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Pseudo-Hermitian realization of the Minkowski world through DLF theory

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Abstract

Segal's chronometry is based on a space–time D, which might be viewed as a Lie group with a causal structure defined by an invariant Lorentzian form on the Lie algebra $u(2)$. There are exactly two more non-commutative four-dimensional Lie algebras that admit such a form. They determine space–times L and F. The world F is based on the Lie algebra $u(1,1)$, in terms of which the pseudo-Hermitian realization of the Minkowski space–time is introduced and studied. The world L is based on the oscillator Lie algebra. The main idea of the DLF approach to modeling particles and interactions is that there are three Hamiltonians (the 'Russian Troika') to drive the evolution of a physical system.

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1. Introduction

This paper is dedicated to the DLF approach, which is based on Segal's chronometric theory. The latter theory (see surveys [Le-93, Le-95]) has been presented in dozens of articles (see [JFA-02] for the complete list of Segal's publications).

It is hardly possible to concisely state the DLF picture before introducing substantial mathematical background. In terms of physics-related topics, a summary of the findings and suggestions is presented in the last section.

Let M denote the Minkowski space–time (in its *Hermitian realization*, see section 5, where the Caley map formula is given). Let D stand for the unitary group $U(2)$. The image of M under the Caley map (refer to [Se-76, p. 68] or [Le-95], as well as to section 7, for more details) is a dense open subset in D.

Let us view M as a vector group. It is commutative: each left translation is the respective right translation, too. The family $\{C_y\}$ of subsets in M forms a bi-invariant cone field; $C_y = y + C$, where C is a light cone at the origin of M. Due to the Caley map, there is the respective cone field on D, too.

On the universal cover $D^\sim$ of D, one can introduce *future sets* in a canonical way. These sets are determined by the above cone field and by the choice of orientation in time; they form the causal structure on $D^\sim$.

Let G denote the conformal group $SU(2,2)$. Recall the well-known linear-fractional G-action on D,

$$g(z) = (Az + B)(Cz + D)^{-1}, \qquad (1.1)$$

where an element g is determined by 2×2 blocks A, B, C and D. This action is canonically lifted to the $G^\sim$-action on $D^\sim$ (the latter action preserves the causal structure). Proofs of the above statements can be found in [Se-76, PaSe-82a].

Theorem 1 ([Al-76, Se-76]). *If a bijection f of $D^\sim$ preserves its causal structure, then f is an element of the transformation group $G^\sim$, determined by the action (1.1).*

It is known ([PaSe-82a, PaSe-82b]) that to model particles on $D^\sim$, one can start with the world D, a compact one. The respective property is called an 'automatic periodicity'.

Recall that the universal cover $P^\sim$ is a two-cover of the Poincaré group P (by which we here understand the respective 11-dimensional (11D) group that includes scaling). The formula for the Caley map C_D from M to D will be given in section 5. The image of this map is an open dense subset of U(2). The group $P^\sim$ acts on $D = U(2)$, too.

Theorem 2 ([PaSe-82a]). *The isotropy subgroup of an event x from D is isomorphic to $P^\sim$. $P^\sim$-action on D, $P^\sim$-action on M and the Caley map determine a commutative diagram.*

Remark. The Caley map is designated by C_D in order to allow a change in C_F when the world F is taken instead of D. The map C_F is defined in section 6, where the pseudo-Hermitian realization of the Minkowski world will be introduced.

2. Lie algebras *d*, *f* and *l*

Recall that invariance of a symmetric form on a Lie algebra n means skew-adjointness (relative to this form) of all 'ad_x' operators. Such an operator maps **y** from n into $[\boldsymbol{x}, \boldsymbol{y}]$; here [,] is a Lie bracket on n.

The following statement holds (see [Mi-76]):

Theorem 3. *A metric on a connected Lie group N is bi-invariant iff the respective form on its Lie algebra n is invariant.*

Remark 1. It is well known that a non-degenerate invariant form in a simple Lie algebra has to be proportional to the Cartan–Killing form.

Theorem 4 ([Le-85, GuLe-84]). *There are exactly three 4D non-commutative Lie algebras that admit invariant non-degenerate form of Lorentzian signature, namely* $d = u(2)$, $f = u(1, 1)$, $l =$ osc.

Remark 2. The first two cases are of no surprise since there are no other non-commutative reductive Lie algebras in dimension 4. The third one is a solvable Lie algebra that can be defined by the following commutation table: $[l_2, l_3] = l_1$, $[l_2, l_4] = l_3$, $[l_4, l_3] = l_2$.

3. Conformally covariant wave operators

Quantum-mechanical states can be viewed as sections of (certain) vector bundles over $D^\sim$; these are the so-called 'induced bundles', since they are determined by $G^\sim$-representations induced from finite-dimensional Poincaré group representations. For a scalar particle, the fiber is complex, 1D, etc (see section 7 for more details).

In Segal's chronometry the entire list of known particles is mathematically derived. One chronometric particle ('exon') has not been detected, yet (more details can be found in [Se-91] or in the survey [Le-95]). It has been conjectured in [Le-10a] that Segal's exon is just the proton.

In chronometry, the 'architecture' of the scalar bundle is determined by a certain conformally covariant second-order differential operator (the so-called 'curved wave operator', compared to the standard 'flat wave operator'), see [PaSe-82a].

The scalar bundle (together with known $P^\sim$ finite-dimensional representations) determines higher spin bundles (see [PaSeVo-87, Se-98, SeVoZh-98]).

The following result is useful for describing the curved wave operator explicitly. Proofs of stronger versions of this statement can be found in [Ør-81].

Theorem 5. *In a 4D conformally flat Lorentzian manifold of constant scalar curvature S, if T is the Laplace–Beltrami operator, then* $T + S/6$ *is conformally covariant.*

We will use (see [PaSe-82a]) a certain basis $\{\boldsymbol{L}_{ij}\}$ (with $\boldsymbol{L}_{ij} = -\boldsymbol{L}_{ji}$) in su(2,2). Here is the commutation table:

$$[\mathbf{L}_{\mathrm{im}}, \mathbf{L}_{\mathrm{mk}}] = -e_{\mathrm{m}}\mathbf{L}_{\mathrm{ik}},$$

$(e_{-1}, e_0, e_1, e_2, e_3, e_4)$ stands for $(1, 1, -1, -1, -1, -1)$.

Remark 3.2. Consider a Lie group N corresponding to one of the Lie algebras in question. There is a bi-invariant metric of Lorentzian signature on N. In terms of a certain basis of left-invariant (or right-invariant) vector fields on N, the Laplace–Beltrami operator is the quadratic Casimir operator (see sections 3.1–3.3 below).

3.1. The world D

Here are a few more chronometric settings. The G-action results in vector fields L_{ij} (non-boldface) on D. Their commutators all have the minus sign as compared to the above commutators of (abstract) generators $\boldsymbol{L}_{ij}$.

Vector fields

$$X_0 = L_{-10}, X_1 = L_{14} - L_{23}, X_2 = L_{24} - L_{31},$$
$$X_3 = L_{34} - L_{12}$$

form a left-invariant orthonormal basis on $D = U(2)$. Let us keep the same symbols to denote respective vector fields on $D^\sim$. Globally, $D^\sim$ is $R^1 \times S^3$, where S^3 is represented by the group $SU(2)$. In a cosmological model based on $D^\sim$, there is a conformal invariant R, interpreted as the radius of a 3D (spherical) space. Segal has put it for the (long wanted by Dirac and others) third fundamental constant—additionally to the speed of light and to the Planck's constant. If (for mathematical convenience) one takes $R = 1$, then the scalar curvature is 6. Bear in mind that curvature calculations are simplified significantly due to the bi-invariance of the metric (see [Mi-76] and section 8). The conformally covariant wave operator is

$$X_0^2 - X_1^2 - X_2^2 - X_3^2 + 1,$$

as shown in [PaSe-82a].

Remark 3.1.1. In the general relativity theory (GRT), $D^\sim$ is known under the name of Einstein's static universe (see [Kr-80, p 122]). The respective solution (of the GRT Einstein equations) is interpreted as an ideal fluid. If we do not assume $R = 1$, then the scalar curvature is $6/(R^2)$. Energy density and pressure both equal $1/(R^2)$. Energy conditions hold. See section 7 for proofs.

3.2. The world F

$F^\sim$ is R^4, topologically. It is the universal cover of the Lie group $U(1,1)$; the latter group is introduced more formally in section 6 . A relatively compact group F (being a 4D orbit in D) is determined by an orthonormal basis of vector fields H_0, H_1, H_2 and H_3 on $U(2)$. Here $H_0 = L_{-10} - L_{12}$, $H_1 = -L_{-12} - L_{01}$, $H_2 = L_{02} - L_{-11}$, $H_3 = L_{34}$. These fields generate a $u(1,1)$-subalgebra of $su(2,2)$. The scalar curvature is now negative 6; hence

$$(H_0)^2-(H_1)^2-(H_2)^2-(H_3)^2-1$$

is another conformally covariant wave operator.

Remark 3.2.1. For more details, see sections 5 and 6.

Remark 3.2.2. Treated as the solution of Einstein equations, it is interpreted as a tachionic fluid [Kr-80, p 57]. In the expression for the corresponding bi-invariant metric, there is a parameter a related to a choice of an invariant form on the simple su(1,1)-subalgebra of u(1,1). If not to choose $a=1$, then the scalar curvature is the negative of $6/a^2$. Energy density and pressure equal $-1/(a^2)$. The parameter a is a conformal invariant. Energy conditions do not hold, which is why the world F plays a special role in what is called the DLF approach (see section 7).

3.3. The world L

Topologically, $L^{\sim}$ is R^4. Its relatively compact form L (being a 4D orbit in D) is determined by a basis of vector fields l_1, l_2, l_3, l_4 on $U(2)$, where

$$l_1=-(L_{-10}+L_{04}+L_{-11}+L_{14}),$$

$$l_2=(1/2)(L_{-12}+L_{24}+2L_{30}+2L_{31}),$$

$$l_3=(1/2)(L_{-13}+L_{34}+2L_{02}+2L_{12}),$$

$$l_4=(1/8)(-5L_{-10}-3L_{-11}+3L_{04}+5L_{14}+4L_{23}).$$

One can prove that they generate an *oscillator Lie algebra* (its commutation table has been given in section 2). The expression for the invariant form (which determines the bi-invariant metric) follows from the formula for the respective wave operator (see below).

The scalar curvature is 0 (as shown in [Le-86b] where this world $L^{\sim}$ has been studied separately; the three worlds have been studied together in [Le-86a]).

The expression for the conformally covariant wave operator is $2l_1l_4-(l_2)^2-(l_3)^2$.In terms of GRT, we now have an isotropic electromagnetic field determined by a covariantly constant lightlike vector (see [Le-86b, p 123]). Energy conditions hold.

In [NaWi-93], a conformal field theory model is based on L. The model is an ungauged Wess–Zumino–Witten model. In [CaJa-92, CaJa-93], the oscillator Lie algebra l is used to formulate string-inspired lineal gravity as a gauge theory. However, the last two publications have mathematical errors. Namely, the upper left corner **h** (see expression (36) of [CaJa-92] and formula (3.41) of [CaJa-93]) has to be an identity matrix, rather than a diagonal matrix with 1, -1, entries. Not surprisingly, the authors cannot believe in one of their own conclusions (see [CaJa-93, p 249]). In [NaWi-93] (which refers to [CaJa-92, CaJa-93]), the invariant form in question is introduced correctly: expression (6) on p 3751.

The group L has been called the *oscillator* one in [Str-67]. Its important property of admitting a non-degenerate bi-invariant metric was only noted in the early 1980s: [GuLe-84, Le-85, MeRe-85].

4. The three worlds together

Each of the three is a symmetric space since the covariant derivative of the curvature tensor vanishes. This is guaranteed by the bi-invariance of the metric (see [Gr-71, p. 121]). The same holds for the Minkowski world, which is related to its vector group. That is why we can speak of four, rather than of just three worlds.

In [Sv-95] the oscillator Lie algebra has been embedded as a subalgebra into $su(2,2)$. However, the following important condition has not been satisfied: the light cone field (emerging due to the choice of the world L determined by the choice of the basis in section 3.3) has to coincide with the 'D-system' of light cones introduced in section 1.

The following statement says that such a condition is now satisfied for both 3.2- and 3.3-orbits.

Theorem 6. *The L-, F- and D-light cone systems coincide (over some open region in $U(2)=D$).*

Proof. At the $U(2)$ neutral element (which is identified, via the Caley map, with the origin of the Minkowski space–time M) vector fields H_0, H_1, H_2 and H_3 coincide with the standard basis e_0, e_1, e_2 and e_3 in M. The same holds for vector fields X_0, X_1, X_2 and X_3 from section 3.1. At the same space–time event vector fields l_1, l_2, l_3 and l_4 have values $-2(e_0+e_1)$, e_2, e_3, $(1/4)$ (e_1-e_0), respectively. It is thus shown that at that neutral element all four light cones coincide (to be convinced, check with table 1 of [SeJa-81]). The linear-fractional action of the entire conformal group G leaves that cone system invariant. As a result, this system is the same for the worlds considered since the four generators for each of the three groups are linear combinations of L_{ij} with constant coefficients. This holds for the entire orbit in question, in each of the three cases. Theorem 6 is thus proven.

Remark 4.1. This open region (from the statement of Theorem 6) cannot be the entire $D=U(2)$ because of different global topological structures of U(2), Osc, and U(1,1). It is proven in [Le-10b] that the largest region with the above properties is a set obtained from D by deleting a certain 2D torus.

Remark 4.2. Theorem 6 seems to be of importance in theoretical physics. Namely, the system of light cones in a space–time determines its causal structure. Now, the question is what are the most fundamental space–times? The most accepted model is the Minkowski world M. However, in terms of the quantum field theory, M is not the best choice.

Its isometry group is too large and its physical space is non-compact. That leads to (different types of) divergences and it implies the absence of invariant vacuum. Segal's chronometric theory is based on D (with physical space being a $dim = 3$ sphere). According to the DLF-approach, the three worlds co-exist as a single entity. Isometry groups are of dimension 7, whereas the role of M is reduced to that of a tangent space–time. To model particles and interactions, there are now three Hamiltonians (the 'Russian Troika' of [Le-05]) to drive evolution of a physical system. Each of the Hamiltonian vector fields is the center of the corresponding Lie algebra (when it is considered as the totality of certain left invariant fields on D, as discussed in subsections 3.1, 3.2 and 3.3).

5. Pseudo-Hermitian realization of the Minkowski world *M*

Let us first recall the well-known Hermitian model for M (see [PaSe-82a, Le-95]). Each event (or an element of M) is represented by a 2×2 Hermitian matrix h. All skew-Hermitian matrices ih form a Lie algebra $u(2)$. A typical element (t, L, f) of the simply connected 11D (scaling included) Poincaré group $P^{\sim}$ maps h into $e^t LhL^* + f$:

$$h \to e^t LhL^* + f. \tag{5.1}$$

In equation (5.1), t is a real number, L is a matrix from $SL(2,C)$, f is a Hermitian matrix. It is a well-known action of $P^{\sim}$.

The Caley map C_D (which has already been mentioned in section 1) is defined as follows:

$$C_D(h) = (1 + \mathrm{i}h/2)(1 - \mathrm{i}h/2)^{-1}. \tag{5.2}$$

The image of this map is an open dense subset of $U(2)$. The group $P^{\sim}$ acts on $D = U(2)$, too. The Caley map intertwines respective actions (see theorem 2). The possibility of the following pseudo-Hermitian picture seems to have gone unnoticed in the literature.

Recall the following description: a 2×2 matrix h (with complex entries allowed) is in $u(1,1)$ iff

$$h^*s + sh = 0,$$

where s is a 2×2 matrix diag$\{1, -1\}$.

Theorem 7. *There is a linear bijection Q of the Lie algebra u(2) onto u(1,1), and there is such a $P^{\sim}$-action on u(1,1) that one gets a commutative diagram (in other words, Q intertwines respective $P^{\sim}$-actions).*

Proof. Choose the bijection Q, which maps a Hermitian matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ into a pseudo-Hermitian matrix $\begin{bmatrix} a & -\mathrm{i}c \\ -\mathrm{i}b & -d \end{bmatrix}$. Q is a bijection between two real 4D subspaces in C^4. If a matrix L is from $SL(2,C)$, then it maps a pseudo-Hermitian matrix h into $A^*\bar{L}B^*hAL^{\mathrm{T}}B$:

$$h \to A^*\bar{L}B^*hAL^{\mathrm{T}}B, \tag{5.3}$$

where $\bar{L}$ is a complex conjugate of L (not a transpose!), $A = \mathrm{diag}\{1, i\}$, $B = \mathrm{diag}\{-i, -1\}$, and L^{T} is the transpose of L. Scaling and parallel translations both act like before, see the law (5.1). It is a straightforward exercise to verify that the two actions commute with Q. □

Remark 5.1. One can easily verify that Q is an isometry between the two pseudo-Euclidean vector spaces. There is nothing special in this particular choice of the map Q, since it can be combined with linear causal automorphisms of u(2) and of u(1,1).

Let us now introduce an analogue of the Caley map, C_F, from u(1,1) into U(1,1):

$$C_F(h) = [1-(shs)/2][1+(shs)/2]^{-1}. \tag{5.4}$$

Similarly to the original Caley map C_D, its analogue C_F is globally defined. It follows from (5.4) and from the definition of u(1,1) that the determinant of $[1+(shs)/2]$ cannot be zero. More details of the pseudo-Hermitian picture are provided in the next section.

6. *F*-represented *SU*(2,2)

As part of the DLF approach, consider the following matrix representation of the Lie group $G = SU(2,2)$. It is conjugate to the D-representation (the latter has been originally introduced in Segal's chronometry; see [PaSe-82a, Le-95] for a survey). This conjugation is performed by the following 4×4 matrix W: W is the direct sum of 1 with a certain 3×3 matrix. The only non-zero entries of the latter matrix are 1s on the auxiliary diagonal. Clearly, W^2 is the unit matrix.

The D-represented $SU(2,2) = G$ (call it DG, in brief) was composed of a certain set of pseudo-unitary matrices. Overall, DG has been defined with the help of a distinguished diagonal matrix, diag$\{1, 1, -1, -1\}$. Under the conjugation by W, we obtain $S = \mathrm{diag}\{1, -1, -1, 1\}$, which determines another copy of $SU(2,2)$ (denoted by FG). Clearly, an isomorphism between DG and FG is carried out (via conjugation in $SL(4,C)$) by the matrix W.

The group FG is composed of those matrices g (with unit determinant), which satisfy

$$g^*Sg = S \tag{6.1}$$

Similarly to the D-case, it is convenient to build each g of 2×2 blocks A, B, C and D. The maximal (essentially) compact subgroup K in D-representation consisted of block-diagonal matrices g, that is, $B = C = 0$. There is an analogue of K in FG, call it H. Formally, H is determined by the same condition as K was. Recall that the world F has been defined in subsection 3.2 as the Lie group $U(1,1)$, see below, equipped with a certain bi-invariant metric.

The above matrix S is the following direct sum of 2×2 matrices:

$$S = \mathrm{diag}\{s, -s\},$$

where $s = \mathrm{diag}\{1, -1\}$. Define $U(1,1)$ as the totality of all 2×2 matrices satisfying

$$z^*sz = s. \tag{6.2}$$

Lemma (It is an analogue of lemma 2.1.4 of [PaSe-82a]). *A matrix g from SL(4,C) belongs to FG if and only if the following conditions hold:*

$$\begin{aligned} &A^*sA - C^*sC = s, \quad D^*sD - B^*sB = s, \\ &D^*sC - B^*sA = 0. \end{aligned} \tag{6.3}$$

Based on (6.1) straightforward proof is omitted.

Let us now introduce the following FG-action on $\boldsymbol{F}$:

$$gz = (Az + B)(Cz + D)^{-1}. \tag{6.4}$$

Theorem 8. *Equation (6.4 defines a (formal) left action on* $F = U(1,1)$*, that is,* $(g'g)z = g'(gz)$*. If the matrix* $Cz + D$ *is non-degenerate, then gz belongs to F.*

Proof. The left action law can be verified in a straightforward manner. Or, one can notice that (6.4) is, formally, the same as the linear-fractional DG-action on U(2). The latter action is known to be the left one.

To prove the remaining part of the statement, assume that gz is defined and show that for $w = gz$ the condition (6.2) holds. To do so, start with $w^*sw = s$ and then transform it into an obvious identity. Here are the suggested steps:

$$\begin{aligned} &(z^*C^* + D^*)^{-1}(z^*A^* + B^*)s(Az + B) = s(Cz + D), \\ &(z^*A^*s + B^*s)(Az + B) = (z^*C^* + D^*)(sCz + sD), \\ &z^*A^*sAz + z^*A^*sB + B^*sAz + B^*sB \\ &\quad = z^*C^*sCz + z^*C^*sD + D^*sCz + D^*sD. \end{aligned}$$

The last equality is verified by applying equations (6.2) and (6.3). □

Remark. It is shown (see below) that for an arbitrarily chosen z from U(1,1), formula (6.4) is well defined in a certain neighborhood of z, and for elements g from a certain neighborhood of a neutral element in FG. Such an action is called a local one.

Let us show that (6.4) is not always defined. Take

$$z = \begin{bmatrix} r & b \\ \bar{b} & r \end{bmatrix}; \ g \text{ is defined by } A = D = \begin{bmatrix} ch & 0 \\ 0 & ch \end{bmatrix},$$

$$B = C = \begin{bmatrix} -sh & 0 \\ 0 & -sh \end{bmatrix};$$

where $ch = ch(t/2)$, $sh = sh(t/2)$ are hyperbolic cosine and sine of a real parameter t, and r is a real number. The matrix $Cz + D$ is singular if and only if $e^{2t}(r-1) = r + 1$. When $t \neq 0$, r is uniquely determined. The numerical value of b is determined by (6.2). Namely, $r^2 = 1 + b\,\bar{b}$, which means that such a choice is possible.

Theorem 9. *Equation (6.4) defines a local FG-action on* $F = U(1,1)$*. The subgroup H acts globally. The orbit of the neutral element (as well as the orbit of any other element of F) is the entire U(1,1).*

Proof. Obviously, $Cz + D = \mathbf{1}$, a neutral element of H, when g is a neutral element of FG. As a result, there is a neighborhood V of z and a neighborhood R of $\mathbf{1}$ in FG such that $Cv + D$ is non-degenerate for all v from V and for all g from R.

For an element g from H, $B = C = 0$ holds, which results in

$$gz = AzD^{-1}. \tag{6.5}$$

Since $\det g = (\det A)\ \det D = 1$, equation (6.5) is well defined. If in (6.5) we choose $g = p\,\mathrm{diag}\{A, 1\}$ with $p^4 = (\det A)^{-1}$, A from F, then (6.4) itself reduces to the left $U(1,1)$-action on itself (namely, by left shifts).

Theorem 9 is thus proven. □

7. More on Segal's chronometry and the DLF approach

Irving Segal (USA, 1918–1998) was one of the greatest mathematicians of the 20th century (see [AMS] and [JFA-02]). After World War II he spent two years at the Institute for Advanced Study, where he held the first of the three Guggenheim Fellowships that he was to win. Other honors included election to the National Academy of Sciences (USA) in 1973 and the Humboldt Award in 1981. At the University of Chicago (1948–1960) he had 15 doctoral students, and at MIT, where he was professor from 1960 onwards, he had 25.

'The chronometric theory by I. Segal is the crowning accomplishment of special relativity' is the title of the survey article [Le-93]. Let us adjust that claim by discussing briefly the main aspects of that theory.

Its world $D^\sim$ consists of Einstein's static universe E as the underlying conformal manifold. E is supplied with a (standard, general relativistic) metric. A future direction of time being chosen, this determines future causal cones in each tangent space of $D^\sim$. 'Future sets' are defined in $D^\sim$ itself [Se-76]. This causal structure gives rise to the symmetry group $G^\sim$, which is the universal covering of the (15D) matrix group $G = SU(2,2)$. The group G acts (without singularities) on $D = U(2)$. This action lifts canonically to the $G^\sim$-action on $D^\sim$. These and other notions can be found in greater detail in many of Segal's articles ([JFA-02] is dedicated to the memory of I Segal and it lists all his publications) as well as in [Le-95].

The Minkowski world is conformally embedded into D via the Caley transform. The radius R of the (physical, 3D) spherical space in D does not depend on the chosen metric from this conformal class, that is, from the metric in which it is calculated. In other words, R is a conformal invariant. From [Se-82, p. 854]: 'This radius R (in laboratory units) provides a natural third fundamental constant, in addition to $\hbar$ and c, which is required for fundamental physical theory to complete the program suggested by Minkowski (1908) of replacing limiting cases (as the Galilean group is of the

Poincaré group, when c goes to infinity, or classical physics as $\hbar$ goes to zero) by less degenerate and mathematically more natural structures.'

Designate by K the 7D Einstein isometry group. It is a so-called 'maximal essentially compact' subgroup of G. It consists of translations in time and rotations in space. Designate by P (respectively P_0) the 11D (respectively 10D) Poincaré group. The group P acts in M, P_0 being a subgroup. P_0 is generated by Euclidean rotations, Lorentz transformations and parallel translations. To obtain P, one has to add scaling transformations.

The chronometric energy $\mathbf{H}$ is the generator of time in E. Relative to each point of observation in D, the Minkowski world M is embedded P-covariantly, and the relativistic (or Minkowski) energy $\mathbf{H_0}$ is the generator of time in M relative to the Lorentz frame in M, which, at the point of observation, osculates the frame defined by the space–time splitting in E. For each unitary positive-energy representation of G, the corresponding chronometric energy exceeds the Minkowski energy by an amount that vanishes infinitesimally but increases with the spatial support of the state in question in terms of the appropriate quantum mechanical consideration. The inertial mass of a cosmologically long-lived particle is represented in accordance with Mach's principle as its interaction energy with the cosmic background and is correspondingly only K-invariant, implying approximate local P_0-invariance of its rest mass.

Additional background on chronometry is given in Segal's book [Se-76] and in many other publications (see [JFA-02, pp. 1–13]). In these articles the physical particles have been modeled, in accordance with the thrust of decades of theoretical investigation in this area, by induced bundles over causally oriented space–times.

Let us now conclude with the justification of the expression 'crowning accomplishment of special relativity'. Firstly, the conformal group G is semisimple in contrast with the Poincaré group. Hence, G cannot be regarded as resulting through a contraction process from a non-isomorphic Lie group of the same dimension. Secondly, it arises as the maximal local causal group of the special relativistic world M (see Theorem 1) in which only the 11D Poincaré group P can be globally (without singularities) realized. When compared with other theories based on the world M or on a particular space time of general relativity (GR), chronometry has other preferable features; let us mention a few:

– the absence of the unique Lorentzian structure (such a structure arises when a particular 'metric observer' is chosen);
– a better unification of elementary particles (let us mention a fundamental notion of 'stability' here, 'stable representations' describe stable particles);
– the existence of 'leaking' ([Se-91, Le-95, sections 6.1 and 6.3]), which gives a kinematic explanation of several decays;
– its application to extragalactic astronomy ([NiSe-86, DS-01, Da-05] and many references therein) has shown that it is capable of precise and detailed predictions regarding the cosmic redshift and other directly measured quantities, in spite of its lack of adjustable cosmological parameters.

As stated above, there are exactly four Lie algebras (from an infinite list in dimension 4), which admit an invariant non-degenerate form of Lorentzian signature. Such a form is known to correspond to a bi-invariant metric on the Lie group in question (M and D are among them, the remaining two being L and F).

Chronometry is derived from very general considerations of causality, stability and symmetry. It is an effective point of departure for elementary particle physics.

To model particles (in a given world), the Hamiltonian is fundamental. In the context of this paper, each Hamiltonian is the image of the central element of the Lie algebra in question. Now, when we have F- and L-Hamiltonians (in addition to the D-Hamiltonian), it is quite a new situation in the 'Particles and their Interactions' theory (the 'Russian Troika', [Le-05]). The world remains, however, a single (not many-fold) unity of events. To specify it as D (or F or L) means to choose a specific mode of the quantum-mechanical measurement. Here are the mathematical specifications.

Parallelization (of a vector bundle over space–time, see [PaSe-82a, Le-95]) is an important part of the chronometric approach. It is even more important in the DLF picture. Recall a few quantum-mechanical features, first.

According to quantum mechanics, each object is assigned its *state* (or *wave function* but this latter notion we better reserve for a more specialized situation, namely after a parallelization has been applied). An *elementary particle* (it 'lives' in a certain world $\mathbf{W}$ of events) is described by the totality of its possible states. The latter set is a certain subspace of the section space (sections can later be specified as smooth, or square integrable, etc—this is not the main concern here) of a certain vector bundle over $\mathbf{W}$. At this point, states are not yet number-valued (for a *scalar particle*) or vector-valued with numerical components (for particles of non-zero *spin*). One way or the other, we then need to convert to parallelized sections (to wave functions, in other words).

The respective Hilbert space can then be determined. It has become an acknowledged way of modern theoretical physics to describe elementary particles and their interactions in terms of induced representations of the (respective) symmetry group. As it is stated in [Se-82], 'the main philosophical point of these developments is perhaps the importance of induced representations, not purely as representations, but as actions on the homogeneous vector bundles that naturally emerge from the induction process. This additional structure provides a spatio-temporal labelling of the vectors in the group representation space that is absolutely essential for the formation of local nonlinear interactions, and relatedly, for causality considerations.'

Conventional quantum mechanics uses representations of the Poincaré group, which are induced from its Lorentz subgroup as in Wigner's seminal work [Wi-39]. The underlying space–time is the Minkowski world M (the one of special relativity). There was no formal parallelization involved since it was unthinkable of a better group than M's vector group (*flat parallelization*, or M-parallelization, according to the current chronometric terminology). Almost always in the literature, physicists merely start with sections having values in a fixed spin space.

In general, the parallelization procedure is essentially defined by parallelizing (4D but not necessarily commutative) subgroup N of the symmetry group G (see [Le-01, Le-03a, Le-04], where some of original chronometric parallelizations of [PaSe-82a] have been discussed from a more geometric viewpoint). Typically, N is a finite cover of the original space–time **W**. In Segal's (with co-authors) publications the mostly used parallelizations were the M- and the D-ones. It is now suggested to deal with two more important parallelizations (based on L and F).

It seems plausible that space–times D, L and F can be used as 'main building blocks' to mathematically develop an *effective theory of gravity* (see, for example, [Vo-03]). The latter theory was initiated by [Sa-67]. Its development and application to certain quantum liquids ([Vo-03, p 15]) resulted in models where the metric field emerges as the low-energy collective mode of the quantum vacuum. From this point of view, gravity is not the fundamental force, but is determined by the properties of the quantum vacuum.

The Bogoliubov theory of the weakly interacting Bose gas is discussed on pp 21–24 of [Vo-03]. The Hamiltonian in question admits $SU(1,1)$-symmetries, which is indicative of a possible relation to the above-discussed world F. The above Hamiltonian is then related to a set of uncoupled harmonic oscillators ([Vo-03, p 22]), which means involvement of the L-properties of the model. The importance of D-properties follows from rotational symmetries of both the Minkowski world M and Einstein's static universe D. In the context of quantum liquids these last symmetries are also admitted by (certain) states of the ^{3}H-condensate (see [Vo-03, p 78]).

Presumably, a more specific mathematical realization of the interconnection of the D-, L- and F-properties and of their applications in certain models can be achieved in terms of Lie groups and Lie algebra *contractions*. In [LeSv-08], some contractions of Lie algebras $d = u(2)$, $f = u(1, 1)$ to the oscillator Lie algebra l and to an abelian Lie algebra m are discussed. These contractions can be implemented via inner automorphisms of the conformal group. Levichev and Sviderskiy [LeSv-08] also discuss similar contractions of (7D) isometry groups of the worlds D, L and F.

Let us now provide proofs for some of the section 3 statements about D, L and F. Recall (see [Kr-80, p 71]) the *dominant energy conditions*. They mean non-positivity of the Einstein tensor T for all timelike vectors v as well as the condition for the *energy flux vector* q to be non-spacelike. In such a context the number $-T(v,v)$ is called the *energy density*. Vector q here is the image of v under the *operator* $\boldsymbol{T}$, corresponding to the Einstein tensor. Let us refer to these conditions merely as *energy conditions*. For the metric tensor g, the +,–,–,–, signature is chosen. The constants R and a have been introduced in sections 3.1 and 3.2. In what follows, there is no need to consider universal covers since we now deal with local properties.

Theorem 10. *1) The world D is an ideal fluid, determined by the vector field* X_0 *(see section 3.1). The scalar curvature is* $6/R^2$*. The energy density is* $1/(R^2)$*. The energy flux vector q is always timelike. Energy conditions hold.*
2) F is a tachyonic fluid, defined by the vector field H_3 *(see section 3.2). The scalar curvature is* $-6/a^2$*. The energy density is* $-1/\mathrm{a}^2$*. The energy flux vector q is not always timelike. Energy conditions do not hold.*
3) Using the Gr terminology, the world L is an isotropic electromagnetic field with covariant constant lightlike vector l_1 *(see [Le-86b, p 123]). The scalar curvature is zero. The energy flux vector is lightlike. Energy conditions are satisfied.*

Proof. The above GR-'names' of the three worlds can be determined on the basis of metric and curvature properties (see the table on p 57 of [Kr-80]). The distinguished vector fields have already been identified in sections 3.1–3.3. Each of the three fields is generated by the *central element* of the Lie algebra in question. In what follows, the curvature computations (see section 8) are instrumental. Each time, they are carried out in terms of the (already chosen) left-invariant basis of vector fields.
1) In the D-case, the energy density is $g(v,v)+2(v_0)^2$, which implies its positivity for every timelike vector v. The energy flux vector $q = -v–2v_0X_0$ is timelike.
2) In the F-case, the energy density is $2(v_3)^2-g(v,v)$. It can thus be negative for certain timelike vectors v if the coordinate v_3 is not too big. If this coordinate is big, then the energy flux vector $q = v + 2v_3X_3$ is spacelike.
3) In the L-case, the energy flux vector $q = (-1/2)v_4l_1$ is lightlike. The energy density $-T(v, v)$ is $(1/2)(v_4)^2$. Note that in the L-case the subscripts take integer values from 1 to 4 rather than from 0 to 3. □

8. Curvature computations

There are different choices for a pseudo-Riemannien metric signature and for the respective curvature tensor. This paper uses curvature conventions from [Mi-76]. The metric signature is the same as that in Segal's works: +, –, –, –.

Designate by x and y the vectors in the (parallelizing) Lie algebra n (depending on the context, the same x and y may stand for respective left-invariant vector fields on a group). In the situation of a bi-invariant metric on a Lie group, the curvature transformation R_{xy} is an operator $(1/4)\mathrm{ad}_{[x,y]}$, see [Mi-76, p 105]. Let x_0, x_1, x_2 and x_3 be an orthonormal basis in n. The component R^0_{kij} (of the curvature tensor) is $(1/4)([x_i, x_j], [x_0, x_k])$, where $(.,.)$ is the respective inner product. For the curvature tensor components with superscripts 1, 2 and 3, there is an extra negative one factor. Recall that the Ricci tensor Ric is the trace of the curvature tensor (use the upper superscript and the second subscript for summation).

These formulae are applied to the D-case first ($n = u(2)$, see section 3.1). The following commutation table has to be used:

$$[x_1, x_3] = 2x_2, [x_2, x_1] = 2x_3, [x_3, x_2] = 2x_1. \qquad (8.1d)$$

The computation results in

$$\mathrm{Ric} = \mathrm{diag}\{0, -2, -2, -2\}, \qquad (8.2d)$$

which implies $S = 6$ for the scalar curvature. The *Einstein tensor* T (in terms of which energy conditions are usually defined) is introduced as $\mathrm{Ric}-Sg/2$, where g is the metric. Clearly,

$$T = \mathrm{diag}\{-3, 1, 1, 1\}. \qquad (8.3d)$$

The world F (see section 3.2) corresponds to the following commutation table:

$$[H_0, H_1] = 2H_2, [H_2, H_1] = 2H_0, [H_2, H_0] = 2H_1. \quad (8.1f)$$

Here is the answer for the Ricci tensor:

$$\mathrm{Ric} = \mathrm{diag}\{-2, 2, 2, 0\}, \quad (8.2f)$$

The scalar curvature is now negative: $S = –6$.

$$T = \mathrm{diag}\{1, -1, -1, -3\}. \quad (8.3f)$$

In the L-case indexation is different. Relative to the basis of left-invariant vector fields chosen in section 3.3, both the Ricci and the Einstein tensors equal diag$\{0, 0, 0, -1/2\}$. The scalar curvature is 0.

Values of curvature invariants, when there is no assumption that $R = 1$ (section 3.1) or $a = 1$ (section 3.2), can be found either by the same methods or by using their behavior regarding scaling (see [Kr-80, p 55]). Some of the resulting values have been already listed in section 3.

9. Conclusions

There are exactly three 4D non-commutative Lie algebras that admit the invariant non-degenerate form of the Lorentzian signature: $d = u(2)$, $f = u(1, 1)$, $l =$ osc. Segal's chronometry is based on D. The main tenet of the DLF theory is to start with all three worlds rather than with a single Minkowski space–time M as special relativity does. It is plausible that this approach may be applied to describe both micro- and macroscopic phenomena. The space–time D takes the place of M (say, instead of the Klein–Gordon equation where one has to deal with the 'curved wave equation', which becomes the Klein–Gordon equation in R going to infinity, etc; a considerable part of such a program has already been carried out by Segal and his school). Nothing is thus 'lost' from special relativity in its transition to the DLF theory. What can be gained is a mathematical description of the *fundamental process* ([Sta-09, p 93 onwards]). As Stapp says: 'it is unreasonable to impose upon process relativistic demands drawn from the Einstein's static realm of readings' ([Sta-09, p 101]). For example, the F-constituent of the theory can shed light on possibilities of a mathematical description of 'actions that transcend space–time separation, i.e. that can act without attenuation over large spacelike distances' ([Sta-09, p 144]).

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