MA122 In-class Practice Problem Set 5

1. (a) What is a "Differential Equation"?
   Answer: A differential equation includes a function and derivatives of that function. If asked to solve a differential equation, you want to find the most general function that satisfies the equation.

   (b) What is the "order" of a differential equation?
   Answer: The order of the equation is the order of the highest-order derivative.

2. (a) Given a differential equation and a function, how can we tell if the function is a "solution" to the differential equation?
   Answer: Take the indicated derivatives of the function. Plug the function and the indicated derivatives into the equation and simplify. If the left-hand side (LHS) is equal to the right-hand side (RHS), then the function is a solution.

   (b) Determine whether $y(x) = x^2 + 2x$ is a solution to the equation $y'(x) = 2y(x)$.
   Answer: No. $y'(x) = 2x + 2$ but $2y(x) = 2x^2 + 4x$ so $y'(x) \neq 2y(x)$

3. (a) For any constant C, show that the function $y(x) = Ce^{2x}$ is a solution to $y'(x) = 2y(x)$.
   Answer: $y'(x) = 2Ce^{2x}$ and $2y(x) = 2Ce^{2x}$ so $y'(x) = 2y(x)$.

   (b) Find the particular solution that satisfies $y(0) = 3$.
   Answer: $y(x) = 3e^{2x}$

4. What is a "Separable" Differential Equation?
   Answer: A differential equation $y'(t) = F(t, y)$ which can be rewritten as $g(y)y'(t) = h(t)$ (ie: with all of the terms containing t on the LHS and all of the terms containing y on the RHS) is called "Separable".

5. A linear first-order differential equation has the form: $y'(x) = ay(x) + b$.
   Answer:

   (a) Show that this is a separable equation by rewriting it in separated form.
   Answer: $\frac{y'(x)}{ay(x)+b} = 1$

   (b) Integrate both sides of this equation.
   Answer: $\frac{1}{a} \ln |ay + b| = x + c$

   (c) Solve for y to find the general solution.
   Answer: $y = Ce^{ax} - \frac{b}{a}$

(d) Use your answer from part c to solve: $y'(x) = 2y(x) - 2$ with $y(0) = 3$.
Answer: $y = 2e^{2x} + 1$

(e) Check that your answer satisfies both the differential equation and the initial condition.

6. (a) Show that $y' = yx^2 + y$ is separable.
Answer: $\frac{y'}{y} = x^2 + 1$

(b) Find the general solution to $y' = yx^2 + y$
Answer: $y(x) = Ce^{x^3/3+x}$

(c) Check that your answer satisfies the differential equation.

7. When an infected person is introduced into an otherwise healthy population, the number of people who become infected with the disease (in absence of intervention or recovery) may be modeled by the logistic equation

$$\frac{dP}{dt} = kP(1 - \frac{P}{A}); P(0) = P_0$$

where k is a positive infection rate, A is the number of people in the community, and P_0 is the number of people infected at time $t = 0$.

(a) Find the solution of the initial value problem for $k = 0.1, A = 300$, and $P_0 = 1$.
Answer: $P(t) = \frac{300}{1+299e^{-0.1t}}$

(b) Show that the general solution is

$$P(t) = \frac{A}{1 + (\frac{A}{P_0} - 1)e^{-kt}}$$

(c) Describe the long-term behavior of the solutions for fixed k, A, and P_0 with $0 < P_0 < A$. Explain why this makes physical sense. Answer: $\lim_{t\to\infty} P(t) = A$. This makes sense because, assuming no recovery, eventually everyone will be infected.

8. A bacteria culture grows at a rate proportional to its size. If $P(t)$ represents the size of the population after t hours, then $P'(t) = kP$, where k is some positive growth rate.

(a) If $P(0) = 200$ and $P(0.5) = 360$, find the value of k.
Answer: $k = \ln \frac{81}{25}$

(b) Find the population size after 4 hours.
Answer: $P(t) = 200e^{\ln \frac{81}{25} t} = 200(3.24)t$, $P(4) \approx 22,040$