Formula Sheet

- **Differentiation rules**:
  - The product rule: $\frac{\mathrm{d}}{\mathrm{d}x}\big(f(x)\cdot g(x)\big) = f'(x)\cdot g(x) + f(x)\cdot g'(x)$
  - The chain rule: $\frac{\mathrm{d}}{\mathrm{d}x} f\big(g(x)\big) = f'\big(g(x)\big)\cdot g'(x)$
- **Derivatives and integrals**:

$$\frac{\mathrm{d}}{\mathrm{d}x}e^x = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x} \qquad \frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$$

$$\int e^x\,\mathrm{d}x = e^x + C \qquad \int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C \qquad \int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + C, n \neq -1$$

- **Integration by parts**:

$$\int u\,\mathrm{d}v = uv - \int v\,\mathrm{d}u$$

- The standard form of a **first-order linear differential equation** is

$$y' + f(x)y = g(x)$$

  the integrating factor is

$$I(x) = e^{\int f(x)\,\mathrm{d}x}$$

- **Taylor polynomials**: The nth-degree Taylor polynomial at a for a function f is

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

- **Taylor series**: The Taylor seires at a for a function f is

$$f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots = \sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

  Let $a_n = \frac{f^{(n)}(a)}{n!}, n = 0, 1, 2, \ldots$. If $a_n \neq 0$ for $n \geq n_0$, then we can evaluate

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$$

  to determine the interval of convergence.

- **Taylor series at 0**:

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \ldots$	$-1 < x < 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$	$-\infty < x < \infty$
$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \ldots$	$-1 < x < 1$

- **Error estimation of alternating series**: If x_0 is in the interval of convergence of the Taylor series for $f(x)$ and the terms in the series $f(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_k x_0^k + \ldots$ are alternating in sign and decreasing in absolute value, then the error in the approximation $f(x_0) \approx a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n$ is strictly less than the absolute value of the next term

$$|R_n(x_0)| < |a_{n+1} x_0^{n+1}|$$

- **Partial derivatives**:

$$f_x = \frac{\partial z}{\partial x} \qquad f_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) \qquad f_{yx} = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$$f_y = \frac{\partial z}{\partial y} \qquad f_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) \qquad f_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

- **Least Squares**: For a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$, the coefficients of the least squares line $y = ax + b$ are given by

$$a = \frac{n\left(\sum_{k=1}^n x_k y_k\right) - \left(\sum_{k=1}^n x_k\right)\left(\sum_{k=1}^n y_k\right)}{n\left(\sum_{k=1}^n x_k^2\right) - \left(\sum_{k=1}^n x_k\right)^2}$$

$$b = \frac{\left(\sum_{k=1}^n y_k\right) - a\left(\sum_{k=1}^n x_k\right)}{n}$$

- **Double integrals & Regular regions**:
  - Regular x region: R can be covered by a union of vertical lines,

    $$R = \{(x,y) | f(x) \leq y \leq g(x), a \leq x \leq b\}$$

    The double integral of $F(x,y)$ over a regular x region is

    $$\iint_R F(x,y)\,\mathrm{d}A = \int_a^b \int_{f(x)}^{g(x)} F(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

  - Regular y region: R can be covered by a union of horizontal lines,

    $$R = \{(x,y) | h(y) \leq x \leq k(y), c \leq y \leq d\}$$

    The double integral of $F(x,y)$ over a regular y region is

    $$\iint_R F(x,y)\,\mathrm{d}A = \int_c^d \int_{h(y)}^{k(y)} F(x,y)\,\mathrm{d}x\,\mathrm{d}y$$