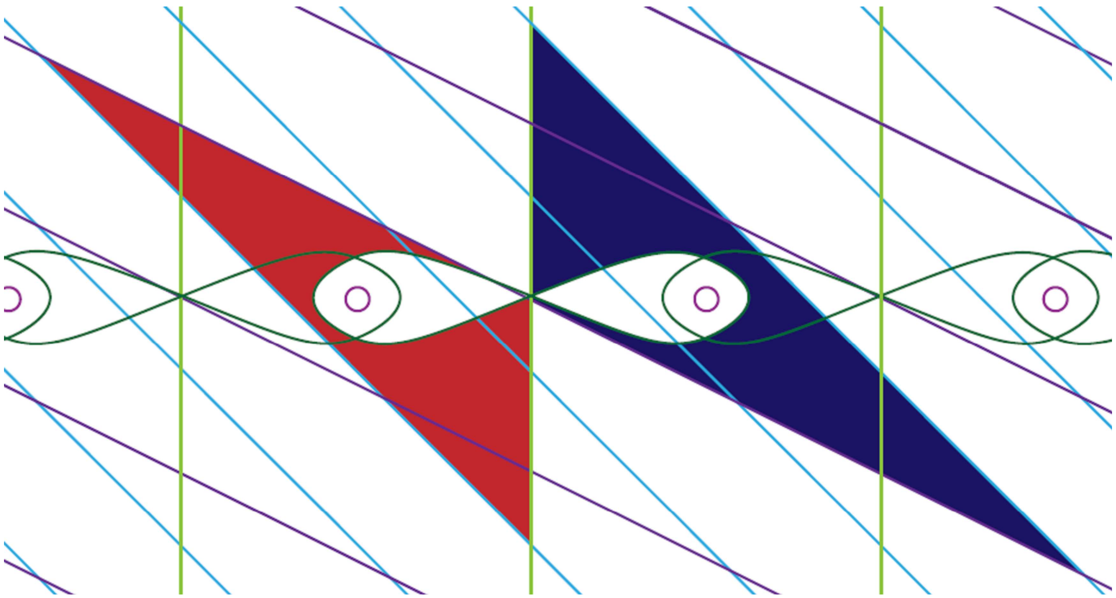
Quiver algebroid stack and its applications

Friday, May 13, 2022 9:13 AM

joint with **Junzheng Nan, Ju Tan**

arXiv:2206.03028.

Quiver representation emerges from Lie theory and mathematical physics. Its simplicity and beautiful theory have attracted a lot of mathematicians and physicists. On the other hand, algebroid stack is a noncommutative generalization of manifold and arises naturally from moduli theory. In this talk, I will explain a version of algebroid stack that can glue together several quiver algebras with different numbers of vertices. I will also explain our original motivation in mirror symmetry and its applications.



I. Quiver algebras and their affine charts
II. Quiver stacks and examples
III. Mirror functor to quiver stacks
IV. Derived equivalence between $\boldsymbol{A}$ and $\mathcal{Y}$

I. Quiver algebras

Q: directed graph. 

$\mathbb{C}Q$: path algebra. trivial paths e_i

R: two-sided ideal of relations.

(Each generator has the same head and same tail.)

Quiver algebra with relations:

$\boldsymbol{A} = \mathbb{C}Q/R$.

Ex. Free algebra $\mathbb{C}\langle x, y\rangle$.

Ex. Noncommutative $\mathbb{C}^2$.

$yx - xy - \hbar$

Ex. A_n resolution.

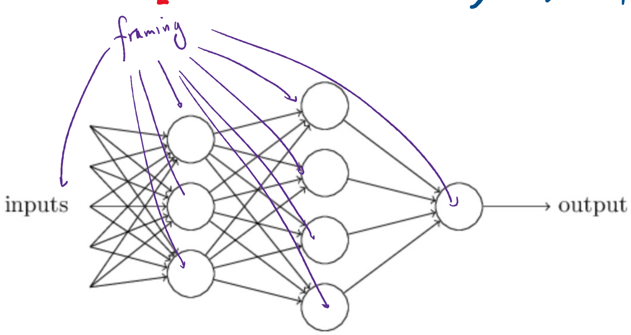
$\mathbb{C}[x,y,t]\#\mathbb{Z}_3$  $y_2x_1 - x_1y_1 - \hbar e_1$ $\widehat{\mathbb{C}^2_\hbar/\mathbb{Z}_3}$

Why quiver?

1. Quiver crepant resolution of Gorenstein singularities.
   [Van den Bergh…]
2. Quiver gauge theory in physics. Deep relation with sheaf theory.
   [Nakajima…]
3. Very interesting representation theory, analogous to Lie theory.
   [Gabriel, Katz…]
4. Framed quiver gives a moduli formulation of computational network!
   [Jeffreys-L., Armenta-Jodoin…]

$\widehat{\mathbb{C}^3/\mathbb{Z}_3} \simeq K_{\mathbb{P}^2}$

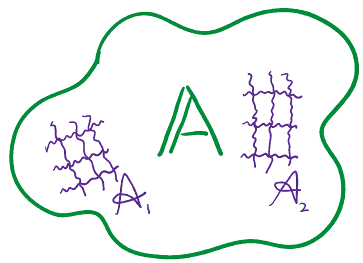
σ: classical analog of quantum measurement.



Understand a quiver algebra $\boldsymbol{A}$ as homogeneous coord. ring graded by Q_0 of the variety

Ex. $\mathbb{C}[x, y, z]$: the homogeneous coordinate ring of $\mathbb{P}^2$.

Want to understand as \`**manifold**':
affine coordinate systems.

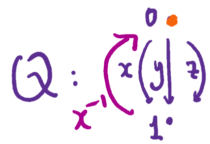


Ex. $\mathbb{P}^2$.
Localize at x: $\mathbb{C}[x^{\pm}, y, z]$ and take:
$\mathbb{C}[x^{\pm}, y, z] \leftrightarrow \mathbb{C}[Y, Z]$:
$Y = x^{-1}y$, $Z = x^{-1}z$;
and
$x = 1$, $y = Y, z = Z$.

Affine charts for quiver algebras?

Ex. Free $\mathbb{P}^2$.

$A = \mathbb{C}Q$.



$$\mathbb{C}Q_{x^{-1}} \leftarrow \mathbb{C}\langle Y, Z\rangle : G_{01}$$

$Y \mapsto x^{-1}y$, $Z \mapsto x^{-1}z$;

G_{10}: $\mathbb{C}Q_{x^{-1}} \to \mathbb{C}\langle Y, Z\rangle$:

$x \mapsto 1$, $y \mapsto Y$, $z \mapsto Z$.

Note:

- G_{10}: $\mathbb{C}Q_{x^{-1}} \to \mathbb{C}\langle Y, Z\rangle$ makes sense as representation or functor, but not as algebra homomorphism;

- $G_{01} \circ G_{10}$: $\mathbb{C}Q_{x^{-1}} \to \mathbb{C}\langle Y, Z\rangle \to \mathbb{C}Q_{x^{-1}}$ is not identity:
  $x \mapsto 1$, $y \mapsto x^{-1}y$, $z \mapsto x^{-1}z$

- Not too bad:
  Set $c(v_0) = e_0$, $c(v_1) = x^{-1}$
  Then
  $G_{01} \circ G_{10}(a) = c(h_a)\, a\, (c(t_a))^{-1}$.

- That is, $G_{01} \circ G_{10}$ is isom. to Id. as rep.

Def. [L.-Nan-Tan] An affine chart of a quiver algebra $\boldsymbol{A}$ is a quiver algebra with a single vertex $\mathcal{A}$ with representations

$G_{01}: \mathcal{A} \to \boldsymbol{A}$, $G_{10}: \boldsymbol{A} \to \mathcal{A}$

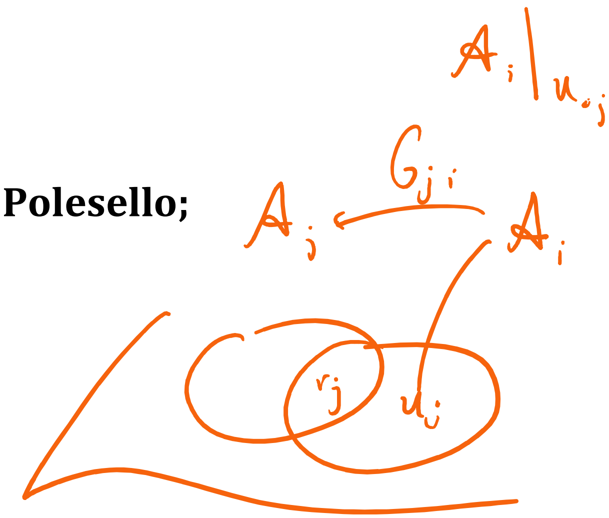
that satisfies

$G_{10} \circ G_{01} =$ Id

and

$G_{01} \circ G_{10}$

$G_{10} \circ G_{01} =$ Id
and
$G_{01} \circ G_{10}(a) = c(h_a) \cdot a \cdot c^{-1}(t_a)$.



II. Quiver stacks and examples

Algebroid stack was defined by **[Kashiwara; O'brian-Toledo-Tong; D'Agnolo-Polesello; Bressler-Gorokhovsky-Nest-Tsygan; Block-Holstein-Wei…]**

We slightly modify the definition to glue quiver algebras.

Def. A quiver stack consists of the following:

(1) *An open cover* $\{U_i : i \in I\}$ *of* B.

(2) *A sheaf of algebras* $\mathscr{A}_i$ *over each* U_i*, coming from localizations of a quiver algebra* $\mathscr{A}_i(U_i) = \mathbb{C}Q^{(i)}/R^{(i)}$.

(3) *A sheaf of representations* G_{ij} *of* $Q_V^{(j)}$ *over* $\mathscr{A}_i(V)$ *for every* i, j *and* $V \overset{\text{open}}{\subset} U_{ij}$.

(4) *An invertible element* $c_{ijk}(v) \in \left(e_{G_{ij}(G_{jk}(v))} \cdot \mathscr{A}_i(U_{ijk}) \cdot e_{G_{ik}(v)}\right)^{\times}$ *for every* i, j, k *and* $v \in Q_0^{(k)}$*, that satisfies*

$$G_{ij} \circ G_{jk}(a) = c_{ijk}(h_a) \cdot G_{ik}(a) \cdot c_{ijk}^{-1}(t_a) \tag{2.13}$$

such that for any i, j, k, l *and* v*,*

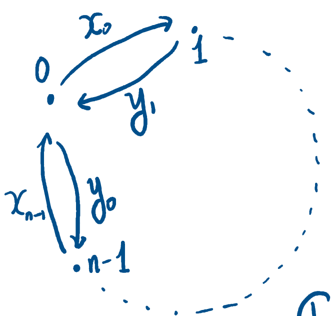
$$c_{ijk}(G_{kl}(v))c_{ikl}(v) = G_{ij}(c_{jkl}(v))c_{ijl}(v). \tag{2.14}$$

In this paper, we always set $G_{ii} = \mathrm{Id}, c_{jjk} \equiv 1 \equiv c_{jkk}$.

c_{ijk} is called gerbe data.

Examples.

<u>Quantum A_{n-1} resolution</u>

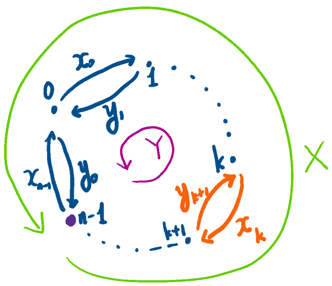


$$\mathbb{C}Q[\hbar] \Big/ \langle y_{i+1}x_i - x_{i-1}y_i - \hbar e_i \ \forall i \rangle$$

This is quiver resolution of quantum $\mathbb{C}^2/\mathbb{Z}_n$.

***k*-th affine chart**: localize at $y_0, \dots, y_k, x_{k+1}, \dots, x_{n-2}$.
$(-1 \leq k \leq n-2)$
Each vertex has a non-zero path from it to v_{n-1}.

$(e_{n-1}, y_0, y_0y_1, \dots, x_{n-2}, x_{n-3}x_{n-2}, \dots)$



$X \circlearrowleft \cdot \circlearrowright Y$

$YX = XY + \tilde{\hbar}$.

$V_{n-1} \longleftarrow$ vertex

$x_{n-2}\cdots x_{k+1}\, x_k\, y_k^{-1}\cdots y_0^{-1} \longleftarrow X$

$y_0\cdots y_k\, y_{k+1}\, x_{k+1}^{-1}\cdots x_{n-2}^{-1} \longleftarrow Y$

Then $\dfrac{YX-XY}{\tilde{\hbar}} \longmapsto n\hbar\, e_{n-1}$.

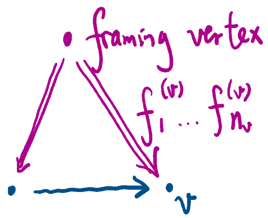
More noncommutative as $n \gg 0$!

$YX \longmapsto y_0\cdots y_k\, y_{k+1}\, x_k\, y_k^{-1}\cdots y_0^{-1}$

$XY \longmapsto x_{n-2}\cdots x_{k+1}\, x_k\, y_{k+1}\, x_{k+1}^{-1}\cdots x_{n-2}^{-1}$

Free version of moduli of framed thin quiver representations.

(free: represent arrows by matrices of any fixed rank)



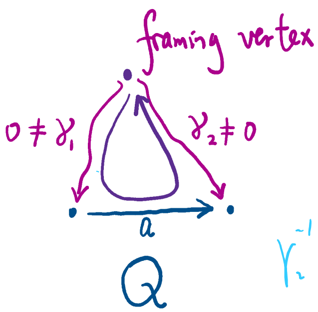
Q (has no oriented cycle)

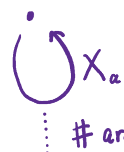
Stability condition:

for every v, has one of incoming arrows a_v of $\hat{Q}$ invertible.

($\Leftrightarrow$ has an invertible path γ from framing to v.)

Affine charts: fix a_v for every v and localize.





$\#$ arrows $= |\hat{Q}_1| - |Q_0|$.

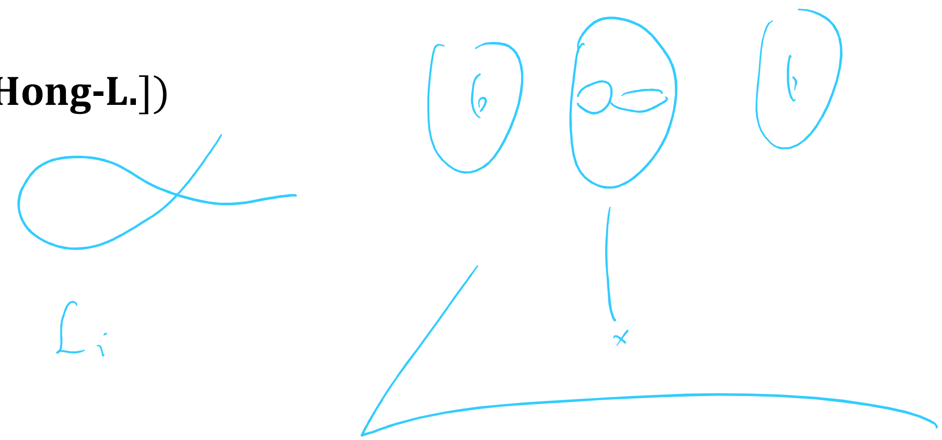
$$\gamma_2^{-1} a \gamma_1 \longleftarrow X_a \quad (a \neq a_v \ \forall v)$$

$$a \longmapsto \begin{cases} 1 & \text{if } a = a_v \\ X_a & \text{otherwise} \end{cases}$$

III. Mirror functor to quiver stacks

We discover affine charts of quiver algebras from mirror construction.

Our recipe of constructing
Symplectic manifold *X* ---> quiver stack $\mathcal{Y}$:
(Extending the previous constructions of [**Cho-Hong-L.**])



1. Take $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$: Lag. imm.
   Typically it comes from: SYZ fibration

2. Construct quiver algebras $\mathcal{A}_i$ of nc deform. of L_i
   (Solve weak MC equations: $m_0^b \sim W \cdot \mathbb{1}_L$)

3. Technical step: (involving gerbe terms c_{ijk})
   Extend Fukaya category $m_k^{\mathcal{Y}}$ over quiver stacks with local charts

4. Choose pre-isomorphisms $\alpha_{ij} \in C^0_{\mathcal{A}_i|_{ij}, \mathcal{A}_j|_{ij}}(\mathcal{L}_i, \mathcal{L}_j)$.

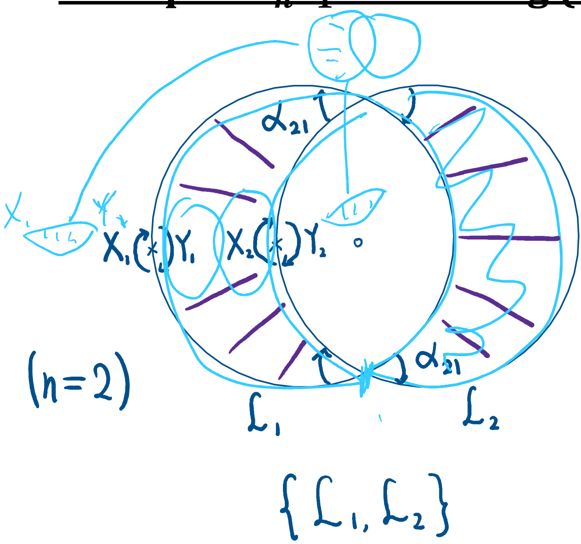
   Key: solve the isomorphism equations for gluing maps:

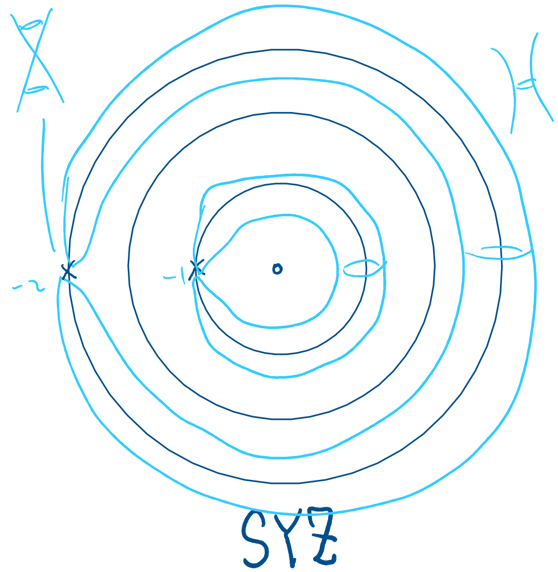
$$m_1^{L_i, L_j, L_i}(\alpha_{ij}) = 0. \quad m_2^{L_i, L_j, L_k}(\alpha_{ij}, \alpha_{jk}) = \alpha_{ik}$$

Theorem: [L.-Nan-Tan] There exists an A_∞-functor:

$F^{\mathcal{L}} : Fuk(X) \to Tw(\mathcal{Y})$

Example: A_{n-1} smoothing (2d).





$w = (z+1)(z+2)$

[**Hong-Kim-L.**]:
(equation for unobstructed deformations)
$\mathcal{A}_i$ are simply $\mathbb{C}[X_i, Y_i]$

Isomorphism equations

$b = b_1$

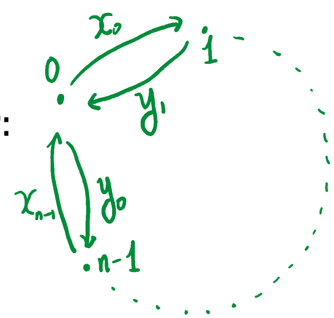
$m_1(\alpha_{21}) = 0$

$$\Rightarrow \begin{cases} Y_1 X_2 = 1 \\ X_1 Y_1 = X_2 Y_2 \end{cases}$$

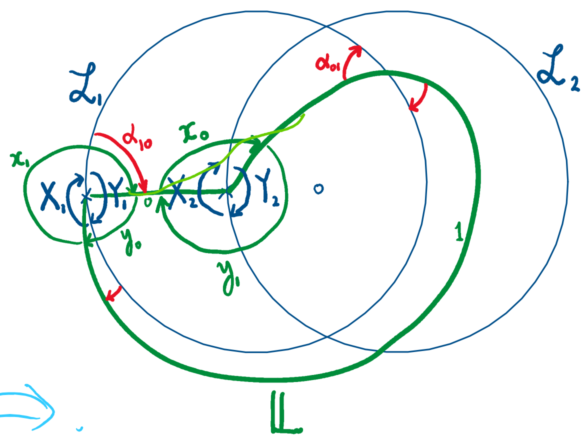
$\mathcal{A}_i$ glue up to A_{n-1} resolution

This local model was applied to construct mirrors of $Gr(2, n)$ **[Hong-Kim-L.]** and del Pezzo surfaces [**L.-Lin-Lee**].

$\mathcal{A}_i$ are affine charts of the A_{n-1} quiver:



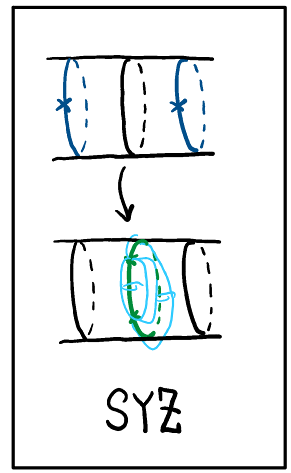
Can be derived from isom. between $(\mathcal{L}_i, b_i)$ and $(\boldsymbol{L}, \boldsymbol{b})$.

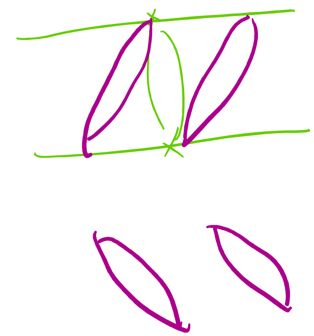


$(n=2)$

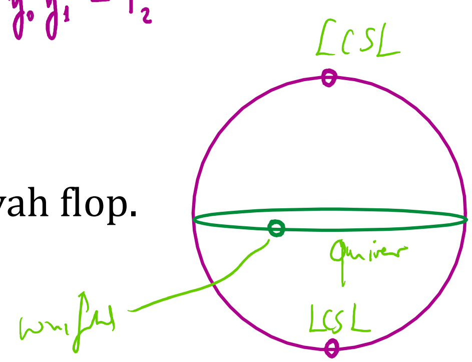
Affine chart A_1 $\begin{cases} x_0 x_1 = X_1 \\ y_0 x_0^{-1} = Y_1 \end{cases}$

Affine chart A_2 $\begin{cases} x_0 y_0^{-1} = X_2 \\ y_0 y_1 = Y_2 \end{cases}$



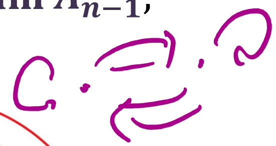


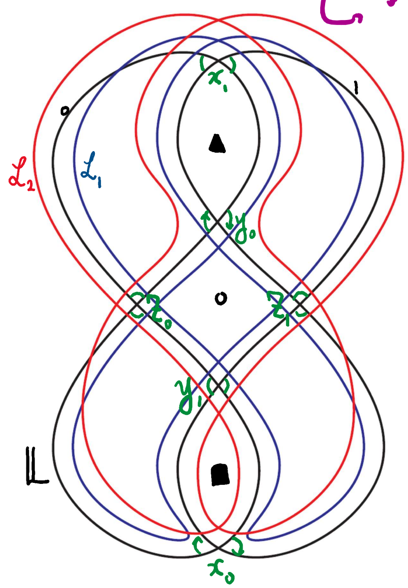
[**Fan-Hong-L.-Yau**]: Use this to understand Atiyah flop.

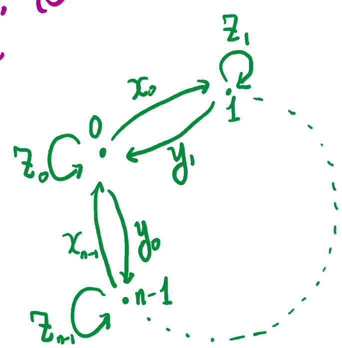


Complex moduli

To obtain **quantum A_{n-1}**,
go to **3d-1d**:

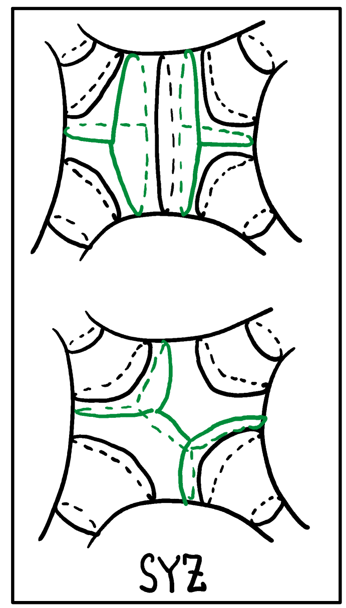




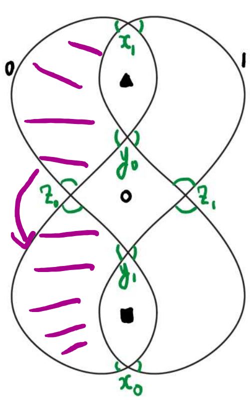


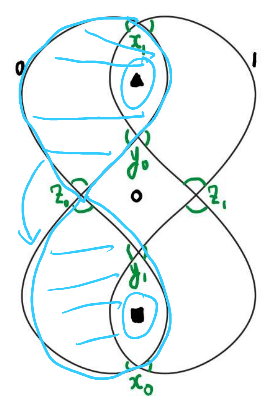
Affine charts derived from
$(\mathcal{L}_i, b_i) \cong (\boldsymbol{L}, \boldsymbol{b})$.

(mirror curve)



$$x_1 y_0 = y_1 x_0 + \underbrace{(T^{\blacksquare} - T^{\blacktriangle})}_{\hbar} e_0 :$$





Summary:

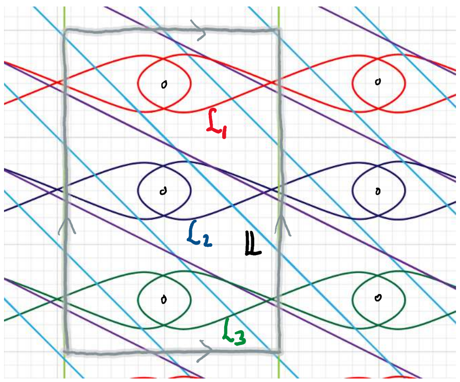
1. We take immersed Lagrangians $\mathcal{L}_0, \dots, \mathcal{L}_n$ and construct the mirror quiver stack $\hat{\mathcal{Y}}$ by using $(\mathcal{L}_i, b_i) \simeq (\mathcal{L}_j, b_j)$.

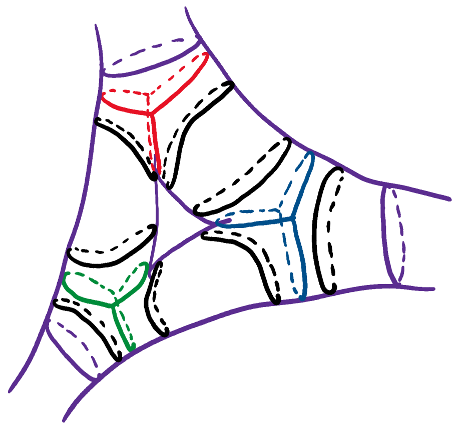
2. Suppose $\mathcal{L}_0 = \boldsymbol{L}$, $\mathcal{A}_i \cong \boldsymbol{A}_{\mathrm{loc}_i}$ $\forall i$ and $\mathcal{A}_i$ has only one vertex.
   $\mathcal{A}_i$ are understood as affine charts of $\boldsymbol{A}$.

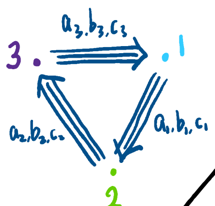
3. Let $\mathcal{Y}$ be glued from $\mathcal{A}_1, \dots, \mathcal{A}_n$.
   We want to compare
   $\mathcal{F}^{\boldsymbol{L}}$: $\mathrm{Fuk}(X) \longrightarrow \text{dg-mod}(A)$
   and
   $\mathcal{F}^{(\mathcal{L}_1,\dots,\mathcal{L}_n)}$: $\mathrm{Fuk}(X) \longrightarrow \mathrm{Tw}(\mathcal{Y})$

Remark: we allow situation that $\mathcal{L}_i$ do not intersect with $\mathcal{L}_j$!
Then $\boldsymbol{L}$ serves as a `middle agent' to find gluing relation between $\mathcal{A}_i$.
Need Novikov ring…

Example: mirror nc local projective plane.

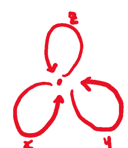




A : 

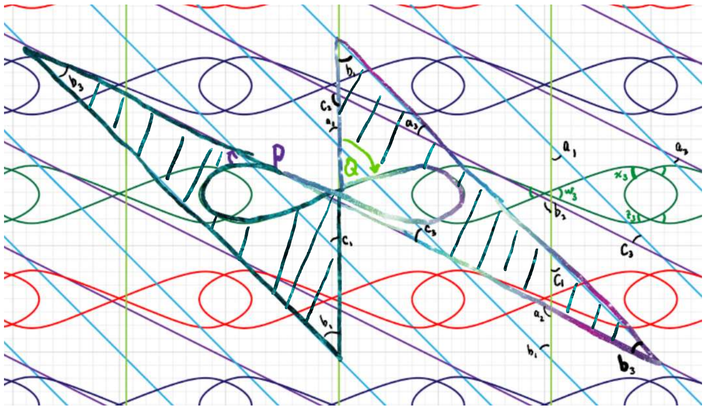
$\Big/ \left\langle \begin{matrix} a_2 b_1 - \hbar\, b_2 a_1 \\ \vdots \end{matrix} \right\rangle$

nc $K_{\mathbb{P}^2}$.

A_i : 

$\Big/ \left\langle \begin{matrix} xy - \tilde{\hbar}\, yx \\ \vdots \end{matrix} \right\rangle$

nc $\mathbb{C}^3$.



Isomorphisms $(\mathbb{L}, b) \simeq (L_i, b_i)$:

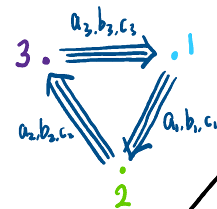
$\alpha_{03} = Q$;

$\alpha_{30} = P \cdot b_3^{-1} b_1^{-1}$.

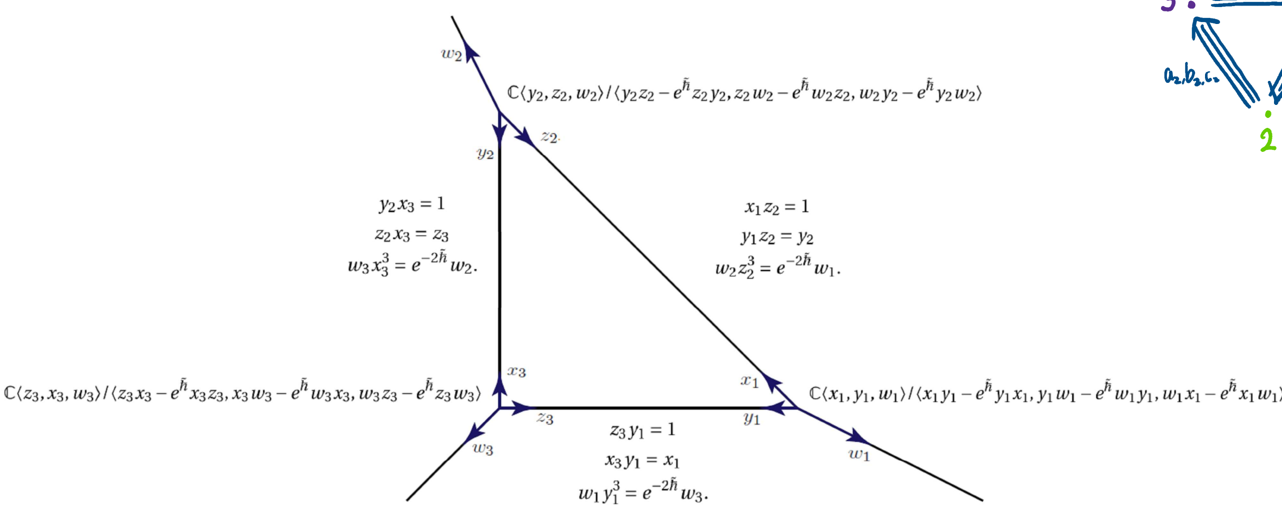
Theorem. [**L.-Nan-Tan**] There exists a quiver stack $\hat{\mathcal{Y}}$ such that
α_{0j}, α_{j0} satisfy the isomorphism equations
$m_1^{\hat{\mathcal{Y}}, b_j, b_k}(\alpha_{jk}) = 0$;
$m_2^{\hat{\mathcal{Y}}, b_j, b_k, b_j}(\alpha_{jk}, \alpha_{kj}) = \mathbf{1}_{L_j}$.

Our method combines SYZ fibers with Seidel's immersed Lagrangian.

`Manifold' description of nc $K_{\mathbb{P}^2}$: $\tilde{\hbar} = 3\hbar$



$\Big/ \Big\langle a_2 b_1 - \hbar\, b_2 a_1 \quad \vdots \Big\rangle$



IV. Derived equivalence between A and $\mathcal{Y}$.

$\boldsymbol{L}$ ------> $\mathbb{A}$

$\{\mathcal{L}_i : i = 1, \dots, n\}$ ------> $\mathcal{Y}$

Have $\mathcal{F}^{\boldsymbol{L}}$ and $\mathcal{F}^{\{\mathcal{L}_i\}}$.

To relate $\mathcal{F}^{\boldsymbol{L}}$ and $\mathcal{F}^{\{\mathcal{L}_i\}}$, consider $\boldsymbol{U} = \mathcal{F}^{\mathcal{L}}(\mathbb{L})$.

$$
\begin{tikzcd}
 & \mathrm{Fuk}(X) \arrow[dl, "\mathcal{F}^{\mathcal{L}}"'] \arrow[dr, "\mathcal{F}^{\mathbb{L}}"] \arrow[d, phantom, "\mathcal{T}"] & \\
\mathrm{Tw}(\mathcal{Y}) \arrow[rr, "\mathrm{Hom}(\mathcal{U}, -)"] & & \mathrm{mod}(\mathbb{A})
\end{tikzcd}
$$

[L.-Nan-Tan]

[Cho-Hong-L.]

[Van den Bergh] for quiver crepant resolutions

Theorem. [L.-Nan-Tan]
There is a natural A_∞ transformation $\mathcal{T}$: $\mathcal{F}^{\mathcal{U}} \circ \mathcal{F}^{\mathcal{L}} \longrightarrow \mathcal{F}^{\mathbb{L}}$.

Assume that we have isomorphism $(\boldsymbol{L}, \boldsymbol{b}) \cong (\mathcal{L}_i, b_i)$.
Then $\mathcal{T}$ has a left inverse.

In concrete cases (ex. conifold), we can compute $\mathcal{F}^{\boldsymbol{U}}$ explicitly and show the equivalence.

We develop a Mayer-Vietoris sequence for algebroid stacks.

$$\ldots \xrightarrow{\partial} H^{\bullet}(C_{\mathscr{A}}) \xrightarrow{j} H^{\bullet}(C_{\mathscr{B}}) \oplus H^{\bullet}(C_{\mathscr{C}}) \xrightarrow{i} H^{\bullet}(C_{\mathscr{D}}) \xrightarrow{\partial} H^{\bullet+1}(C_{\mathscr{A}}) \xrightarrow{j} \ldots$$

THE END