Unimodular Conformal and Projective Relativity and the Compatibility of Causal and Dynamical Space-Time Structures

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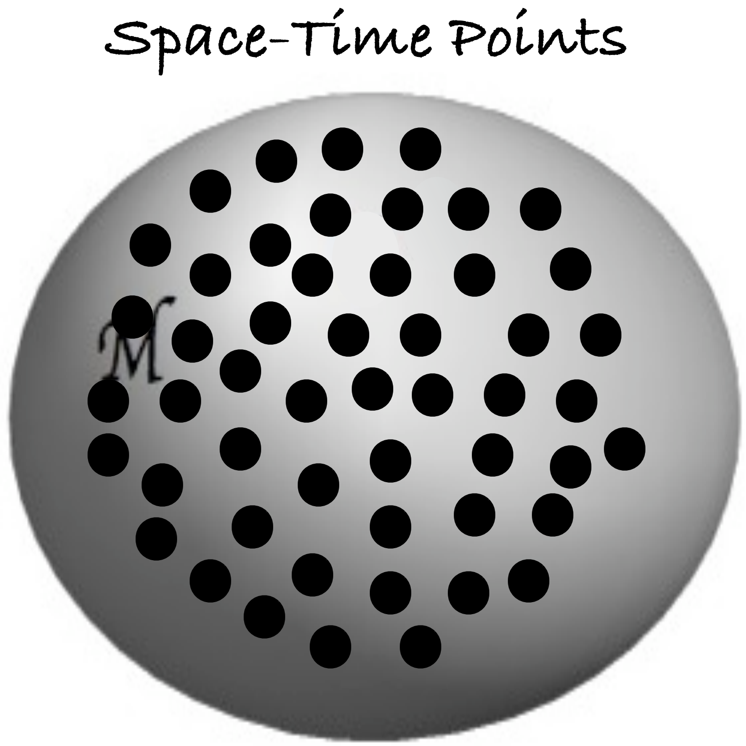
Outline

- Unimodular Conformal and Projective Relativity (UCPR)

- Compatibility of Causal and Dynamical Structures

Standard Formulation of GR

Invariant under the diffeomorphisms group Diff(M) of *all active point transformations*.



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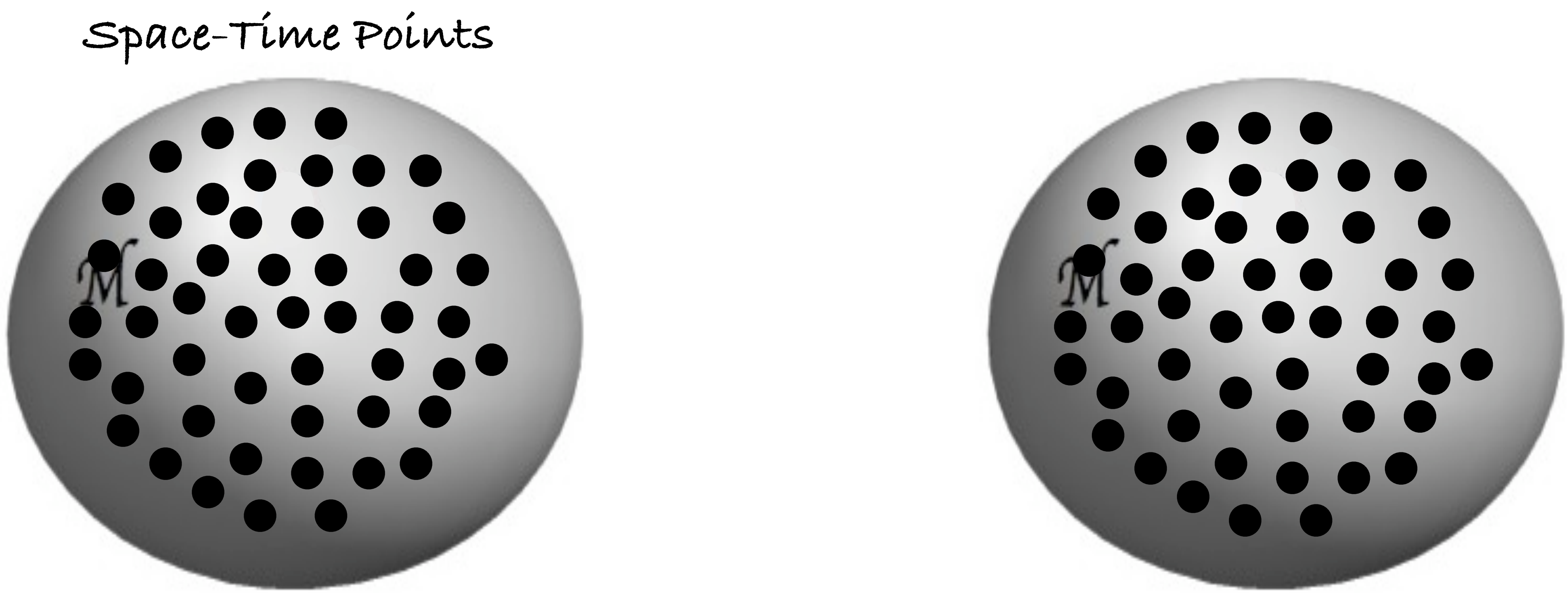

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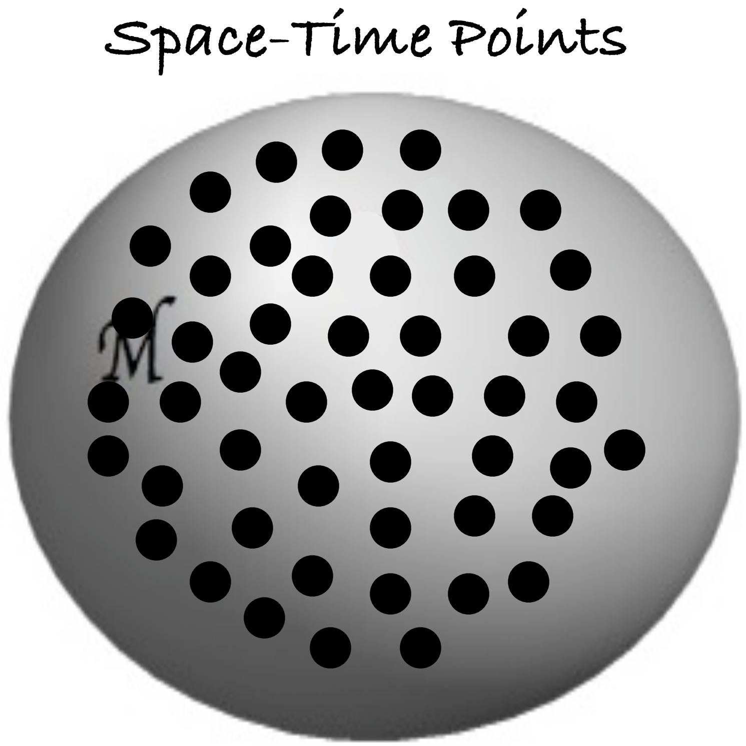
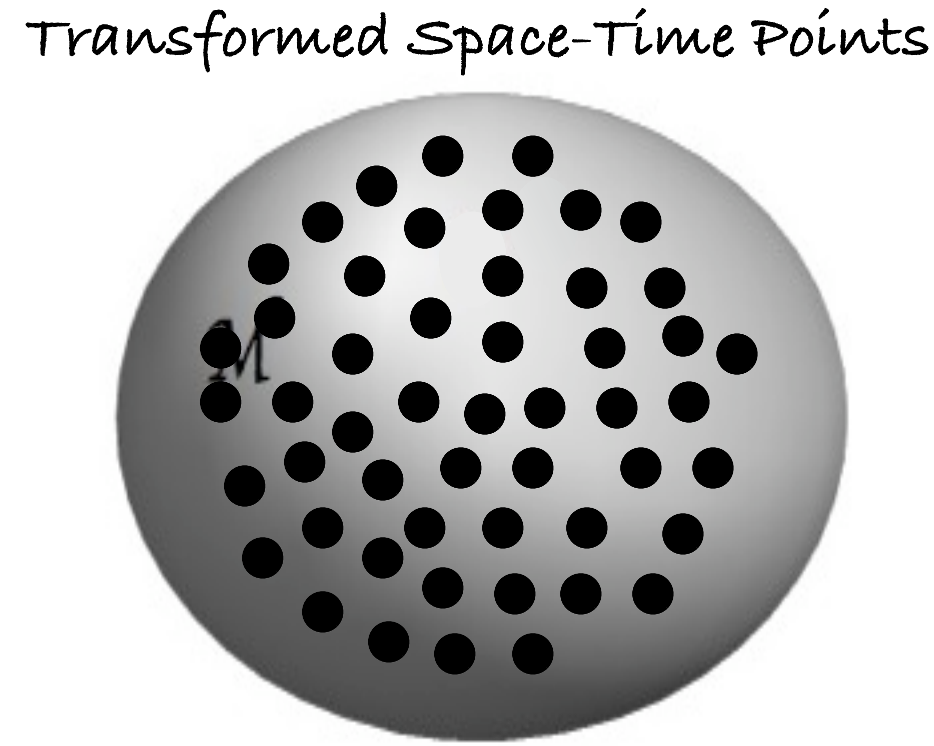
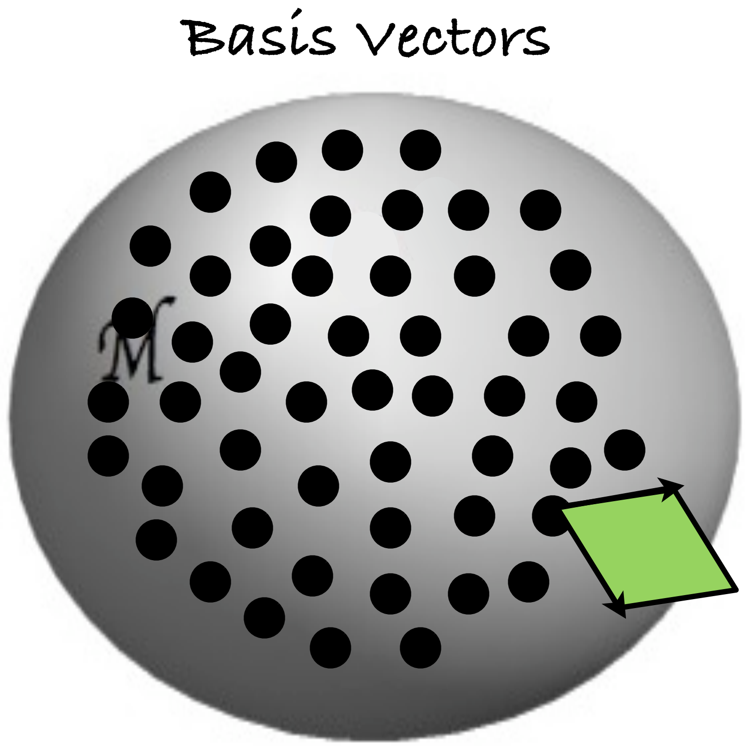
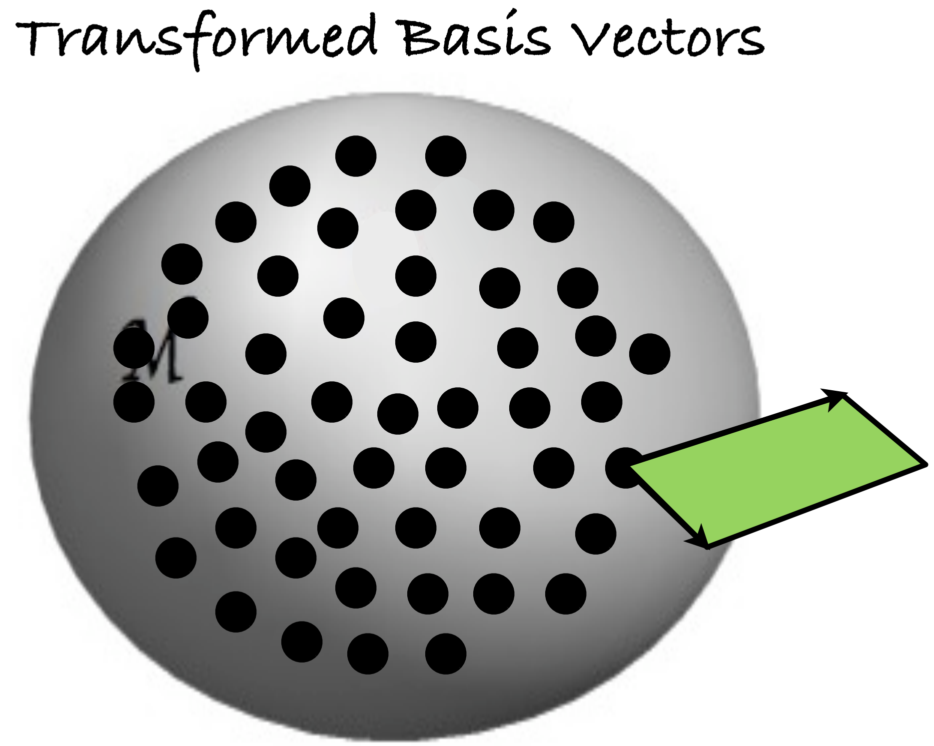




Standard Formulation of GR

Diff(M) induces general linear group GL(n,R) in the the tangent and co-tangent spaces at each point.





Standard Formulation of GR

Metric does it all.

$$\Gamma^{\sigma}_{\mu\nu} = \left\{ {\sigma \atop \mu\nu} \right\} = \frac{1}{2} g^{\sigma\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu})$$

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}$$

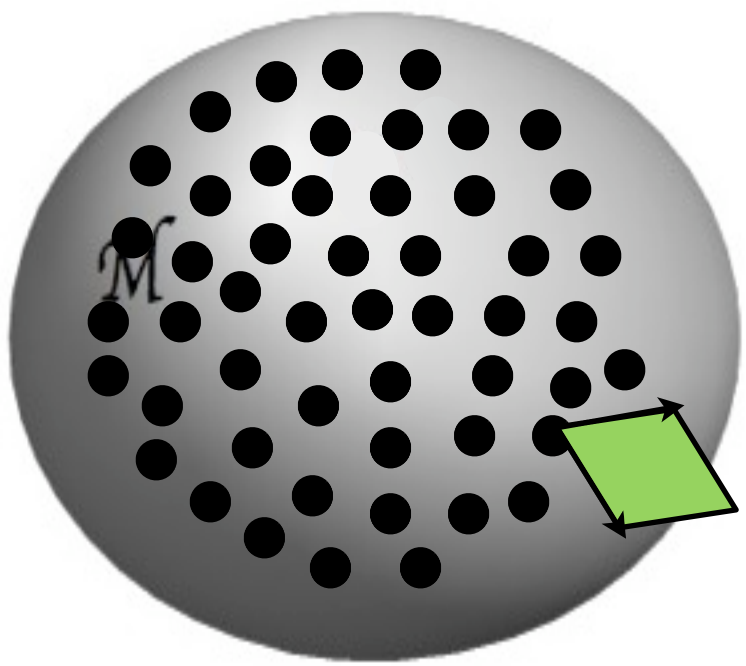
First Order Formalism

It is the *connection* that represents the inertio-gravitational field.

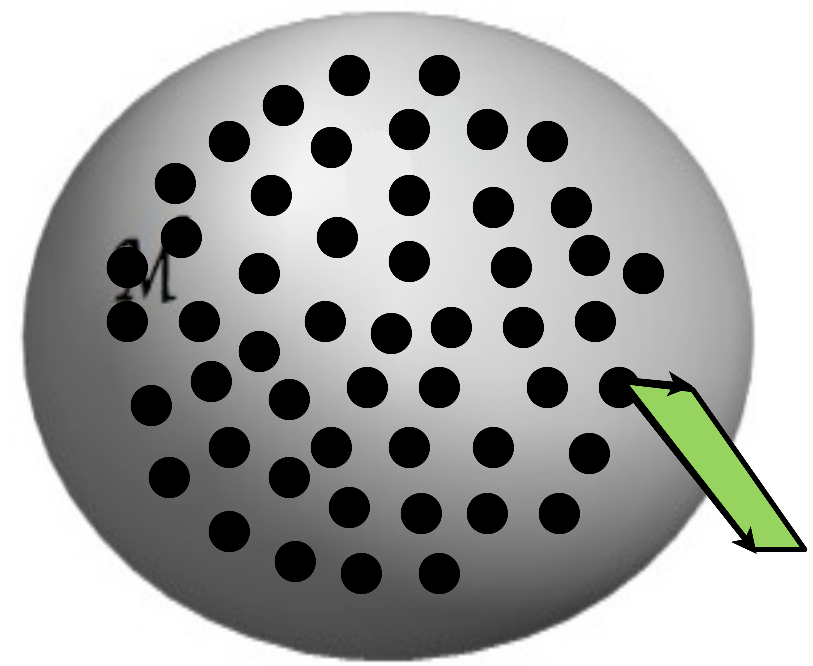
Unimodular Relativity

Invariant under the *unimodular group, which induces a special linear group* SL(n,R) *in the tangent and co-tangent spaces, which preserves the volume element*

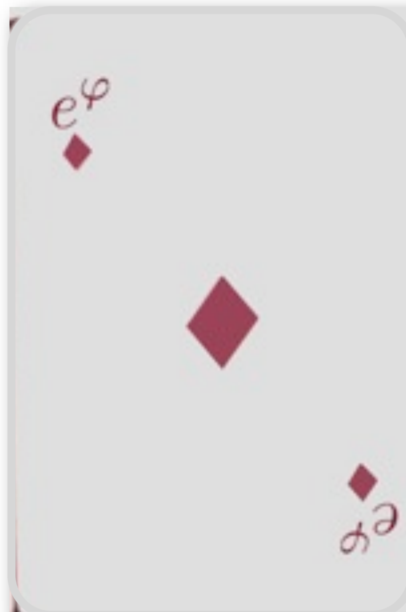
Basis Vectors



Transformed Basis Vectors



Unimodular Conformal and Projective Relativity



Conformal Metric

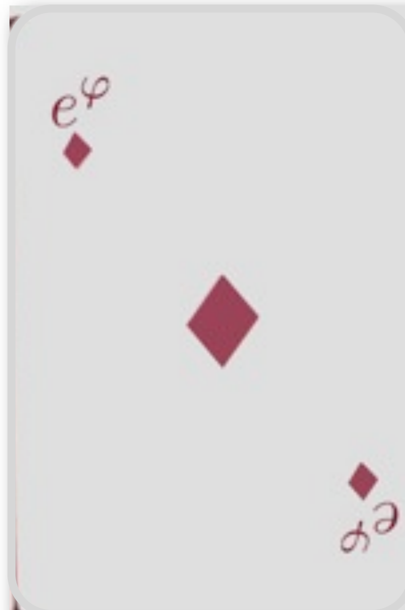
4-Volume Element

Projective Connection

Affine One-Form

Unimodular Conformal and Projective Relativity

Conformal Metric



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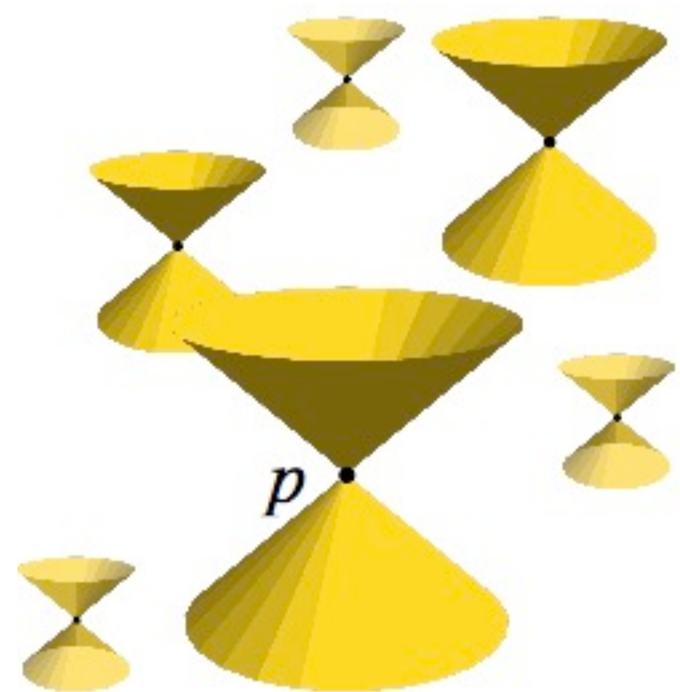
Affine One-Form

Conformal Structure

$$\tilde{g}_{\mu\nu}$$

$$|\tilde{g}_{\mu\nu}| = -1$$



Determines a null-cone at each point, and hence a causal structure on $\mathcal{M}$.

Conformal Structure

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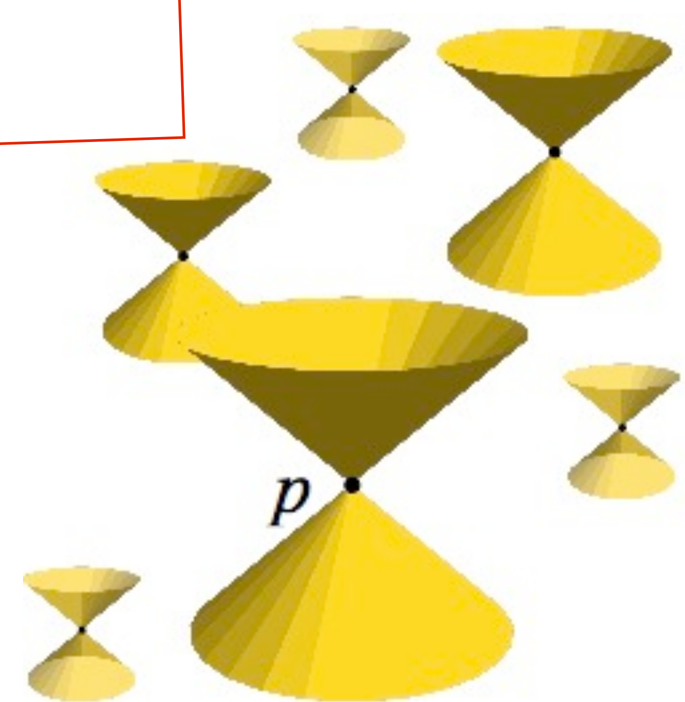
$\boxed{\text{Tensor density in GR, but a tensor in UR}}$



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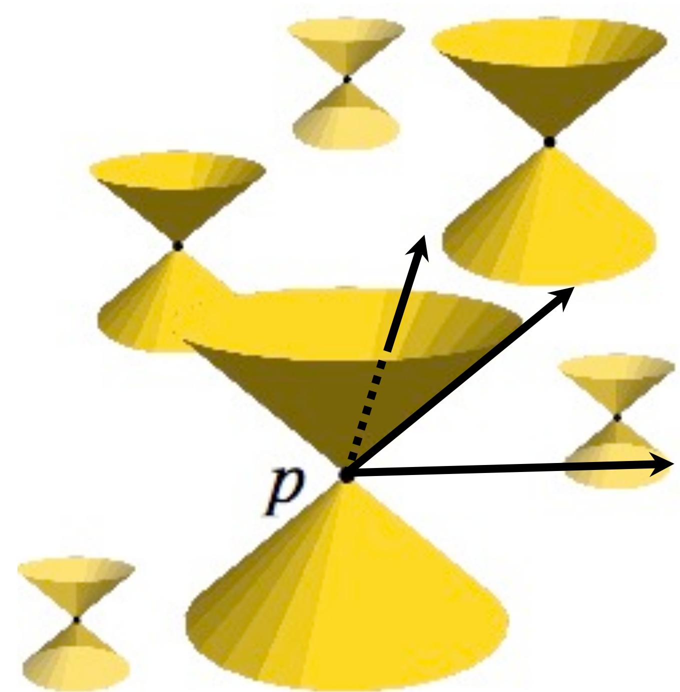
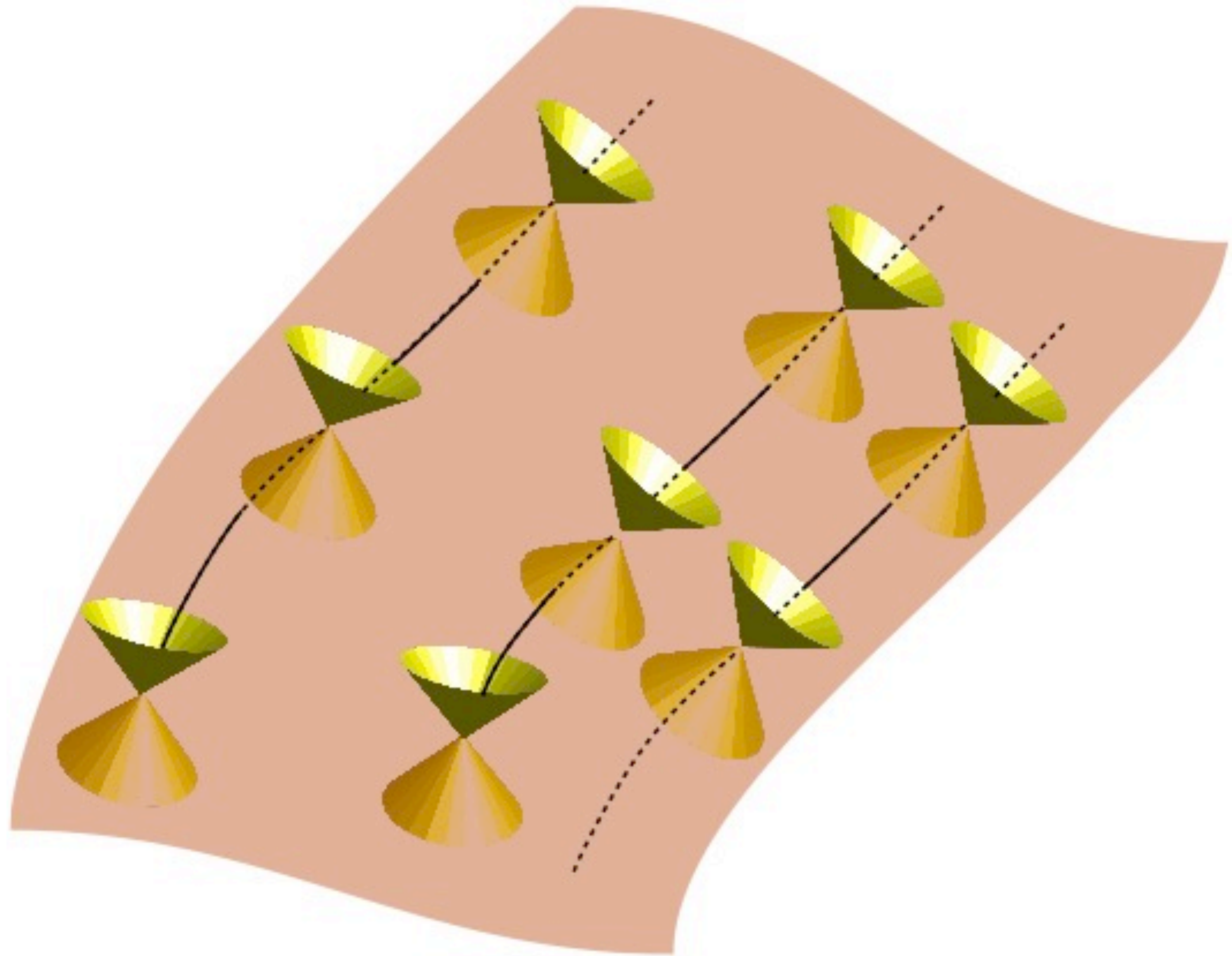


It picks out space-like, time-like, and null-vectors in the tangent space at each point.

Conformal Structure



Determines the propagation of zero rest-mass fields, including gravitation, and hence determines null hypersurfaces.

Conformal Structure

Conformal Christoffel symbols

$$\left\{ {\widetilde{\sigma} \atop \mu\nu} \right\} = \frac{1}{2}\tilde{g}^{\sigma\rho}\left(\tilde{g}_{\rho\nu,\mu} + \tilde{g}_{\mu\rho,\nu} - \tilde{g}_{\mu\nu,\rho}\right)$$

Not a connection in GR,
but it is in UR

Conformal Structure

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Conformal Covariant Derivative

$$\widetilde{\nabla}_{\mu} v^{\sigma} = \partial_{\mu} v^{\sigma} + \widetilde{\left\{ {\sigma \atop \mu\nu} \right\}} v^{\nu}$$

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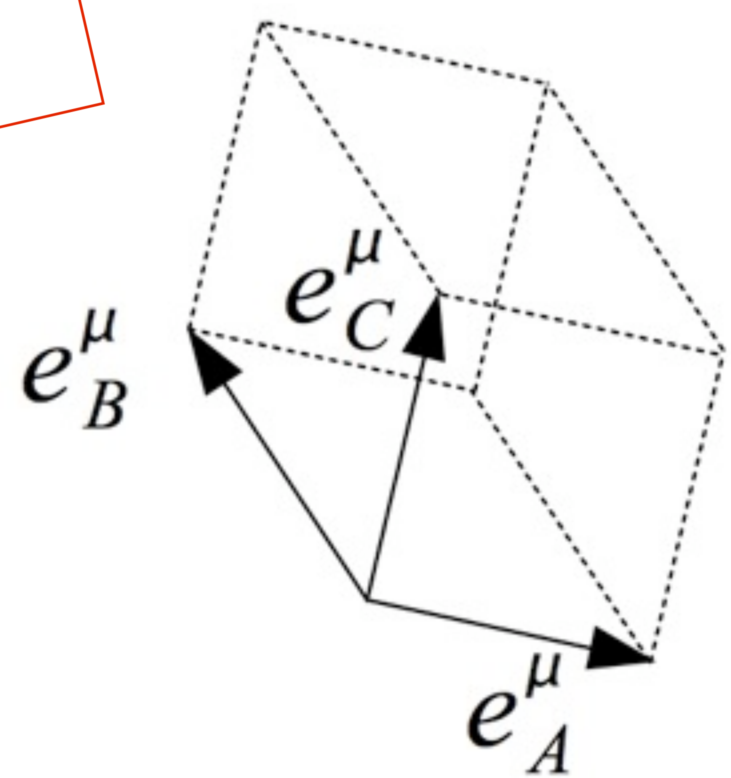
$$\widetilde{\nabla}_{\mu} v^{\sigma} = \partial_{\mu} v^{\sigma} + \widetilde{\left\{{\sigma \atop \mu\nu}\right\}} v^{\nu}$$

$$\widetilde{\nabla}_{\mu}\tilde{g}^{\sigma\rho} = 0$$

Four-volume element

$$e^{\varphi}, \varphi > 0$$

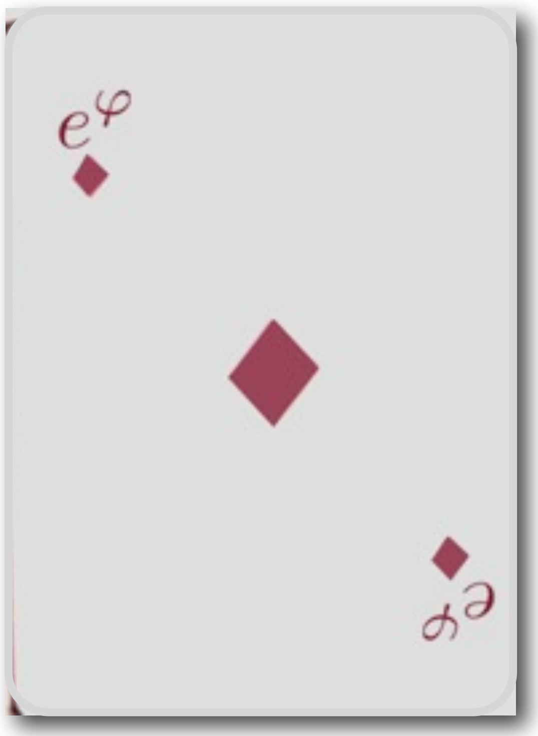
A classical limit of
a quantum of volume?



4th dimension suppressed.

A scalar quantity which weights the volume of a 4-D parallelepiped formed by a set of basis vectors at each point.
Necessary for carrying out integration over volumes.

Metric



$$g_{\mu\nu} = e^{\varphi}\tilde{g}_{\mu\nu}$$

$$\overset{m}{\nabla}_{\mu}\, g^{\sigma\rho} = e^{\varphi}\widetilde{\nabla}_{\mu}\tilde{g}^{\sigma\rho}$$

Projective Structure

$$\Pi^{\kappa}_{\mu\nu}$$

Describes the geometrically same unparametrized curves (paths).

$$\frac{d^2x^\mu}{du^2} + \Pi^{\mu}_{\sigma\rho}\frac{dx^\sigma}{du}\frac{dx^\rho}{du} = \lambda\frac{dx^\mu}{du}$$

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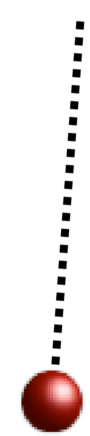

u=0

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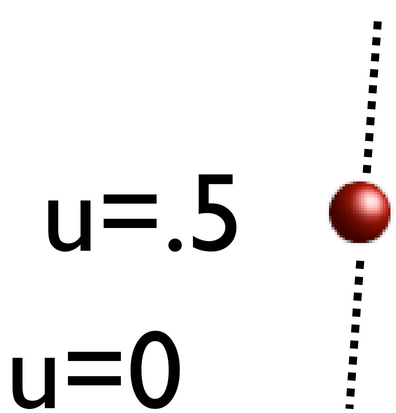

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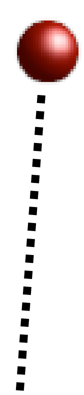
u=1

u=.5

u=0



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Not a connection in GR, but it is in UR

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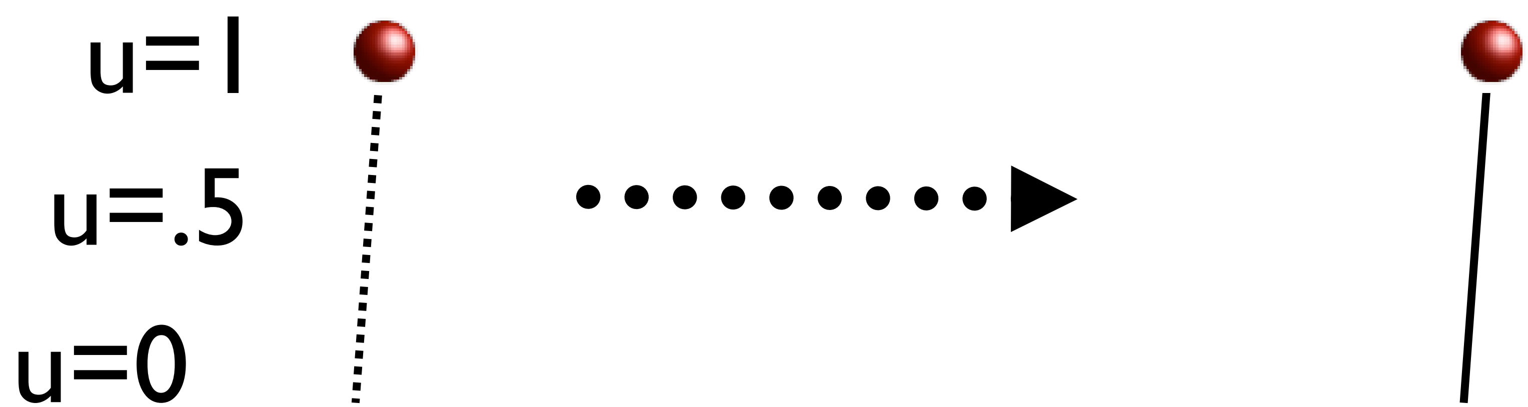

Projective Structure

$$\Pi^{\kappa}_{\mu\nu}$$

Transforms as a Connection under SL(n,R)

Projective Covariant Derivative

$$\bar{\nabla}_{\mu} v^{\sigma} = \partial_{\mu} v^{\sigma} + \Pi^{\sigma}_{\mu\nu} v^{\nu}$$

Projective Structure

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Projective-Connection Curvature Tensor

$$\Pi^{\cdots\kappa}_{\nu\mu\lambda} = \partial_{\nu}\Pi^{\kappa}_{\mu\lambda} - \partial_{\mu}\Pi^{\kappa}_{\nu\lambda} + \Pi^{\kappa}_{\nu\rho}\Pi^{\rho}_{\mu\lambda} - \Pi^{\kappa}_{\mu\rho}\Pi^{\rho}_{\nu\lambda}$$

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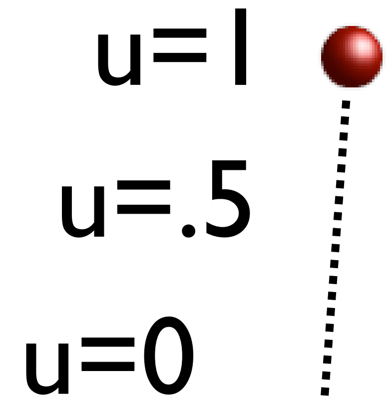
Projective Curvature Tensor

$$P^{\cdots\kappa}_{\nu\mu\lambda} = \Pi^{\cdots\kappa}_{\nu\mu\lambda} + \frac{2\delta^{\kappa}_{\lambda}\Pi_{[\nu\mu]}}{(n+1)} - \frac{\delta^{\kappa}_{\nu}\left(n\Pi_{\mu\lambda} + \Pi_{\lambda\mu}\right) - \delta^{\kappa}_{\mu}\left(n\Pi_{\nu\lambda} + \Pi_{\lambda\nu}\right)}{(n^{2}-1)}$$

Affine One-Form

$$\Gamma_\mu$$

Determines the preferred affine parameter along the paths defined by $\Pi^\sigma_{\mu\nu}$, which *can* later be interpreted as proper time.



Affine Connection

$$\Gamma^\sigma_{\mu\nu} = \Pi^\sigma_{\mu\nu} + \frac{1}{5}(\delta^\sigma_\mu \Gamma_\nu + \delta^\sigma_\nu \Gamma_\mu)$$

Field Equations in UCPR

e^{φ}

$$\widetilde{R} = -\frac{2}{(n-2)}\kappa\widetilde{\mathfrak{T}}.$$

$\tilde{g}_{\mu\nu}$

$$R_{\lambda\kappa} - \frac{1}{n}\widetilde{g}_{\lambda\kappa}\widetilde{R} = \kappa e^{\frac{(4-n)}{2}\varphi}\widetilde{g}_{\mu\lambda}\widetilde{g}_{\nu\kappa}\left(\mathfrak{T}^{\mu\nu} - \frac{1}{n}\widetilde{g}^{\mu\nu}\widetilde{\mathfrak{T}}\right).$$

$\Pi^{\kappa}_{\mu\nu}$

$$\widetilde{\widetilde{Q}}_{\kappa}^{\;\cdot\,\mu\nu} - \frac{2\delta_{\kappa}^{(\mu}\widetilde{\widetilde{Q}}_{\sigma}^{\;\cdot\,\nu)\sigma}}{(n+1)} = \left[\frac{(n-1)}{(n+1)}\Gamma_{\sigma} - \frac{(n-2)}{2}\partial_{\sigma}\varphi\right]\left[\widetilde{g}^{\mu\nu}\delta_{\kappa}^{\sigma} - \frac{2}{(n+1)}\delta_{\kappa}^{(\mu}\widetilde{g}^{\nu)\sigma}\right], \qquad \widetilde{\widetilde{Q}}_{\sigma\mu\nu} \equiv -\bar{\nabla}_{\sigma}\tilde{g}_{\mu\nu}$$

Γ_{μ}

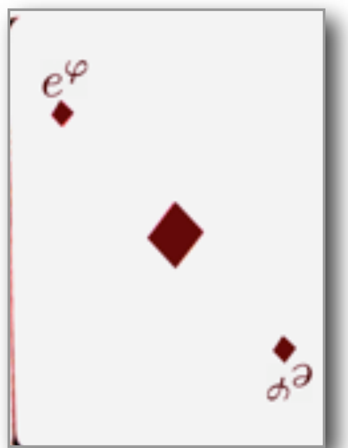
$$\widetilde{\widetilde{Q}}_{\nu}^{\;\nu\mu} = -\widetilde{g}^{\nu\mu}\left[\frac{(n-2)}{2}\partial_{\nu}\varphi + \frac{2\Gamma_{\nu}}{(n+1)}\right]$$

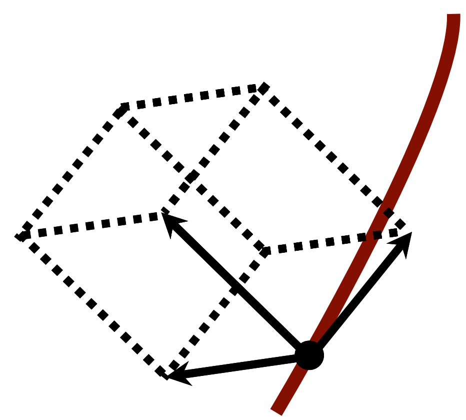
Compatibility in UCPR

In UCPR we can approach the compatibility in steps:

1. Equi-affine condition
2. Weyl condition
3. Conformal-Projective compatibility
4. Metric-Affine compatibility
5. Intermediate compatibility

Equi-Affine Condition





$$\Gamma_\mu = 2\partial_\mu\varphi$$

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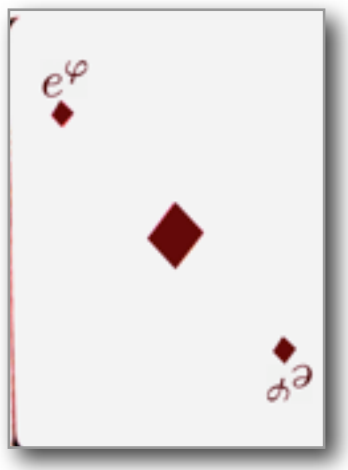

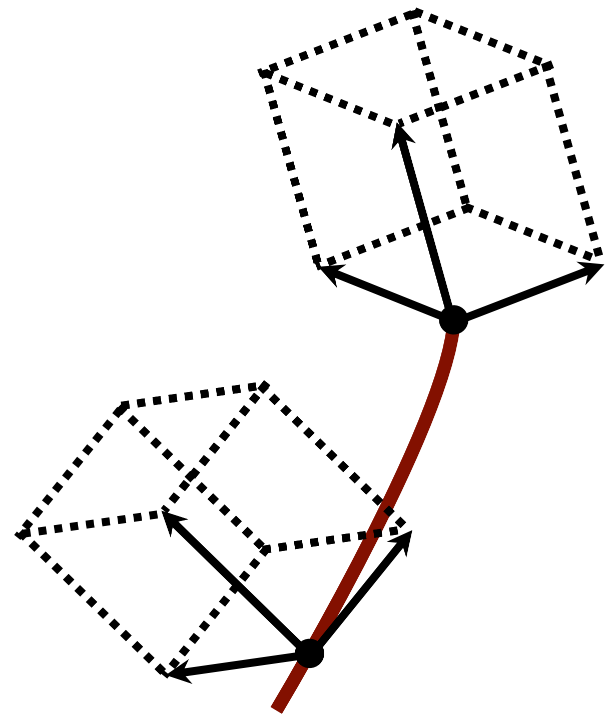

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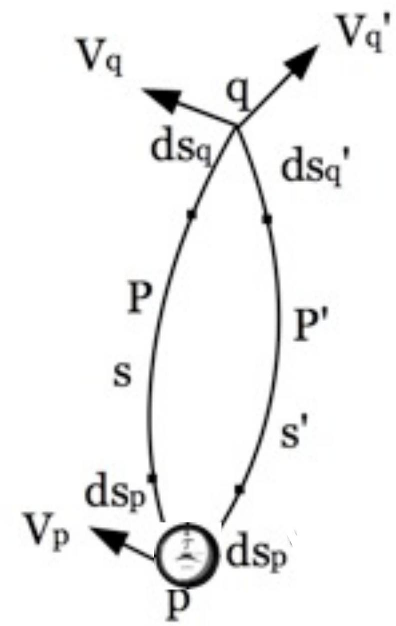
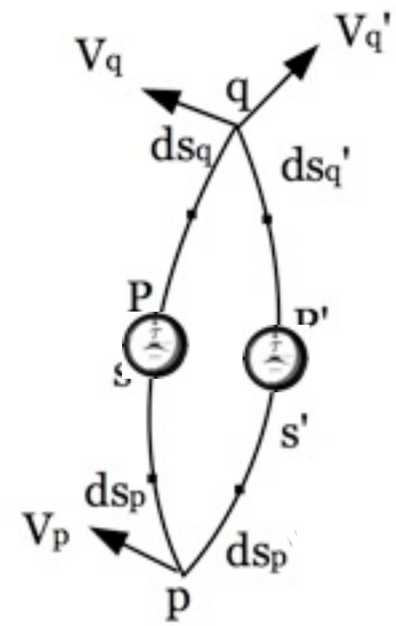
Weyl Condition



$$\Pi^{\kappa}_{\mu\nu} - \widetilde{\left\{ {\kappa \atop \mu\nu} \right\}} = \frac{1}{4}\left(\frac{2}{5}\delta^{\kappa}_{(\mu}\Gamma_{\nu)} - \tilde{g}_{\mu\nu}\tilde{g}^{\kappa\sigma}\Gamma_{\sigma}\right)$$

One physical consequence of the Weyl condition
is that the ticking rate of a clock depends on its history.

Weyl Condition



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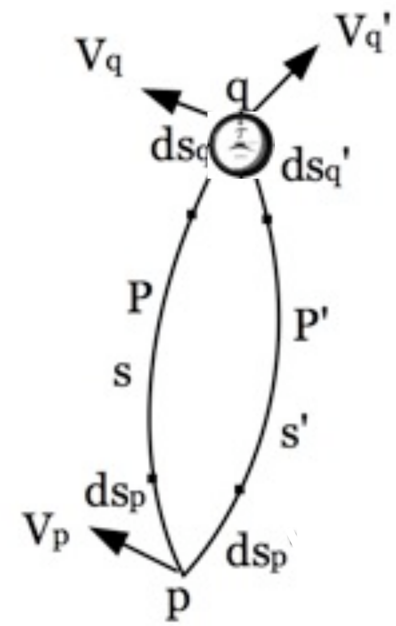

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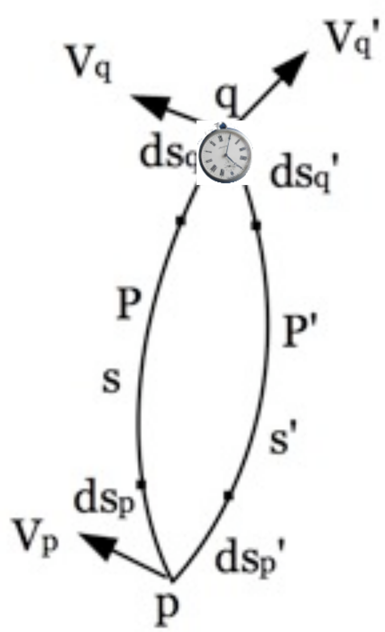
Conformal-Projective Compatibility

$$\Pi^{\kappa}_{\mu\nu} = \widetilde{\left\{ {\kappa \atop \mu\nu} \right\}}$$

Metric-Affine Condition



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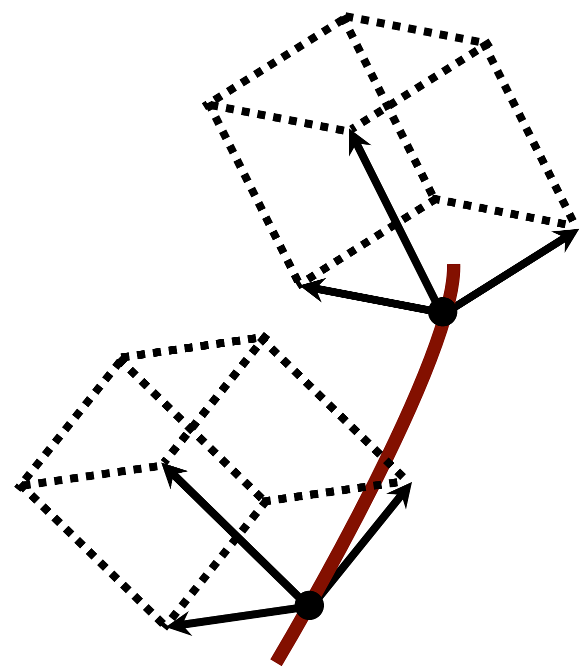

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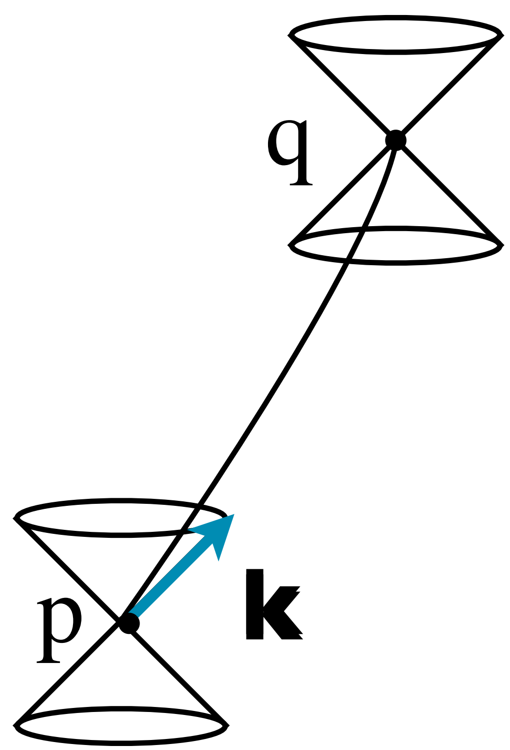
Compatibility of Causal and Dynamical Space-Time Structures

Causal structure is determined by the conformal metric.

Once a dynamical connection is introduced, in general the parallel transport with respect to the connection need not be compatible with the causal structure.



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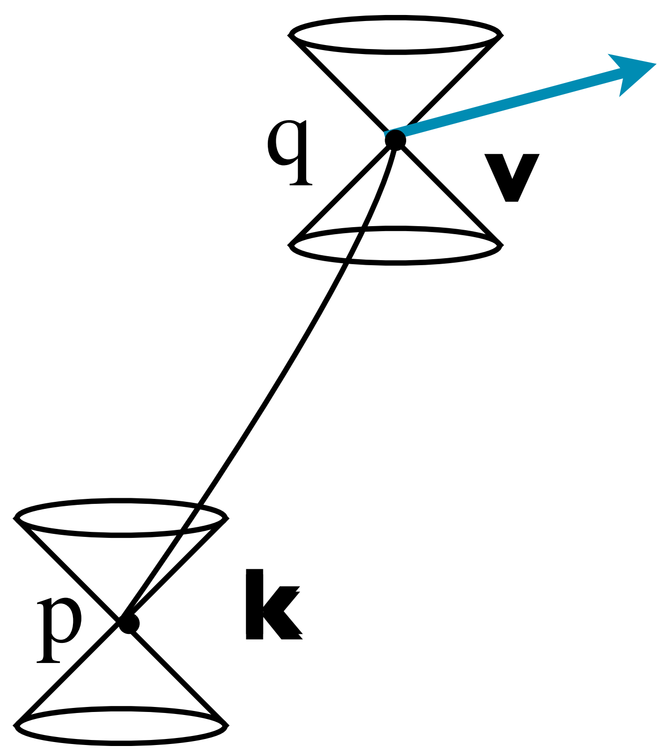
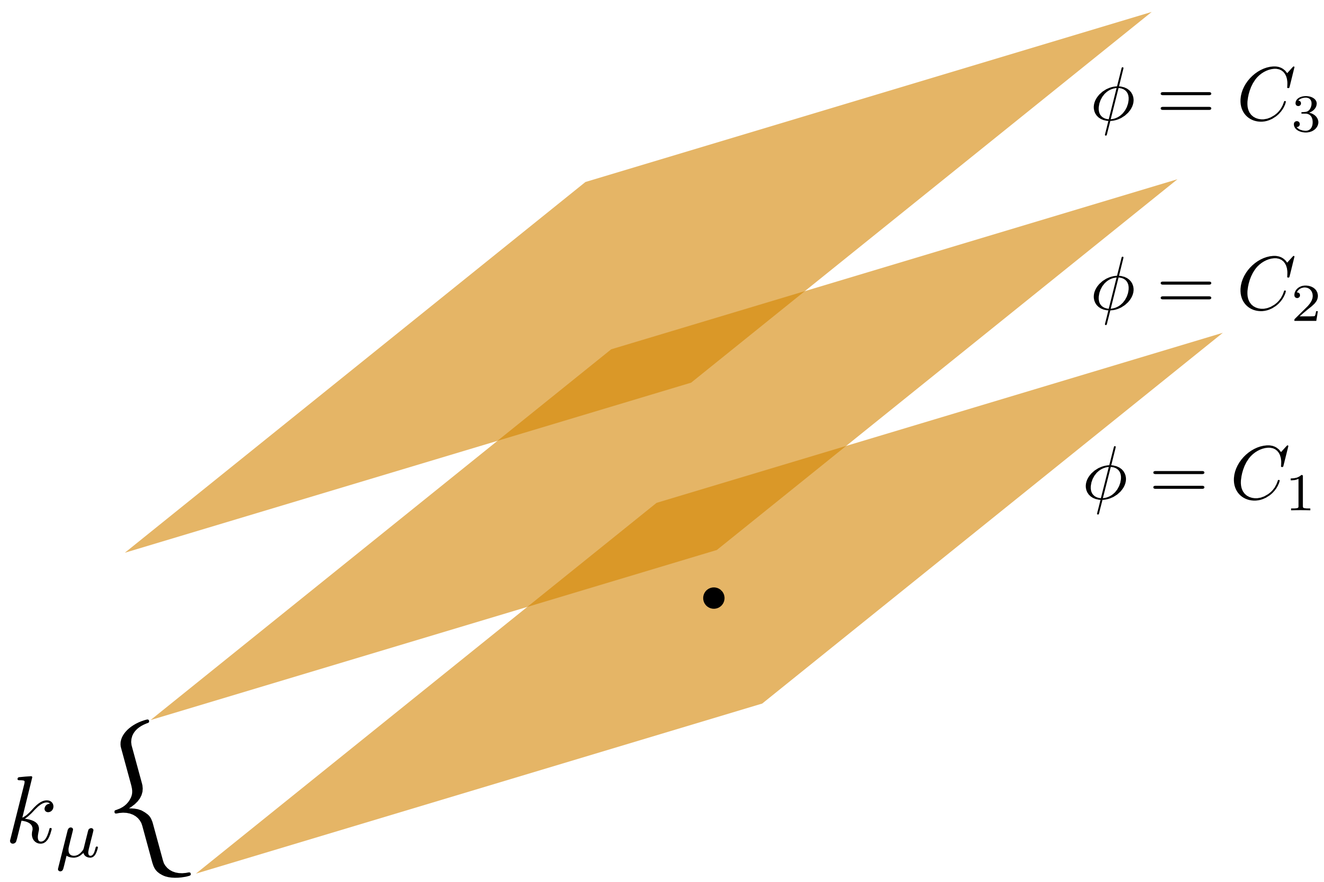


Compatibility of Causal and Dynamical Space-Time Structures

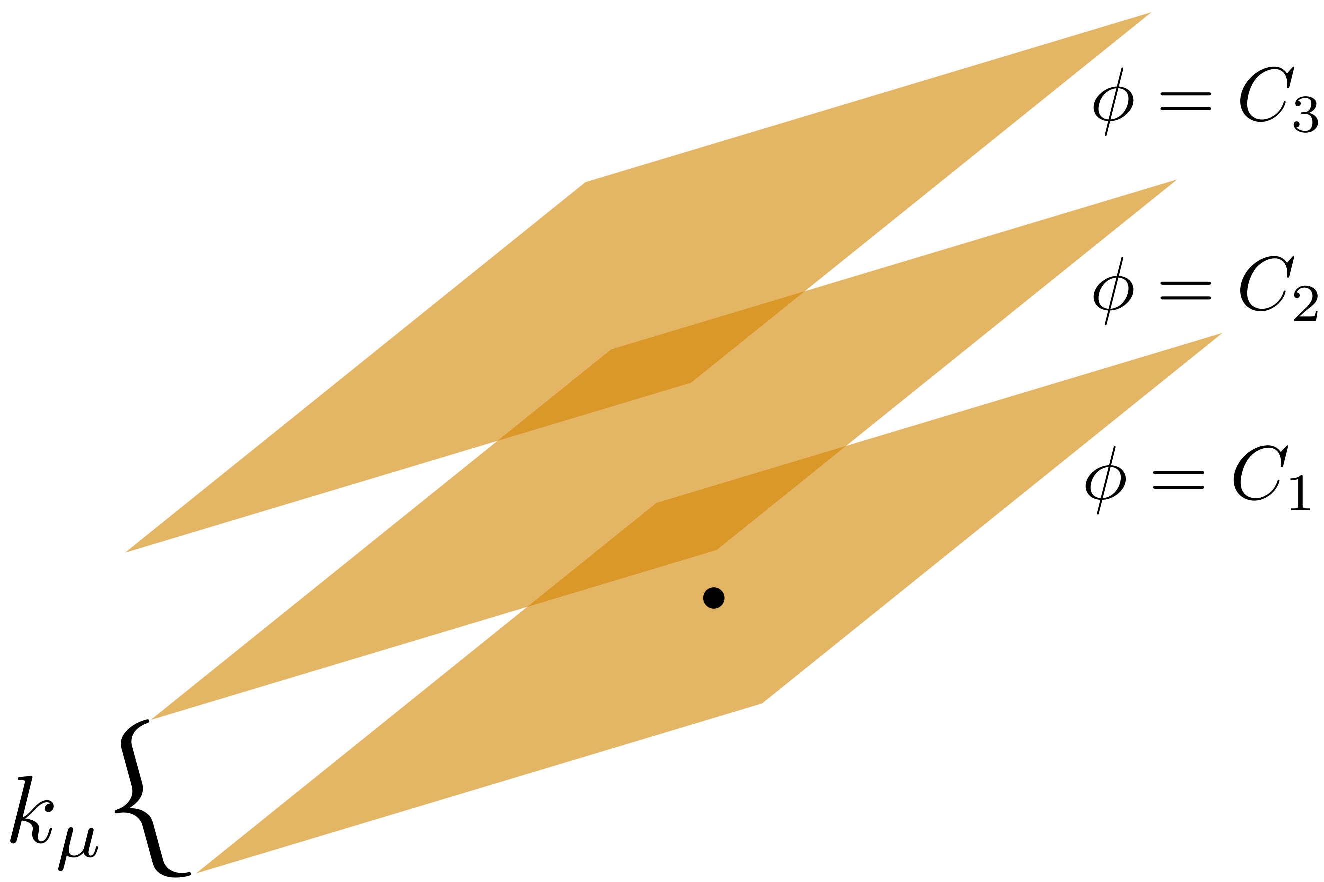
In UCPR, we can study the compatibility of the causal structure and the dynamical space-time structures by studying the compatibility of the conformal metric and the projective connection.

Null co-vectors and vectors



Null Co-vector $\quad k_\mu = \partial_\mu \phi$

Null co-vectors and vectors



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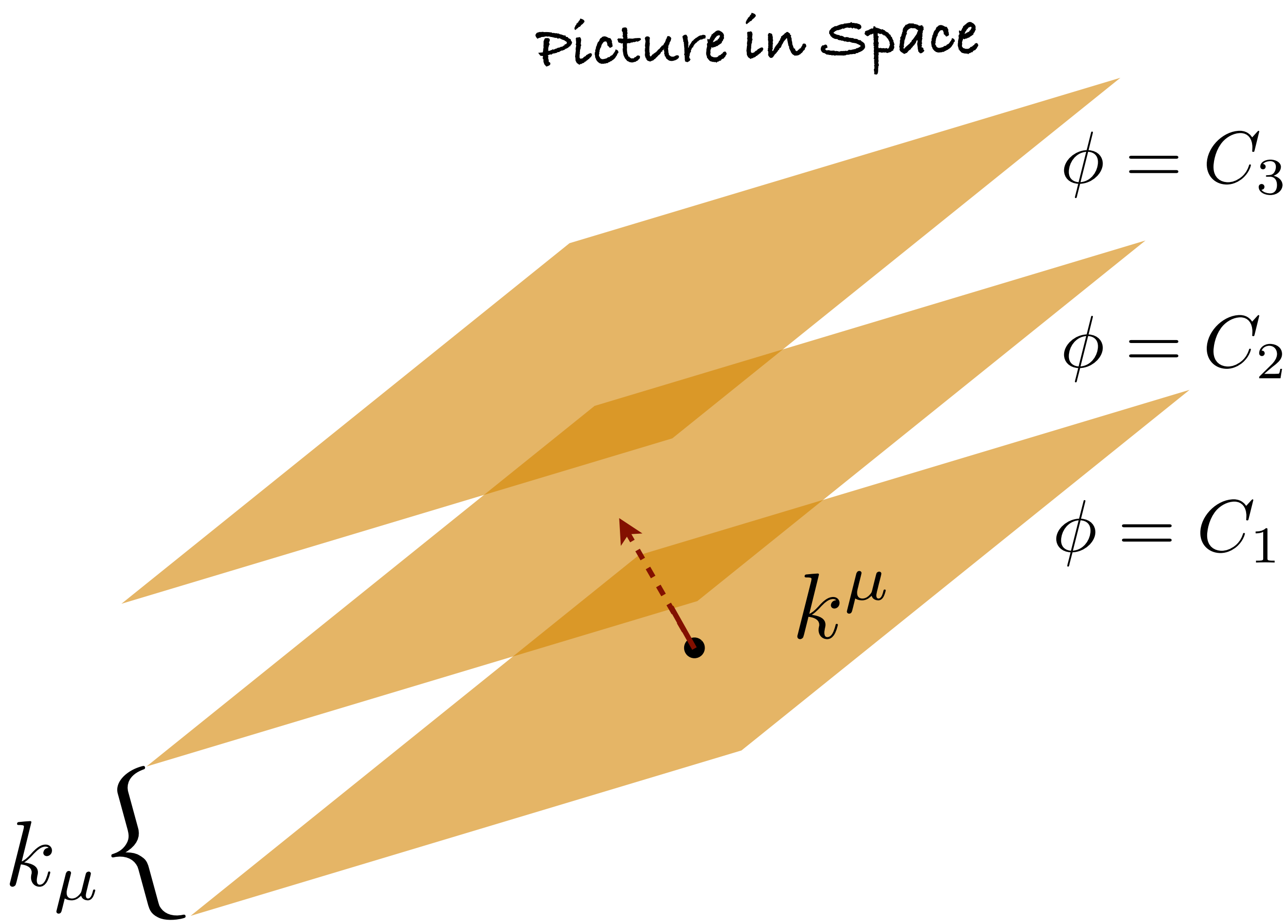
Huygen's Principle → Eikonal Equation $\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = 0$

Null co-vectors and vectors



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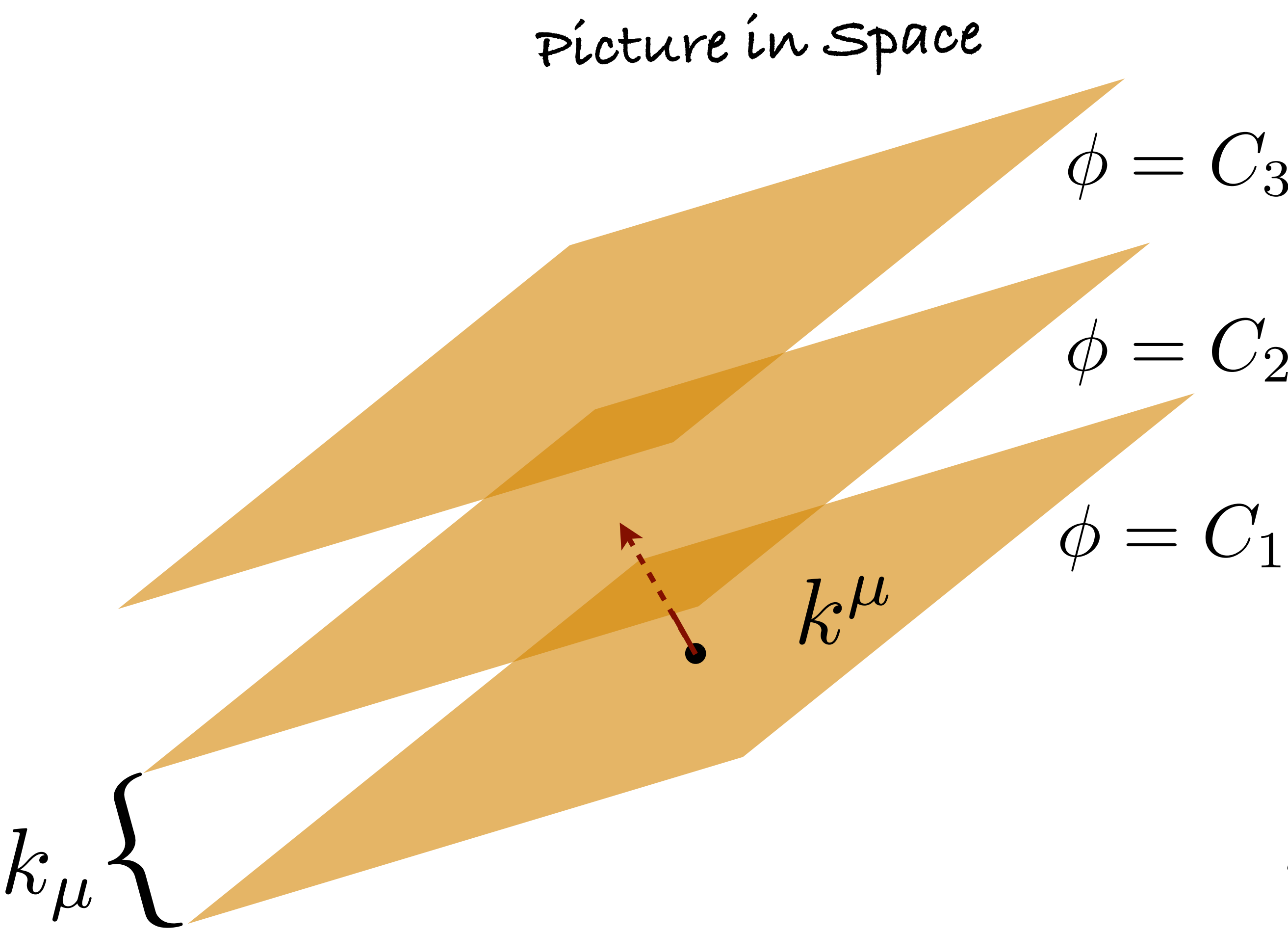

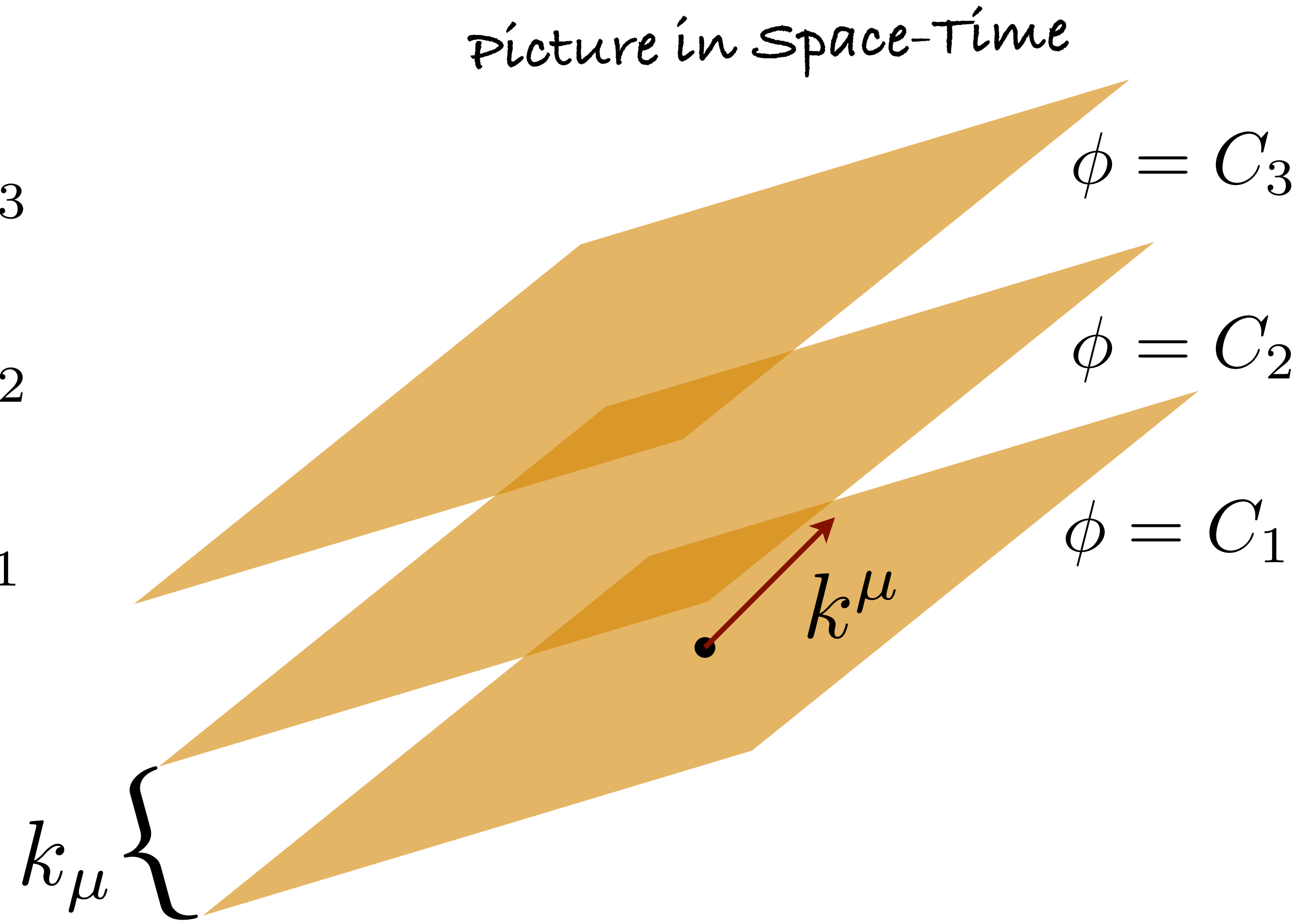

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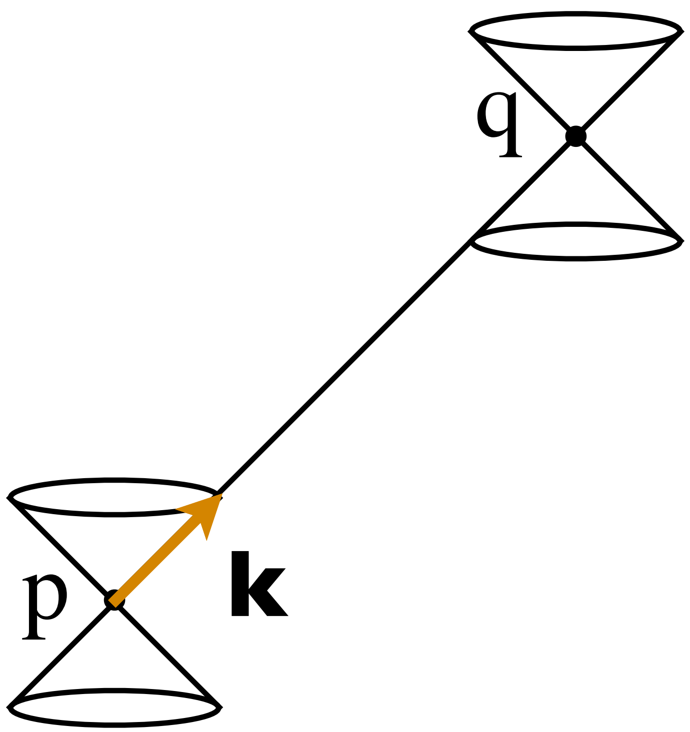
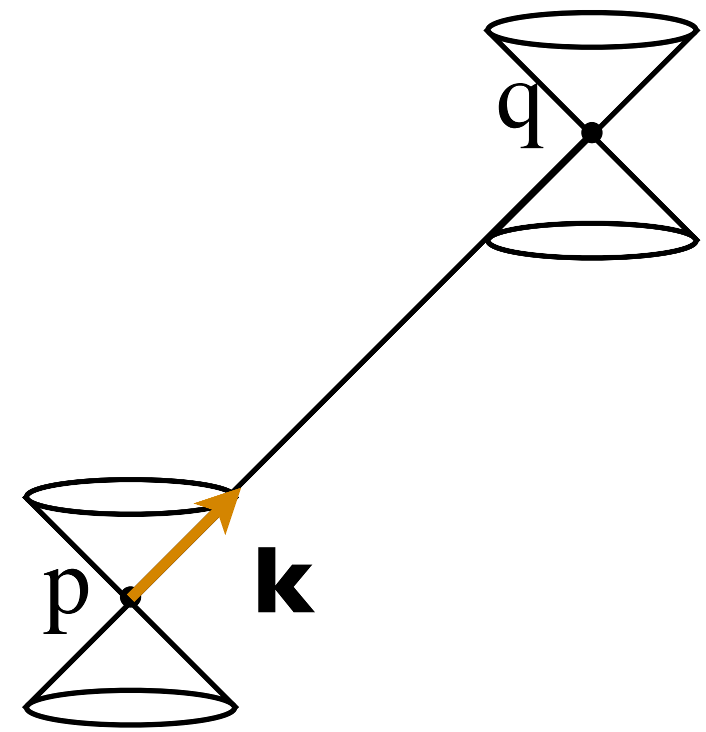
Null Geodesic Paths & Curves



$$k^\mu \widetilde{\nabla}_\mu k^\nu = \alpha k^\nu$$



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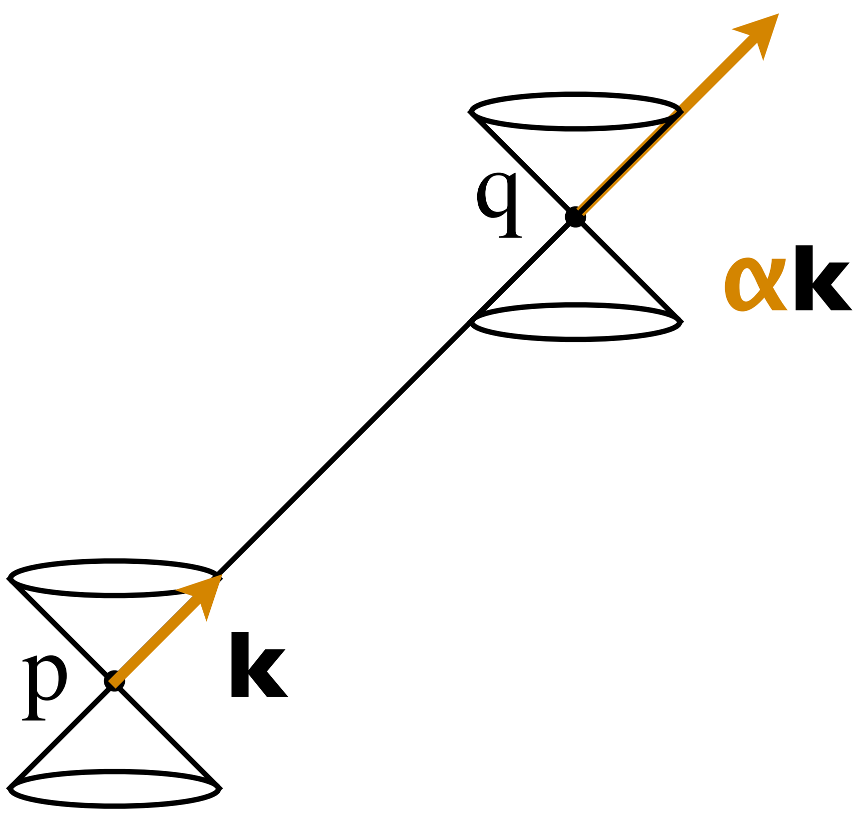
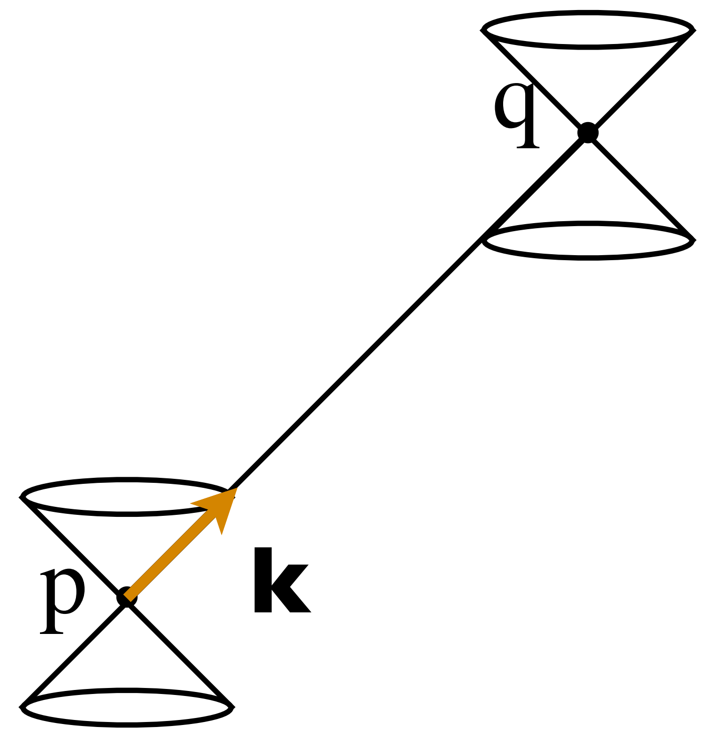
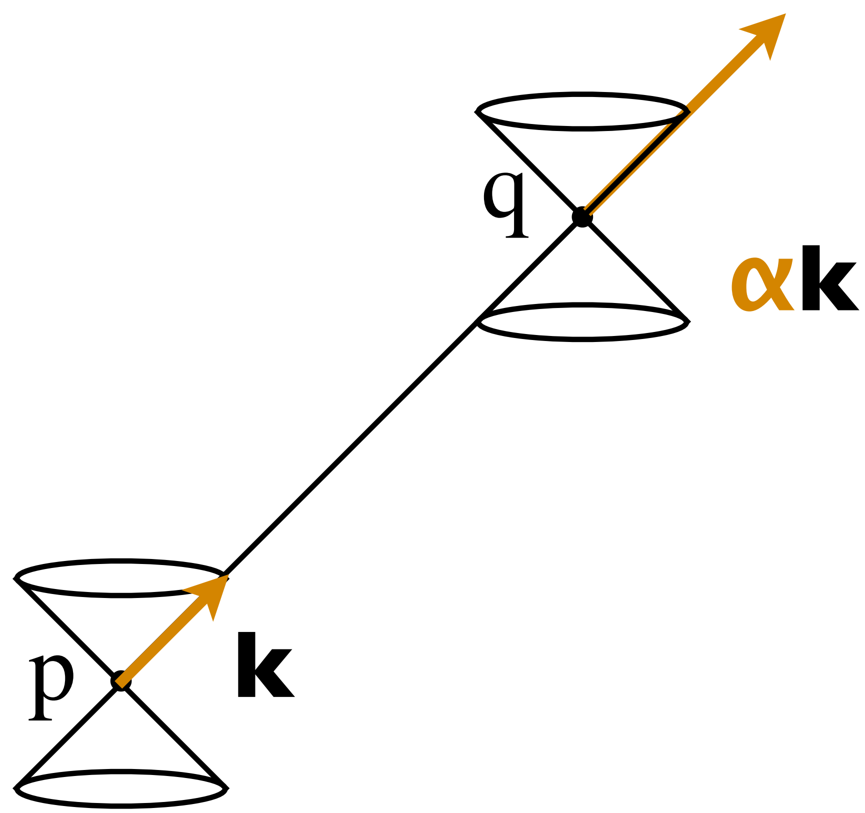


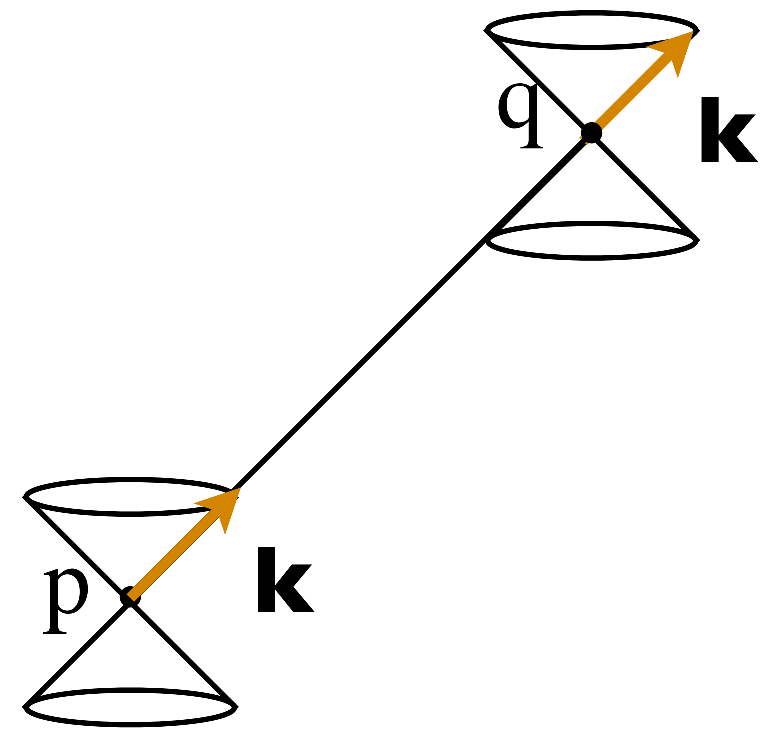
$$k^{\mu}\widetilde{\nabla}_{\mu}k^{\nu} = \alpha k^{\nu}$$



$$k^{\mu}\widetilde{\nabla}_{\mu}k^{\nu} = 0$$

Null Geodesic Paths & Curves

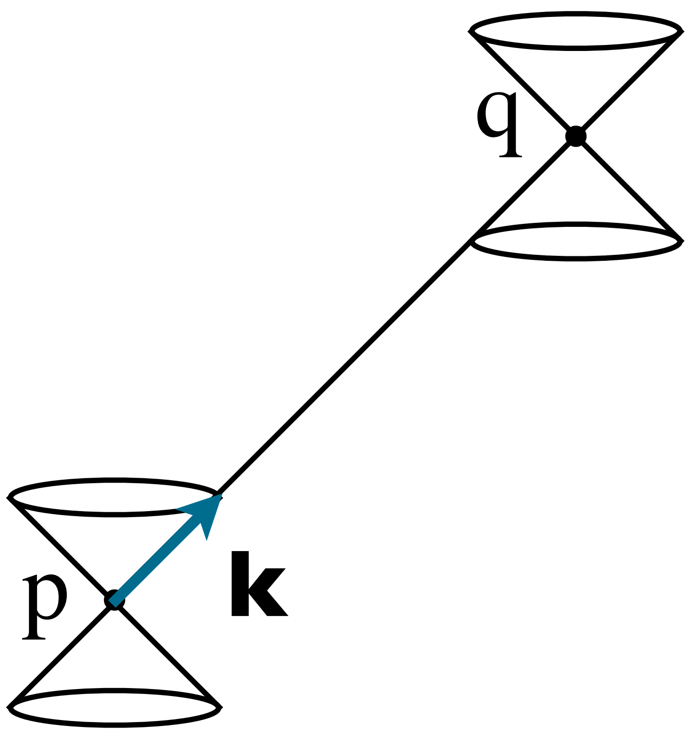




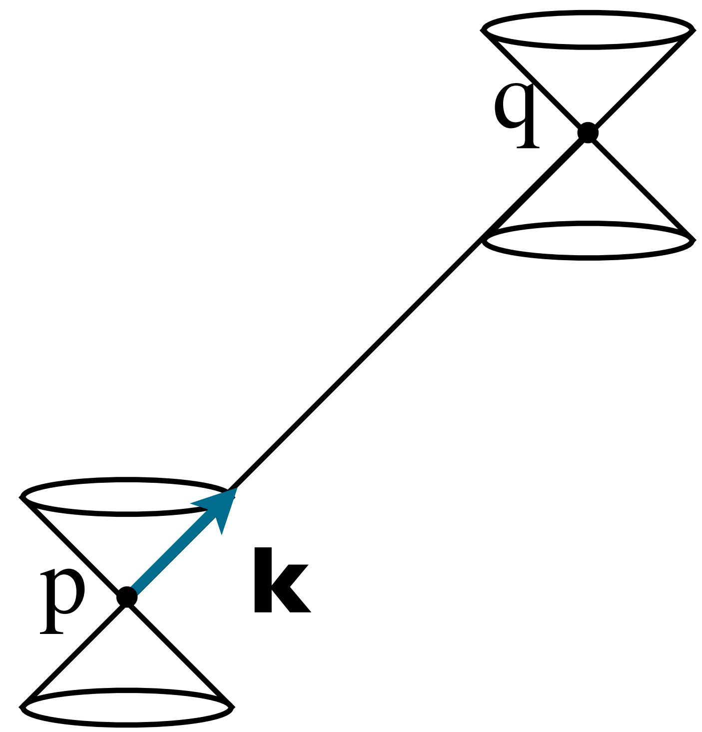
$$k^\mu \widetilde{\nabla}_\mu k^\nu = \alpha k^\nu$$

$$k^\mu \widetilde{\nabla}_\mu k^\nu = 0$$

Null Auto-Parallel Paths & Curves

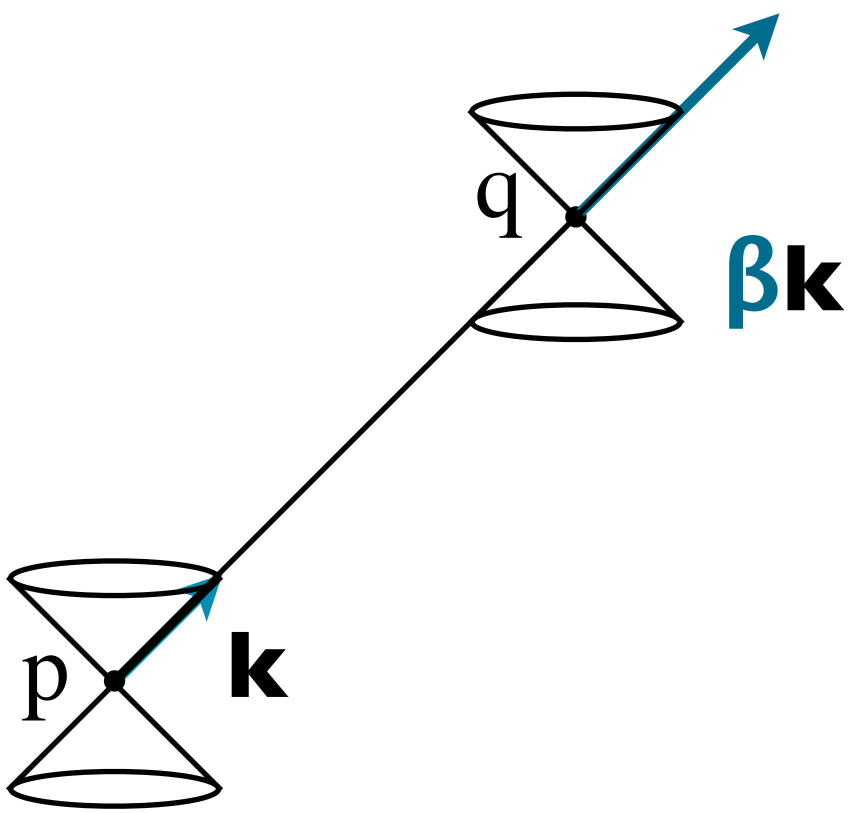


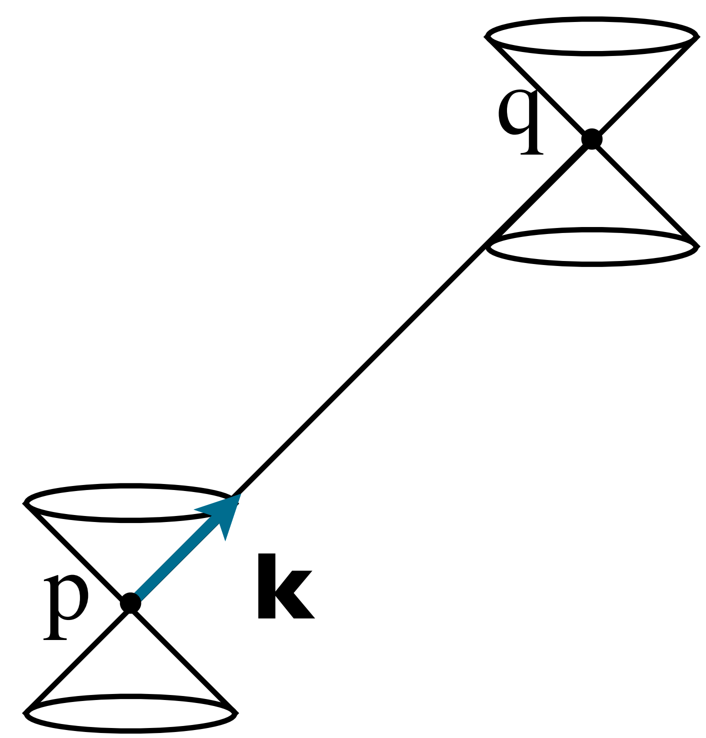
$$k^\mu \bar{\nabla}_\mu k^\nu = \beta k^\nu$$



$$k^\mu \bar{\nabla}_\mu k^\nu = 0$$

Null Auto-Parallel Paths & Curves

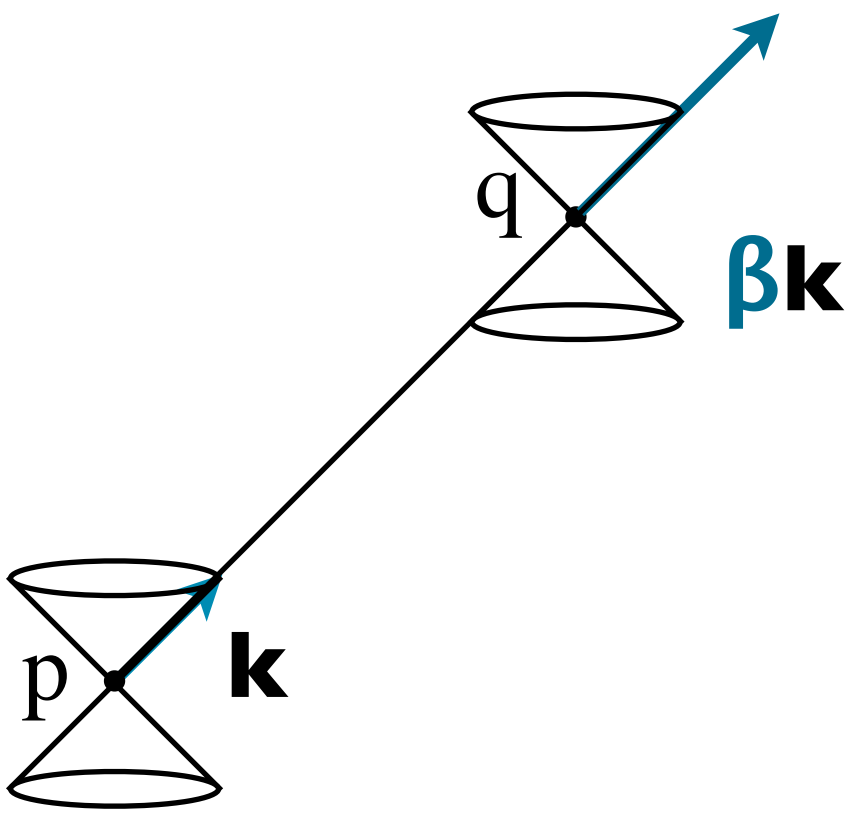


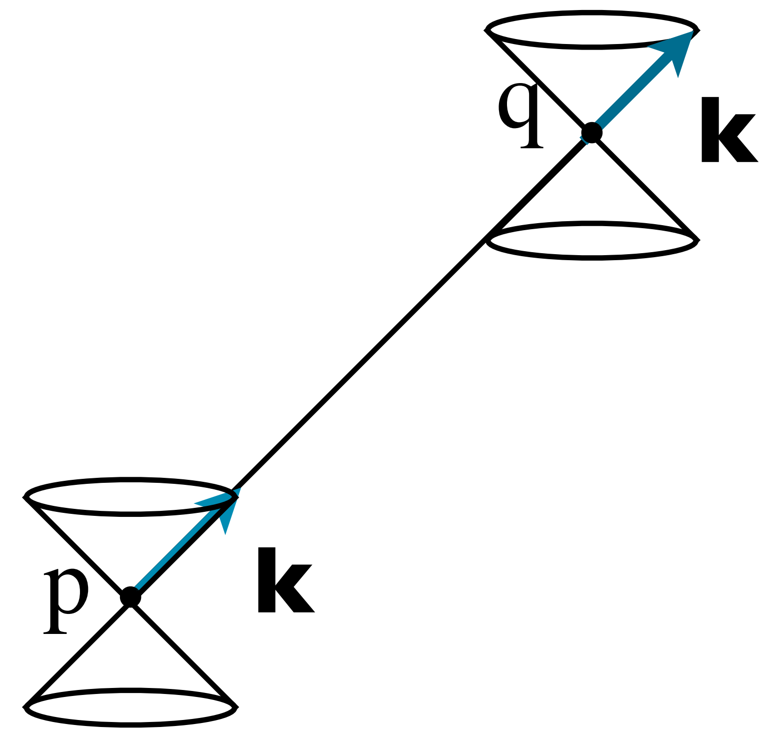


$$k^{\mu}\bar{\nabla}_{\mu}k^{\nu} = \beta k^{\nu}$$

$$k^{\mu}\bar{\nabla}_{\mu}k^{\nu} = 0$$

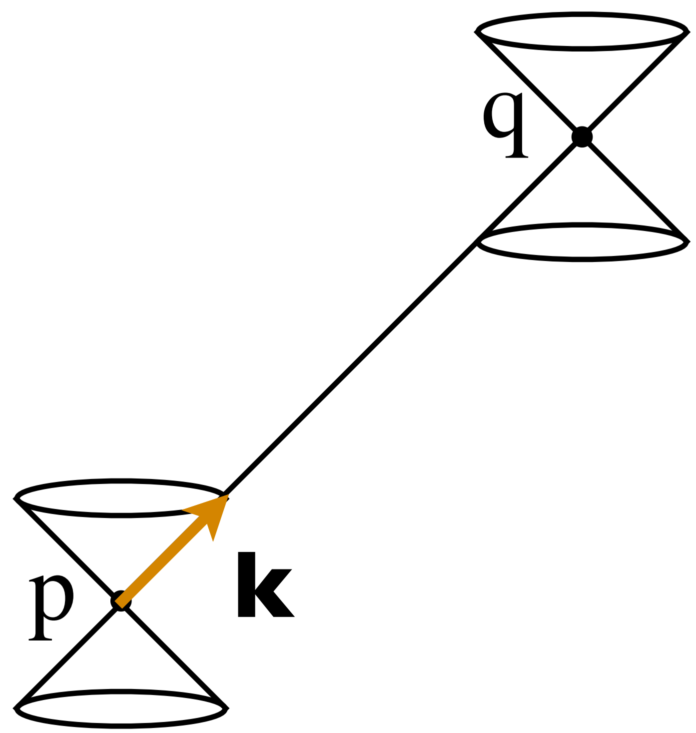
Null Auto-Parallel Paths & Curves





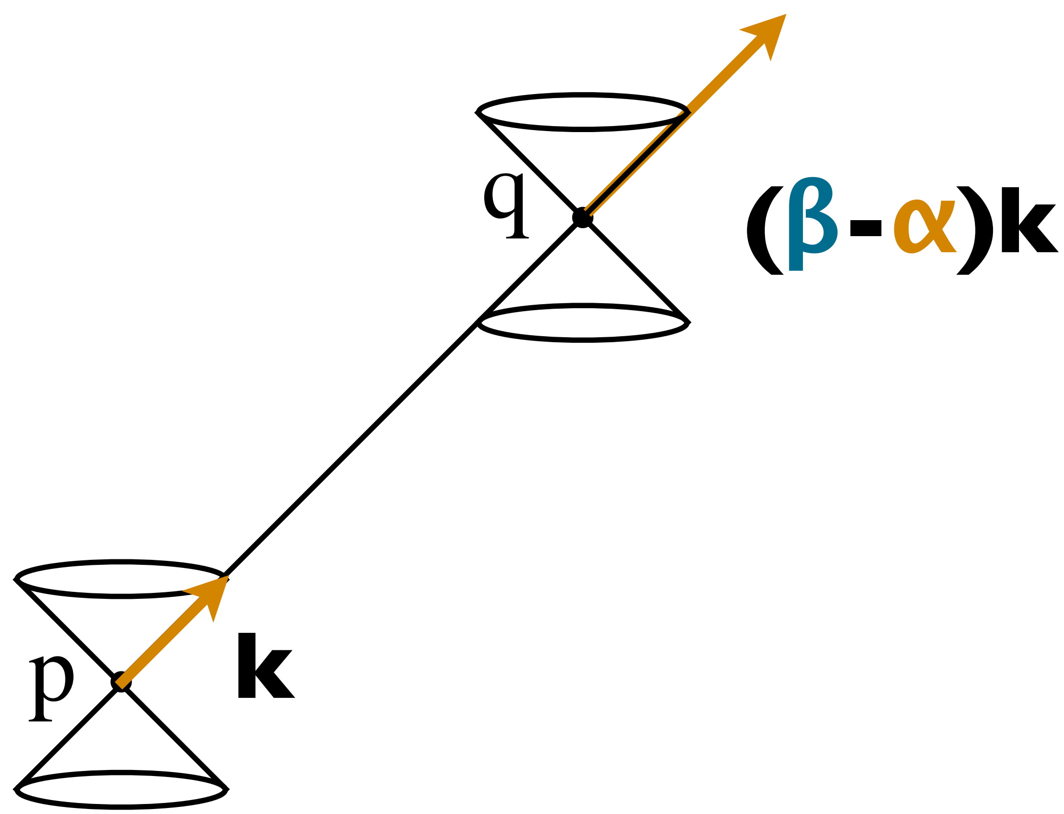
$$k^\mu \bar{\nabla}_\mu k^\nu = \beta k^\nu \qquad\qquad k^\mu \bar{\nabla}_\mu k^\nu = 0$$

Compatibility of Causal & Projective Structure



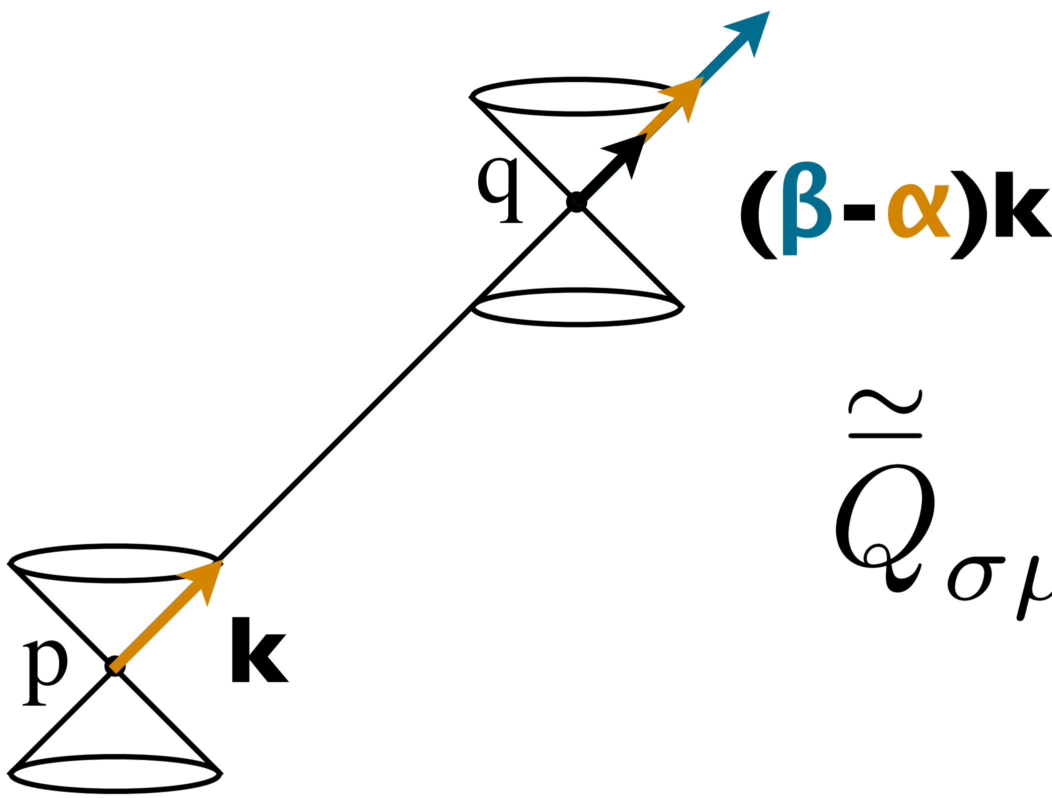
$$k^{\mu}(\bar{\nabla}_{\mu} - \widetilde{\nabla}_{\mu})k^{\nu} = (\beta - \alpha)k^{\nu}$$

Compatibility of Causal & Projective Structure



$$k^\mu(\bar{\nabla}_\mu - \widetilde{\nabla}_\mu)k^\nu = (\beta - \alpha)k^\nu$$

Conformal-Projective Non-Metricity Tensor



$$\overset{\cong}{Q}_{\sigma\mu\nu} \equiv -\bar{\nabla}_\sigma \tilde{g}_{\mu\nu}$$

$$\overset{\cong}{T}_{\mu\nu}{}^{\cdot\cdot\sigma} \equiv \Pi^{\sigma}_{\mu\nu} - \widetilde{\left\{ {\sigma \atop \mu\nu} \right\}} = \frac{1}{2}\tilde{g}^{\sigma\lambda}\left(\overset{\cong}{Q}_{\mu\lambda\nu} + \overset{\cong}{Q}_{\nu\mu\lambda} - \overset{\cong}{Q}_{\lambda\nu\mu}\right)$$

Compatible Causal and Dynamical Structures

Ehlers, Pirani, and Schild

Conformal and projective structures are compatible if conformal null-geodesics are also geodesics of projective structure.

A manifold with compatible conformal and projective structures is called a Weyl space.

Compatible Causal and Dynamical Structures

Ehlers, Pirani, and Schild

Weyl Space

$$\Pi^{\kappa}_{\mu\nu} - \widetilde{\left\{ {\kappa \atop \mu\nu} \right\}} = \frac{1}{4}\left(\frac{2}{5}\delta^{\kappa}_{(\mu}\Gamma_{\nu)} - \tilde{g}_{\mu\nu}\tilde{g}^{\kappa\sigma}\Gamma_{\sigma}\right)$$

Conformal and projective structures are compatible if conformal null-geodesics are also geodesics of projective structure.

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Compatible Causal and Dynamical Structures

Weyl Space

$$\Pi^{\kappa}_{\mu\nu} - \widetilde{\left\{{}^{\kappa}_{\mu\nu}\right\}} = \frac{1}{4}\left(\frac{2}{5}\delta^{\kappa}_{(\mu}\Gamma_{\nu)} - \tilde{g}_{\mu\nu}\tilde{g}^{\kappa\sigma}\Gamma_{\sigma}\right)$$

UCPR

Compatible Causal and Dynamical Structures

$$\Pi^{\sigma}_{\mu\nu} - \widetilde{\left\{{}^{\sigma}_{\mu\nu}\right\}} = \frac{1}{2}\left(\tilde{g}^{\sigma\lambda}\tilde{g}_{\mu\nu}(2Y_{\lambda} - Z_{\lambda}) + \delta^{\sigma}_{\nu}Z_{\mu} + \delta^{\sigma}_{\mu}Z_{\nu}\right)$$

Conclusions

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- Framework in terms of four irreducible fields with clear physical and mathematical interpretations.

- Allows us to approach the "metric-affine" compatibility in steps

- It can be used to study the compatibility of causal and dynamical structures in a more general way than the metric-affine formalism

- It can be used to formulate a variety of theories with possible applications to cosmology, quantum gravity, etc.

Let’s Play!

Thank You.