Topics in Chronogeometry

(BU 2010 course package)

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DEFINITION. A *group* G is a set whose elements can be *multiplied* (so that the product is also an element of G). The multiplication has to satisfy the following three properties (*group axioms*):

(u times v) times w is always the same as u times (v times w)   (ASSOCIATIVE LAW)

There exists an element e with the property e times u (and u times e) is the same as u; this holds for each element u of the group; this e is called the *neutral element* (or just the *unit*) of G;

The third axiom states the existence of the *inverse* element v for any u: the product of u and v is independent of the order and this product is just e. A standard notation for v is u^{-1}.

   The simplest example of a group is G consisting of number 1. Another example (with an ordinary multiplication between elements) is the set of all real numbers except zero: 0 has to be excluded since the inverse for it does not exist.

EXAMPLE 1 (of a non-abelian group, it later goes under the name BASIC AFFINE group).

   The totality $G = \{\, x\,\}$ of its elements is parameterized as $x = (x_1,\ x_2,\ x_3)$, x_s being any real number. Fix

$$1 = (0, 0, 0) \tag{1}$$

Define multiplication as

$$z = xy = (x_1 + y_1 e^{x_3}, x_2 + y_2 e^{x_3}, x_3 + y_3). \tag{2}$$

EXERCISE 1. Prove G is a group.

   Part of the proof is to verify the associative law, $(xy)z = x(yz)$, as well as (given any x) to prove existence of v to satisfy $xv = vx = 1$. Such a v is usually denoted as x^{-1}, and it is referred to as an INVERSE of x. Answer:

$$x^{-1} = (-x_1 e^{-x_3}, -x_2 e^{-x_3}, -x_3) \tag{3}$$

On the notion of an ISOMORPHISM of two groups (From Wikipedia): a **group isomorphism** is a function between two groups that sets up a one-to-one correspondence between the elements of the groups in a way that respects the given group operations. If there exists an isomorphism between two groups, then the groups are called **isomorphic**. From the standpoint of group theory, isomorphic groups have the same properties and need not be distinguished.

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Introduce $\tilde{G}$ as the totality of all three by three matrices of the form

$$x = \begin{pmatrix} e^{x_3} & 0 & x_1 \\ 0 & e^{x_3} & x_2 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

EXERCISE 2. Prove $\tilde{G}$ is a group (w.r.t. a matrix multiplication and matrix inversion).

One of the possible proofs is to notice that the above (1), (2), (3) are satisfied for $\tilde{G}$. As a corollary, that proves the existence of an isomorphism f between G and $\tilde{G}$ (an example of such an f is one which takes $x = (x_1, x_2, x_3)$ into the matrix (4)).

Or, one can say that $\tilde{G}$ is a MATRIX REALIZATION of the original group G. Mostly, we will be using one and the same letter to denote two isomorphic groups.

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We have introduced the right shift R_y as a transformation of G; it takes each x into xy. The left shift L_x takes an element y of G into xy. Both transformations are invertible:

$(R_y)^{-1} = R_u$ where $u = y^{-1}$, $(L_x)^{-1} = L_v$ where $v = x^{-1}$.

The differential $df|_a$ (also known as tangent map) of a mapping

f: $x = (x_1, x_2, x_3) \rightarrow z = (z_1, z_2, z_3)$ of G into itself (think of G being a surface parameterized by x_1, x_2, x_3) is defined as the linear mapping from T_a into $T_{f(a)}$ determined by the square matrix of size 3 with elements $\partial z_i / \partial x_m$ (it is a matrix of size n where $dimG = n$). Partial derivatives $z_{i,m} = \partial z_i / \partial x_m$ are evaluated at $x = a$. Given an element b in G, T_b stands for the tangent space to G at b.

Figure: a dim=2 surface G with points $a, f(a), \boldsymbol{1}$ on it; tangent plane T_a at a, tangent plane $T_{f(a)}$ at $f(a)$.

A finite-dimensional linear space **L** is said to be a Lie algebra iff a bilinear, antisymmetric operation [,] in it satisfies the Jacobi identity: $[\mathbf{a},[\mathbf{b},\mathbf{c}]]+[\mathbf{b},[\mathbf{c},\mathbf{a}]]+[\mathbf{c},[\mathbf{a},\mathbf{b}]]=\mathbf{0}$ for arbitrary **a,b,c** in **L**. A vector $[\mathbf{a},\mathbf{b}]$ is referred to as the Lie bracket (or the commutator) of **a**, **b**.

Recall the following three matrices:

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix};$$

recall the series $\exp(t)=1+t+t^2/(2!)+...$

Notice that $\exp(x_1\mathbf{e}_1) = (x_1,0,0)$, $\exp(x_2\mathbf{e}_2) = (0,x_2,0)$, $\exp(x_3\mathbf{e}_3) = (0,0,x_3)$ are three respective elements in G (when G is realized as a set of matrices).

Also, $(x_1,0,0) \cdot (0,x_2,0) =(0,x_2,0) \cdot (x_1,0,0) = (x_1,x_2,0)$ and $(x_1,x_2,0) \cdot (0,0,x_3) = (x_1,x_2,x_3)$. Introduce the Lie algebra L(G) as the span of the vectors (=matrices) $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Our G is thus the "image of L under the exponential map" (in physics literature $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are referred to as "generators" of G). (It goes without saying that) a commutator in that case is a commutator of the two matricies: $[A,B]=A\cdot B-B\cdot A$

(In such a case the Jacobi identity is satisfied; clearly, it is bilinear and antisymmetric; it can be proved that each Lie algebra has a MATRIX REALIZATION).

Namely, $[\mathbf{e}_3,\mathbf{e}_1]=\mathbf{e}_1$, $[\mathbf{e}_3,\mathbf{e}_2]=\mathbf{e}_2$ (2-1)

defines (due to bi-linearity and anti-symmetry) the Lie bracket in the span of the respective vectors.

(2-1) is referred to as a "commutation relations" table. It determines the structure of L in question. In general, L with a bracket [,] is said to be ISOMORPHIC to an L' with a bracket [,]' iff there is a linear isomorphism f between L, L' which takes any $[\mathbf{a},\mathbf{b}]$ into $[f\mathbf{a},f\mathbf{b}]'$ (so this holds for arbitrary **a**, **b** from L).

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If an (abstract) L is determined by commutative relations

$$[e_i \quad e_j] = \sum_{k=1}^{n} C_{ij}^{k} e_k$$

between its basic vectors, then coefficients C_{ij}^{k} (="structure constants") define a $\binom{1}{2}$-tensor.

   Exercise. Consider R^3 with canonically defined *vector* (=*cross*) *product*

$$[\mathbf{a,b}]=\mathbf{a} \times \mathbf{b}$$

Prove it to be a Lie algebra. Find commutative relations table. Prove that it is not isomorphic to our L(G) from above.

   Exercise. Define isomorphism of two Lie groups.

Theorem: if L(G) is isomorphic to L($\tilde{G}$) then G is locally isomorphic to $\tilde{G}$.

   In any dimension there is one trivial (=abelian=commutative) Lie algebra (all commutators vanish). A (multiplicative) group of unit complex numbers is a dim=1 Lie group. An (additive) group of all reals is a one-dimensional Lie group. Their Lie algebras are abelian, hence isomorphic. The groups, however, are only locally isomorphic.

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dL_x, dR_x found. We have understood that the columns of the respective matrices

$$dL_x = \begin{pmatrix} e^{x_3} & 0 & 0 \\ 0 & e^{x_3} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad dR_x = \begin{pmatrix} 1 & 0 & x_1 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix}$$

are composed of coefficients in the following decompositions:

$$\boldsymbol{l}_1 = e^{x_3}\partial_1 , \qquad \boldsymbol{l}_2 = e^{x_3}\partial_2 , \qquad \boldsymbol{l}_3 = \partial_3 ;$$
$$\boldsymbol{r}_1 = \partial_1 , \qquad \boldsymbol{r}_2 = \partial_2 , \qquad \boldsymbol{r}_3 = x_1\partial_1 + x_2\partial_2 + \partial_3 .$$

Such a decomposition is carried out at the point x of our group G. The respective operators of partial differentiation are denoted by $\partial_1, \partial_2, \partial_3$. Certain authors use the notation:

$$\bar{e}_1(x), \ \bar{e}_2(x), \ \bar{e}_3(x)$$

for

$$\partial_1, \ \partial_2, \ \partial_3$$

respectively.

The notion of grid curves (on our parameterized surface G) is helpful: for each point $x = (x_1, x_2, x_3)$, there are three grid curves passing through x. $\bar{e}_1(x)$, $\bar{e}_2(x)$, $\bar{e}_3(x)$ are the respective tangent vectors to those curves at x. The three vectors $\bar{e}_1(x)$, $\bar{e}_2(x)$, $\bar{e}_3(x)$ form a ("coordinate") basis for the tangent space $T(G)$.

It is clear that $l_1, l_2, l_3;\ r_1, r_2, r_3$ are two other bases of $T(G)$. The totality of all <u>left-invariant</u> vector fields on G is a linear space of dimension 3. Fields l_1, l_2, l_3 form a basis of this space. Similarly, the <u>right-invariant</u> fields r_1, r_2, r_3 form the other basis.

The <u>Lie algebra</u> $L(G)$ of a Lie group G is defined as the totality of all left-invariant vector fields on G.

Any field

$$\bar{a}(x) = a_1\partial_1 + a_2\partial_2 + a_3\partial_3$$

(where functions a_1, a_2, a_3 depend on x_1, x_2, x_3) can be viewed as a first-order differential operator. Thus, given a function

$$f = f(x_1, x_2, x_3),$$

$\bar{a}f$ is a function on G. Now, $l_1(l_3 f)$, say, can be defined.

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The COMMUTATOR (or a LIE BRACKET) of two vector fields is introduced as:

$$[\mathbf{a},\mathbf{b}]f = \mathbf{a}(\mathbf{b}f) - \mathbf{b}(\mathbf{a}f).$$

The $[\mathbf{a},\mathbf{b}]$ (it can be proved) is a first-order differential operator, that is, a vector field.

Exercise. Prove that $[l_3,\ l_1] = l_1, [l_3,\ l_2] = l_2, [l_2,\ l_1] = 0$.

(Hint: use the fact that operators of partial differentiation commute, $[\partial_i,\ \partial_j] = \mathbf{0}$).

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To be able to calculate a length of a given curve on a surface, one has to know how to calculate the length of a vector in a tangent plane (space). Usually, it is done in terms of introducing a scalar (=inner) product of vectors in each tangent space.

$$\bar{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \bar{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \in T_x, < \bar{a}, \bar{b} >= \sum_{i,j=1}^{n} g_{ij}(x)a_i b_j$$

(here we are using any parameterization $(x_1, x_2, \ldots, x_n)$ of a surface G).

When G is a Lie group, there are metrics (=inner products) which in the most natural way are compatible with a group operation in G (left-invariant metrics, say). To introduce such a metric, it is enough to specify a matrix

$g = (g_{ij})$ at $1 = (0, \ldots, 0)$, and to demand this very g determines inner product in each T_x between LEFT-INVARIANT VECTORS $\langle l_i, l_j \rangle = g_{ij}$.

In classical approach, the metric is mostly expressed in terms of coordinate vector fields. Let us proceed with an example. In a two-dimensional analogue of our main Lie group G introduce

$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ which says: at each point q the vectors $l_1(q)$, $l_2(q)$ form an orthonormal basis of a tangent plane T_q. (To get rid of the indices, we are now using notation $q = (x,y)$, etc. with a straightforward modification of all formulas involved. The metric coefficients $g_{ij}(q) = \langle e_i, e_j \rangle$ become $g_{11} = e^{-2y}$, $g_{12} = q_{21} = 0$, $g_{22} = 1$ (which follows immediately from the way how left-invariant fields l_1, l_2 are expressed in terms of e_1, e_2). Classic books present it as

$$ds^2 = e^{-2y}dx^2 + dy^2$$

(metric coefficients are picked up from such an expression if one compares it with its general form $ds^2 = \sum_{i=1,j=1}^{n} g_{ij}(q)dx_i dx_j$). In our case dx, dy denote basic coordinate one-forms (or, better to say, fields of one-forms). At each point q they form the basis of a CO-TANGENT space T_q (the so-called DUAL BASIS, since as linear functionals on T_q they satisfy $dx(\partial_1) = 1, dx(\partial_2) = 0, dy(\partial_1) = 0, dy(\partial_2) = 1$
In general $dx_1, dx_2, \ldots, dx_n$ satisfy $dx_i(\partial_j) = \delta_{ij}$ where δ_{ij} is 1 if $i=j$, 0 if $i \neq j$ (δ_{ij} is called the *Kronecker symbol*). They form a basis dual to $\partial_1, \ldots, \partial_n$ in T_q. The "product" $dx_i dx_j$ more properly reads $\quad dx_i \times dx_j$ (TENSOR PRODUCT of two one-forms is a two-form).

Given a curve C by its parametric equations $x_1 = x_1(t), \ldots, x_n = x_n(t); a \leq t \leq b$, its length is $\int_a^b \sqrt{\langle \dot{r}, \dot{r} \rangle} dt$ where a tangent vector r has

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components $\dot{x}_i$ w.r.t. a coordinate basis; "dot" signifies differentiation w.r.t. t.

The distance d between two points in a Riemannian space G is defined as the largest lower bound of lengths $l(C)$ where a curve C (being a subset of G) connects the two points. If the two points p, q, are "not too far from one another", then there is unique C from p to q which has d as its length.

CEODESICS of G can be thus defined. They turn out to be solutions to the following system of ordinary 2nd order differential equations:

$$\ddot{x}_1 + \sum_{i,j=1}^{n} G^1_{ij}\, \dot{x}_i \dot{x}_j = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\ddot{x}_n + \sum_{i\,j=1}^{n} G^n_{ij}\, \dot{x}_i \dot{x}_j = 0$$

The above involved functions G^i_{jk} (x) can be found as

$G^k_{ij} = \sum_{m=1}^{n} \frac{g^{km}}{2} (g_{mi,j} + g_{mj,i} - g_{ij,m})$. They are called *connection coefficients* (or *Christoffel symbols*).

In the above, a standard notation g^{km} is used to denote respective entries of g^{-1}; these entries form the so called contravariant metric tensor. In our example, the matrix

$$[g^{km}] = g^{-1} = \begin{bmatrix} e^{2y} & 0 \\ 0 & 1 \end{bmatrix}$$

To specify the geodesic, one has to set up an initial value problem (as well as to specify the tangent vector to C at the initial point). Locally, geodesics behave like straight lines in a flat space: if the direction is specified, then there is a unique geodesic through a given point p.

In our example, the above definitions imply

$G^1_{12} = -1,\ G^2_{11} = e^{-2y}$,

All other connection coefficients G^i_{jk} vanish. From the above it follows $G^k_{ij} = G^k_{ji}$

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The geodesic system now reads $\quad \begin{cases} \ddot{x} - 2\dot{x}\dot{y} = 0 \\ \ddot{y} + e^{-2y}(\dot{x})^2 = 0 \end{cases}$

Let us look for y=y(x) solutions, $\ ^{"\prime\,"}$ is used for the derivative w.r.t. x

$y' = \frac{dy}{dx} = \dot{y}(dt/dx)$ or $\dot{y} = \dot{x}\, y'$

$\ddot{y} = \ddot{x}y' + \dot{x}(dy'/dt) = \ddot{x}y' + \dot{x}y''\dot{x}$ or $\ddot{y} = \ddot{x}y' + (\dot{x})^2 y''$

Expressing $\ddot{x}$ as $2y'(\dot{x})^2$ from the 1st equation, we obtain in the second (after dividing by$(\dot{x})^2$):

$2(y')^2 + y'' + e^{-2y} = 0$

The general solution is $y = \frac{1}{2}\ln[\lambda - (x - a)^2]$ where a constant λ can only be positive, while a is arbitrary. Introducing $z = e^y$, we got equations $z^2 + (x - a)^2 = \lambda$ which describe semicircles $z > 0$ half-planes. "Singular" solutions $x{=}const$ describe vertical half-lines. The sketch below shows them in the respective geometry (which is the 2-dim Lobachevsky geometry). The famous Euclid's 5th postulate does not hold: P$\notin$ L but there are many "lines" (they are represented by geodesics) through P which do not intersect L (any $\tilde{L}$ "between" L_1 and L_2 shown, $\tilde{L}$ has such a property). Since the time of Euclid, mathematicians had been trying to prove the 5th postulate as a theorem. In the below presented geometry all of Euclid's axioms (except the 5th postulate) are demonstrably true.

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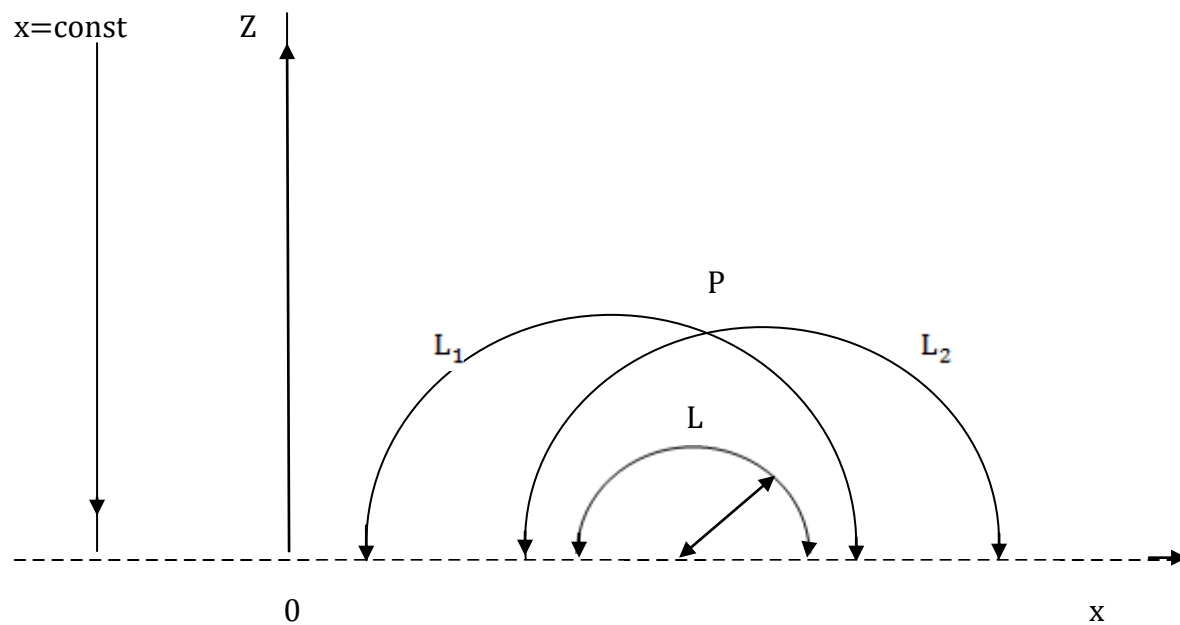
For the definition of an ACTION of a Lie group G on a space M see L-6.

Example 1 (of an action of our G on $M = R^2$). Here G is generated by parallel translation and by scale transformations. Let $g = (t, a, b) \in G$, define the following action

$$(1) \qquad gm = r = (p, q) = (xe^t + a, ye^t + b)$$

on a generic point $m = (x, y)$ of R^2.

One easily verifies it is an action, that is, R^2 is a HOMOGENEOUS SPACE (of G). Remark 1. Given G, M, there, in general, might be different actions of G on M.

A possible way to verify that (1) defines an action, is to notice that (1) can be implemented by matrix multiplication

$$\hat{g}\hat{m} = \begin{pmatrix} e^t & 0 & a \\ 0 & e^t & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}, \text{ where } \hat{g}, \hat{m} \text{ implement } g \text{ and } m.$$

This also shows the group G is our well-known ("basic affine") one. Now it acts not on itself (by earlier introduced left translations) but on a lower dimensional manifold (="surface"). However, the Lie algebra $L(G)$ can also be introduced as a certain totality of vector fields (on M rather than on G).

Define an H-ORBIT of a point (x, y) from M as the totality of ALL points (p, q) of the form $(p, q) = g * (x, y)$ where g goes over a subgroup H of G, " $*$ " denotes the action.

To realize $L(G)$, consider one-dimensional subgroups H of G. It is enough to choose a finite number of those (they should generate the entire G). $\{(t, 0, 0)\}, \{(0, a, 0)\}, \{(0, 0, b)\}$ are natural choices for Hs in Example 1.

H chosen, M becomes a union of (disjoint) orbits. When $\dim H = 1$, a typical orbit is a curve in M. The totality of tangent vectors (="velocities") form a vector field in question.

Example 2 (it goes after Example 1 and it presents fields l_1, l_2, l_3 on $M = R^2$). Namely, start from $H = \{(0, a, 0)\}$ applied to $m = (x, y)$. In general, differentiate $r = (p, q) = g * m$ with respect to the same variable which parameterizes the one-dimensional group H. Take the zero value of the parameter, afterwards. One thus gets the value $l(m)$ of the vector field in

question; $l(m)$ corresponds to that H (as well as $l(m)$ determines the H-orbits in M). The answer for our choice of one-dimensional subgroup in G is

$$(2) \qquad l_1(m) = \partial_x, \qquad l_2(m) = \partial_y, \qquad l_3(m) = x\partial_x + y\partial_y$$

Clearly, the commutation relations are

$$(3) \qquad [l_3, l_1] = -l_1, \qquad [l_3, l_2] = -l_2$$

which differ by signs in the r.h.s. from our "standard" table. Equations (3) determine a so-called DUAL LIE ALGEBRA.

Exercise 1. A Lie algebra is isomorphic to its dual. Or, (3) is referred to as an "ANTI-REPRESENTATION" of $L(G)$.

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Exercise 2. Another way to get the above table (3), is to consider the left action of our G ON ITSELF. In that case, the vector fields l_1, l_2, l_3 are velocity fields of corresponding one-dimensional trajectories.

In the next exercise, C, S stand for $\cos t$, $\sin t$.

Exercise 3. Let the group E_2 consist of all matrices of the form

$$\begin{pmatrix} C & -S & a \\ S & C & b \\ 0 & 0 & 1 \end{pmatrix}.$$

Introduce the following action of E_2 on the plane:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \text{ goes into the product } \begin{pmatrix} C & -S & a \\ S & C & b \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} xC - yS + a \\ xS + yC + b \\ 1 \end{pmatrix}.$$

The Lie algebra $L(E_2)$ is generated by vector fields $\partial_x, \partial_y, v_3 = -y\partial_x + x\partial_y$. Clearly, E_2 as the group of transformations is generated by rotations and parallel translations. The commutation relations table is as follows:

$$[v_3, \partial_x] = -\partial_y, [v_3, \partial_y] = \partial_x.$$

Let us discuss why $L(G)$ and $L(E_2)$ are not isomorphic.

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In a Lie algebra L the following operators are naturally defined:

$$ad_{\vec{a}} : \quad \rightarrow L \qquad \text{where} \qquad ad_{\vec{a}}\vec{} = [\vec{a},\vec{b}]$$

In other words, the first argument in the Lie bracket is now fixed. If a basis $\vec{e_1},\vec{e_2},...,\vec{e_n}$ is fixed, then

$$\vec{a} = \sum_i a_i \vec{e_i}$$, where ad_i stands for the operator corresponding to the basic vector number i.

Remark 1. A matrix D of ANY linear operator $\hat{D}$ can be found as follows: 1) decompose vectors

$\hat{D}\vec{e_1},\mathrm{D}\vec{e_2},...,\mathrm{D}\vec{e_n}$ w.r.t that very basis $\vec{e_1},\vec{e_2},...,\vec{e_n}$; namely, $\hat{D}\vec{e_i} = \sum_k D_i^{\kappa}\vec{e_k}$; 2) the columns

of D are exactly vectors $\hat{D}\vec{e_1},\mathrm{D}\vec{e_2},...,\mathrm{D}\vec{e_n}$ (considered as columns with respective numerical

components).

Our standard notation will be $D=(D_i^k)$ where D_i^k stands both in row number "**k**" and column

number "**i**".

Remark 2. Let a second basis u_1, u_2, ..., u_n be fixed. By the TRANSITION MATRIX we

understand the matrix (w.r.t. the 1st basis) A of the (unique!) linear operator which takes $\vec{e_1}$ into u_1, $\vec{e_2}$

into u_2, ..., $\vec{e_n}$ into u_n. The expression for the new matrix $\tilde{D}$ of the original linear operator $\hat{D}$ is

$$\tilde{D} = A^{-1}DA \qquad (1)$$

It follows from the definition, that a Lie algebra L is isomorphic to a Lie algebra V if and only if

there exist a basis $\vec{e_1},...,\vec{e_n}$ in L and a basis $v_1,...,v_n$ in V, such that the two commutation relations

tables are IDENTICAL. From this and (1), a useful SUFFICIENT CONDITION for L and V to be NON-ISOMORPHIC follows:

If for a (certain) vector $\vec{a} \in L$ there is such an eigenvalue h of $ad_{\vec{a}}$ that h is not an eigenvalue of any $ad_{\vec{v}}$ in V, then L and V are not isomorphic.

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Recall the basic affine Lie algebra L(G) defined by

$$[e_3, e_1] = -e_1 \, , [e_3, e_2] = -e_2; \quad (2)$$

And L(E$_2$) defined by

$$[e_3, e_1] = -e_2 \, , [e_3, e_2] = e_1 \, . \quad (3)$$

From (2) one finds that in the L(G)

$$ad_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ ad_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \ ad_2 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} ;$$

Which implies, $ad_a = \begin{bmatrix} -a_3 & 0 & a_1 \\ 0 & -a_3 & a_2 \\ 0 & 0 & 0 \end{bmatrix}.$

The eigenvalues are the roots of the characteristic equation
$$det(ad_a - h\mathbb{1}) = 0$$
So, for any a in L(G) , they all are real.

For the case of V=L(E$_2$), equation (3) implies
$$ad_v = \begin{bmatrix} 0 & v_3 & -v_2 \\ -v_3 & 0 & v_1 \\ 0 & 0 & 0 \end{bmatrix}$$
There are certain non-zero purely imaginary eigenvalues h of ad_v.

We have thus proved that the two Lie algebras are not isomorphic.
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Minkowski spacetime $\mathbf{M}_0$ is generally regarded as the appropriate area within which to formulate those laws of physics that do not refer specifically to gravitational phenomena. We would like to spend a moment here at the outset briefly examining some of the circumstances which give rise to this belief.

We shall adopt the point of view that the basic problem of science in general is the description of "events" which occur in the physical universe and the analysis of relationships between these events. The term "event" is used, however, in the idealized sense of a "point-event", that is, a physical occurrence which has no spatial extension and no duration in time. One might picture, for example, an instantaneous collision or explosion or an "instant" in the history of some (point) material particle or photon (to be thought of as a "particle of light"). In this way the existence of a material particle or photon can be represented by a continuous sequence of events called its "worldline". We begin with an abstract set W whose elements are called "events". We shall provide W with a mathematical structure which reflects certain simple facts of human experience as well as some rather nontrivial results of experimental physics.

Events are "observed" and we will be particularly interested in a certain class of observers (called "admissible") and the means they employ to describe events. We identify events by their "location in space and time".

DEF.1. *Minkowski world* is a 4-dimensional real vector space $\mathbf{M}_0$ on which is defined a nondegenerate, symmetric bilinear form g of Lorentzian signature. This g is referred to as a *Lorrentz inner product* on $\mathbf{M}_0$. Thus, there exists a basis e_0, e_1, e_2, e_3 for $\mathbf{M}_0$ with the property that is $\mathbf{v} = v^m e_m$ and $\mathbf{w} = w^s e_s$, then

$$g(\mathbf{v},\mathbf{w}) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

REMARK 1. We will sometimes make use of an *affine* Minkowski world M_0. It may be referred to as $\mathbf{M}_0$ devoid of its linear structure. To define M_0 strictly, we need to introduce (our version is a simplified one of [Wo, p.11]) the notion of *action of a group on a manifold*. The

students are supposed to be aware (intuitively, at least) of the notion of a manifold X. The key examples will be later described in detail. Usually, the *tangentspace* to X at $p \in X$ is denoted by T_p.

DEF.2. An *action* of a group G on a manifold X is a map F from the direct product of G and X into X such that $g \to F(g, .)$ is a homomorphism of G into the group of transformations of X. The *kernel* of the action is the kernel of this homomorphism. The action is *effective* if the kernel is trivial. The action is *free* if, for every $x \in X$ the subgroup $\{g \in G : g(x) = x\}$ is trivial. An action is *transitive* if given $x, y \in X$, there exists $g \in G$ such that $g(x) = y$. An *orbit* of a point x is the set $\{g(x) : g \in G\}$.

It is obvious that a free action is effective. In case of a transitive action one may also say that G acts with just one orbit, the latter being the X itself. Other words to label the case: "X is a homogeneous space" (for the group G).

DEF.3. Given a linear space V, the set W is called an *affine space* (with V referred to as an *associated* linear space to W) if V, considered as an abelian group, acts freely and transitively on W.

The affine Minkowski world is one of the simplest examples of "Lorentzian manifolds". To introduce the latter (and a few more sophisticated objects) we recall the following.

FACT 1 (notion of a *tangent map*). Let $f : \mathbf{X} \to \mathbf{Y}$ be a map from one manifold into another. If the neighborhoods of $p \in \mathbf{X}$ and of $q = f(p) \in \mathbf{Y}$ are supplied with coordinate systems then f may be regarded as $f : \mathbf{x} \to \mathbf{y}$, where

$$\mathbf{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y^1 \\ \vdots \\ y^m \end{pmatrix}.$$

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The tangent map (at point p) is denoted by df or f_* (with p as the subscript to be more pedantic) and is exactly the linear map from T_p into T_q defined by the matrix

$$(\partial y^s / \partial x^k). \tag{1}$$

The derivatives in (1) are understood to be calculated at point p. All actions involved below will be *smooth*, that is for each f its tangent map is defined at any point p. We will mostly work with examples of homogeneous manifold **X** with a prescribed smooth action of a group G on it.

The Minkowski linear world $\mathbf{M}_0$ acts on the affine space M_0 to which $\mathbf{M}_0$ itself is an associated vector space.

DEF. 4. This M_0 is called an affine Minkowski world provided its tangent spaces T_p at all pints p are supplied with the Lorentzian inner product invariant under the corresponding action.

REMARK 2. The invariance means exactly that all linear maps (1) are isometries (i.e. inner product conserving linear bijections) between corresponding tangent spaces. One might also think of various tangent spaces to be identified (via parallel translations) with the $\mathbf{M}_0$ itself. Mention that there are no distinguished elements in the affine space. It is not however the case when the linear space is considered.

In case the basis in $\mathbf{M}_0$ is chosen one may introduce in the following manner coordinates in the corresponding affine space M_0. Choose a point O in M_0 (the *origin*). It has (by definition or as a consequence of the ongoing anzats) zero coordinates. Any other point $x \in M_0$ is supplied with exactly those coordinates which does have the (unique) vector $\mathbf{a} \in \mathbf{M}_0$ which (being an element of the abelian group acting on M_0) takes O into x. Such coordinates being introduced, the action $g : x \rightarrow y$ has the well-known simple form

$$y^m = x^m + a^m .\tag{2}$$

It follows from (2) that in such coordinates all matrices (1) are units. It is not however the fact when the coordinates are curvilinear (spherical, cylindrical, etc.). It is rather clear that the above introduced coordinates (they may also be referred to as *affine coordinates*) are most natural, they make the rule for the action look very simple.

DEF. 5. A manifold X each tangent space of which is supplied with a Lorentzian inner product is called a *Lorentzian manifold*. A map $f: X \rightarrow Y$ between two Lorentzian manifolds is an *isometry* if each its tangent map is a linear isometry of the corresponding Lorentzian vector spaces.

One can express it differently, in terms of a *metric tensor* g on X and of a metric tensor h on Y. Each tangent map $f_* : T_p \rightarrow T_q$ thus has the property $h(f_*\mathbf{a}, f_*\mathbf{b}) = g(\mathbf{a}, \mathbf{b})$ for any two vectors $\mathbf{a}$, $\mathbf{b}$ from T_p; $p \in X$, $q = f(p) \in Y$.

In many cases we will deal with examples when X is a homogenous space with regards to an action of the group G acting by isometries.

2. CAUSALITY ISSUES

DEF. 1. Let $W \subset \mathbf{M}_0$ be a linear subspace. The *causal character* of W is: (a) *spacelike* if g is positive definite on W; (b) *lightlike* if g is positive semi-definite but not positive definite on W; (c) *timelike* otherwise. Suppose $\mathbf{v} \in \mathbf{M}_0$; the *causal character* of $\mathbf{v}$ is that of span $\mathbf{v}$; $\mathbf{v}$ is defined as *causal* if $\mathbf{v}$ is not spacelike.

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Note: Vectors are shown in **bold** (like **a**).

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Earlier (in L-4) we have learned how to introduce a left-invariant metric on a group. There, we have introduced a positive-definite matrix $g = (g_{ij})$ at the neutral element of the group G. In terms of g ("transported" to each point x of G by the respective left translation), the inner product of vectors is defined in each tangent space T. A pair (G,g) was an example of a Riemannian space.

If one starts from a non-singular matrix g with one negative eigenvalue (and n-1 positive ones), such a procedure results in a (homogeneous) Lorentzian space. Most important is the n=4 case; such pairs (G,g) are referred to as SPACETIMES. The equation $g(\mathbf{a},\mathbf{a}) = 0$ determines a (double) cone C in the respective tangent space. C at x goes into C at y under the differential (= the tangent map) of the respective left translation. C consists of LIGHTLIKE (LL) vectors (the zero vector is usually not considered to be a LL vector). Vector **b** is referred to as a TIMELIKE (TL) vector iff $g(\mathbf{b},\mathbf{b}) < 0$, SPACELIKE (SL) iff $g(\mathbf{b},\mathbf{b}) > 0$. Vectors **a**, **b** are said to be ORTHOGONAL iff $g(\mathbf{a},\mathbf{b}) = 0$. An orthogonal complement to a TL vector **Z** is a (dim=3) plane S which is interpreted as an (instant) physical SPACE of an (instant) OBSERVER **Z**. It is convenient to have **Z** normalized: $g(\mathbf{Z},\mathbf{Z}) = -1$.

To time-orient (G,g) means to fix a (continuous) field of CAUSAL CONES K^+_x in G. K^+ at the neutral element is naturally fixed by a choice of an observer **Z** at **1**: K^+ consists of 0 as well as of all those non-spacelike vectors **a** which satisfy $g(\mathbf{Z},\mathbf{a}) \leq 0$. Again, K^+_x is the image of K^+ under the differential of the respective left translation. The light future cone C^+ is the boundary of K^+.

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Let int K$^+$ stand for the open future cone ("int" is for *interior*). A curve in (G,g) is said to be a TL one, iff a tangent vector to it at any point x belongs to int K^+_x. Such curves (as well as those with LL tangent vectors) are of most importance in (Special and General) Relativity. A **Time Machine**, geometrically, is a closed timelike curve. Given a certain M = (G,g), it is one of the most intriguing questions: is there a time machine in M? The famous K. Gödel was, seemingly, the first to discover a time machine in a (topologically trivial) homogeneous space-time (his article was published in 1949).

Following publications by R. Penrose, S. Hawking and others, J. Beem and P. Ehrlich introduced the hierarchy of **Causality Conditions** (see their book Global Lorentzian Geometry, 1981). In 1988, A. Levichev showed that in the **Class of Homogeneous Lorentzian Spacetimes** six (of the most widely used ten causality conditions) conditions are equivalent. The (thus surviving) five conditions are pair-wise non-equivalent in this class ("On causal structure of homogeneous Lorentzian manifolds", Gen.Rel.&Gravit., v.21, n.10, 1989, pp.1027-1045). Other problems still exist in that subject of **Homogeneous Chronogeometry**. Some of them have been solved by A. Levichev's students in late 80s and early 90s.

To give an idea how (more or less) general theorems might look like, one can say the following. Of importance is the way how the causal cone K$^+$ is located w.r.t. the derived algebra D = [L,L]. Namely, if D does not contain any LL vector (which means D is SL), then the space-time M is perfect from the point of view of causality: it is **stably causal**. The last condition says that you may change the metric in such a way that an 'old' K$^+$ is inside of the new one (so you get more TL curves) but even in the new metric there is no time machine. A possibility of time machine increases in the case when D is TL. There is no time machine when D is LL but such spaces are usually not stably causal (since w.r.t. the new metric D becomes time-like). On the other hand, there are the so called **energy conditions** which come into interplay (they are introduced in [SaWu]. Those are mostly violated in cases of a space-like D (such spaces are physically unacceptable). So to say, energetically acceptable space-times 'almost' possess time machines (or, in some cases, actually do, e.g. Gödel's universe).

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is dedicated to <u>curvature</u>. Let M be a (u, v) – parameterized surface in $\mathbb{R}^3$. M is, thus, a Riemannian space (with an induced metric: we can view each tangent vector as a vector in $\mathbb{R}^3$, the dot-product of any two vectors can be calculated). A curvature of M is determined by a single function $K = K(u, v)$, a Gaussian curvature of M. It is a vanishing constant for a plane, for a cylinder, for a cone; $K = 1/a^2$ for a sphere of radius a. Better understanding is achieved by viewing K as a product of (the two) extreme values of the curvatures of intersections with normal planes at a given point of M: a normal plane to M is determined by choosing a direction in the tangent to M plane. When M is a plane, each intersection is a straight line with a vanishing curvature k. For a sphere each intersection is a circle of radius a which has curvature $1/a$, etc. When the two curves bend into different half spaces (at a point of a hyperbolic paraboloid or of a hyperboloid of one sheet) the numbers k_1, k_2 are of opposite signs; their product $K = k_1 k_2$ is negative. A pseudo-sphere (for geodesics on which the Lobachevski geometry is valid) is a surface with a constant negative curvature.

On any Riemannian (or Lorentzian, etc.) space, the <u>curvature tensor</u> $\mathcal{R}$ can be introduced. Originally, it is a (1,3)-type tensor field, so R^i_{jkl} stand for its components. We introduce the <u>covariant derivative</u> $D_X Y$ first. It is a certain vector field, given fields X, Y. Technically, a covariant derivative is determined by (earlier introduced) Christoffel symbols Γ^i_{jk}. Namely, the i-component of $D_X Y$ equals

$$(D_X Y)^i = \frac{\partial Y^i}{\partial x^m} X^m + \Gamma^i_{jm} Y^j X^m$$

where the Einstein summation rule is presumed. Such a construction, applied to a tangent vector field X of a given curve C, determines the notion of a <u>parallel vector field Y</u> along C: the covariant derivative of Y along C should vanish at each point of C. As a corollary, a curve is a geodesic iff the covariant derivative of its tangent vector field w.r.t. the same tangent vector field vanishes (as a vector field on C).

Now, $\mathcal{R}_{XY}$ are linear operators (determined by a choice of X, Y): $\mathcal{R}_{XY}(Z)$ is a vector field

$$D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]} Z$$

Components R^i_{jkl} are coefficients in the decomposition of the above field w.r.t. a basis $e_1, e_2, \ldots, e_n$ when $Z = e_j$, $X = e_k$, $Y = e_l$. Technically, they are certain functions of coordinates. There are a lot of symmetries: it turns out that for $n = 4$ we need to know 20 components. When $n = 2$, one component is enough.

Exercise. For a dim=2 Lobachevski geometry, calculate the component R^1_{212}.

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Introduction to DIFFERENTIAL FORMS

It is based on the important notion of DUALITY. A dual to a vector space V is the totality W of all linear functions on V. Of course, W, itself, is a vector space, since the sum $f + g$ of two linear functions is a linear function; and, if s is a scalar, sf is a linear function. W contains a zero element (which assigns a number zero to each vector **v** from V). All usual properties of vector spaces take place for W. Always, they are of one and the same dimension: dim W = dim V.

Consider the x-axis as a one-dimensional vector space V, "**i**" is its basic vector. By "dx" mathematicians denote such a basic element of the dual space W, that the value of dx on **i** is the number 1. Since dx is a linear function, this determines dx completely. "dx" is defined everywhere on V.

Let V be a three-dimensional vector space with a standard basis **i**, **j**, **k**. By dx, dy, dz we denote the "dual basis" in W. Namely, dx maps **i** into 1, **j** into 0, **k** into 0; similarly, dy maps **i** and **k** into 0, dy maps **j** into 1; dz maps **i**, **j** into zeros, dz assigns 1 to **k**. Having the linearity in mind, the above defines dx, dy, dz as real-valued functions on V. Any other linear function on V is a unique linear combination of dx, dy, dz.

Multiple integrals will be introduced as integrals of a one-form (dim=1), of a two-form (double integral), etc. The "d" can now be interpreted as an OPERATOR of DIFFERENTIATION. It maps functions into one-forms, one-forms into two-forms, etc. All

usual rules apply (linearity, the product rule, etc.). Now, *dx* can also be interpreted as a result of application of an operator "*d*" to a coordinate function "*x*" (depending on the context, "*x*" is a function of a single variable, of two variables *(x,y)*, of three variables *(x,y,z)*, etc.).

Example: d applied to a 1-form *pdx + qdz* (where *p, q* might depend on *x, y, z*) results in:

-pydx∧dy + (qx-pz)dx∧dz + qydy∧dz

dx∧dy can be viewed as a function with two input channels. It is linear with respect to each channel (as well as it is anti-symmetric). From the above it follows that dx∧dy is also completely determined by its value on a pair $\mathbf{e}_1$, $\mathbf{e}_2$ which is 1. Also, *dy∧dx = -dx∧dy*, etc. (∧ denotes an EXTERIOR MULTIPLICATION)

dx∧dy, dx∧dz, dy∧dz is a standard basis of a space of all two-forms on R. If *x, y, z* parametrize a three dimensional surface M then the above notation introduces two-forms on M, linear combinations of those are (DIFFERENTIAL) two-forms on M (coefficients might be point-dependent).

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        If the Christoffel symbols are known, the connection one-forms and curvature two-forms method seems to be the most elegant to calculate curvature. Connection one-form

$$\Gamma_j^i = \Gamma_{jm}^i dx^m.$$

Curvature two-forms are obtained as

$$\theta_j^i = d\Gamma_j^i + \Gamma_k^i \wedge \Gamma_j^k.$$

On the other hand, each

$$\theta_j^i = R_{jkm}^i (dx^k \wedge dx^m)$$

These results of their computation in the case of Lobachevski plane.

$$\Gamma_1^1 = -dx^2, \qquad \Gamma_2^1 = -dx^1, \qquad \Gamma_1^2 = e^{-2x^2} dx^1, \qquad \text{where } x^1 = x, \ x^2 = y;$$

$$\theta_2^1 = -dx^1 \wedge dx^2, \qquad \theta_1^2 = e^{-2x^2}(-dx^1 \wedge dx^2)$$

        It follows from the above that $R_{212}^1 = -1$, $R_{112}^2 = e^{-2x^2}$ is the answer to the Exercise from L-8. To list some of the general symmetry properties of the curvature tensor, it makes sense to view it as a (0, 4) –tensor:

$$R_{ij\,km} = g_{ip} R_{j\,km}^p$$

One is getting a negative of the original component if; 1) 1st pair of indices is interchanged 2) 2nd pair is interchange, 3) two pairs are. And there are more symmetries.

The Ricci tensor "Ric" is introduced as the following (0, 2) –tensor:

$$R_{jm} = R^{p}_{jpm}$$

It can be converted into a (1, 1) –tensor:

$$R^{k}_{m} = g^{kj}R_{jm} \quad ,$$

The trace, $S=R^{m}_{m}$, of the last tensor is referred to as the scalar curvature S. Technically, S is a function of coordinates.

Exercise: Find S of the Lobachevski plane. (Answer: S=-2, a constant)

Formally, the curvature tensor can be non-trivial already in dim=2 (but it is determined by the scalar curvature S alone). In dim=3, the curvature properties are completely determined by the Ricci tensor.

In any dimension, the Ric – (gS)/2 is referred to as the Einstein tensor G.

For a Lorentzian case (and dim=4), The Einstein field equations of General Relativity read: G=T+E, where T is the "stress-energy of all mater," and E is the "stress-energy tensor of the electromagnetic field."Mathematically, the r.h.s is viewed as given. Then one has to look for metric g (g_{ij} being unknown functions, solution to the 2nd order PDE system). Geometrically, if metric is given, one calculates the Einstein tensor G which should satisfy certain properties ("energy conditions") in order the metric g be physically acceptable. It is interesting that for homogeneous spaces energy conditions versus causality conditions restrict severely the list of possible metrics (when a space is "too good" from the point of view of causality, then energy conditions are violated in most cases).

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For the Minkowski space-time the Ricci tensor vanishes. That means there is no curvature and there is "no matter." It is the "trivial solution" to Einstein's equations.

The (scale-extended) Poincare group P is the totality of all those transformations of Minkowski space-time M_0 which preserve the system of causal cones. Dim $P = 11$ and P is generated by parallel translations (dim = 4), by Lorentz transformations (with "boosts" and rotations being "typical representatives;" dim $L_0 = 6$). Scaling adds one more dimension. Let the same demand of preserving the causal structure be left only locally:

$$f(K_x^+ \cap D) = K_{f(x)}^+ \cap D$$

In the above, D is a fixed open region in M_0, and f might not be defined on all of M_0.

Now we have a larger group, the CONFORMAL GROUP G. It contains P as a subgroup, with dim $G = 15$. Four parameters are added due to the fact that a typical (not from P) transformation is an inversion about a light cone (similar to an inversion about a sphere in the case of a Euclidean space). To fix such a cone C, we need to fix four numbers (coordinates of the C's center of symmetry). G is the main symmetry group of Segal's Chronometric Theory which leads to the development of Special Relativity. P_0, which is P without scaling, is the main symmetry group of conventional theoretical physics.

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