MA122 Practice Problem Set 7

1. In Section 2-1, do:
   (a) #71
   (b) #73
   (c) #75
2. Find the power series of the following functions at $a = 0$. Give the interval of convergence:
   (a) $f(x) = \frac{1}{1-3x}$
   Answer: $1 + 3x + 3^2x^2 + \cdots + 3^n x^n + \ldots$, converges for $|x| < 1/3$
   (b) $f(x) = \frac{x^3}{1-x}$
   Answer: $x^3 + x^4 + x^5 + \cdots + x^{n+3} + \ldots$, converges for $|x| < 1$
   (c) $f(x) = \frac{x^2}{1+4x}$
   Answer: $x^2 - 4x^3 + 4^2x^4 + \cdots + (-1)^n 4^n x^{n+2} + \ldots$, converges for $|x| < 1/4$
3. Find the power series of the following functions at $a = 0$. Give the interval of convergence:
   (a) $f(x) = \ln(1 - 3x)$
   Answer: $-3x - \frac{3^2}{2}x^2 - \cdots - \frac{3^n}{n}x^n + \ldots$, converges for $|x| < 1/3$
   (b) $f(x) = x^3 \ln(1 - 3x)$
   Answer: $-3x^4 - \frac{3^2}{2}x^5 - \cdots - \frac{3^n}{n}x^{n+3} + \ldots$, converges for $|x| < 1/3$
   (c) $f(x) = x^2 \ln(1 + 4x)$
   Answer: $4x^3 - \frac{4^2}{2}x^4 + \cdots + \frac{(-1)^{n-1}4^n}{n}x^{n+2} + \ldots$, converges for $|x| < 1/4$
4. Find the power series of the following functions at $a = 0$ by differentiating or integrating some known Taylor series. Give the interval of convergence:
   (a) $g(x) = \frac{1}{(1-x)^2}$ using $f(x) = \frac{1}{1-x}$
   Answer: $\sum_{k=1}^{\infty} kx^{k-1}$ or $\sum_{k=0}^{\infty}(k + 1)x^k$, converges for $|x| < 1$
   (b) $g(x) = \frac{1}{(1-x)^3}$ using $f(x) = \frac{1}{1-x}$
   Answer: $\frac{1}{2}\sum_{k=2}^{\infty} k(k - 1)x^{k-2}$ or $\frac{1}{2}\sum_{k=0}^{\infty}(k + 1)(k + 2)x^k$, converges for $|x| < 1$
   (c) $g(x) = \frac{1}{(1-x)^4}$ using $f(x) = \frac{1}{1-x}$
   Answer: $\frac{1}{6}\sum_{k=3}^{\infty} k(k - 1)(k - 2)x^{k-3}$ or $\frac{1}{6}\sum_{k=0}^{\infty}(k + 1)(k + 2)(k + 3)x^k$, converges for $|x| < 1$

(d) $g(x) = \frac{x}{(1+x^2)^2}$ using $f(x) = \frac{x}{1+x^2}$
Answer: $\sum_{k=1}^{\infty}(-1)^{k+1}kx^{2k-1}$, converges for $|x| < 1$

(e) $g(x) = \ln(1-3x)$ using $f(x) = \frac{1}{1-3x}$
Answer: $-3\sum_{k=0}^{\infty} 3^k \frac{x^{k+1}}{k+1}$ or $-\sum_{k=1}^{\infty} 3^k \frac{x^k}{k}$, converges for $|x| < 1/3$

(f) $g(x) = \ln(1+x^2)$ using $f(x) = \frac{x}{1+x^2}$
Answer: $\sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k}}{k}$, converges for $|x| < 1$

5. Find the power series of the following functions at $a = 0$ by substitution. Give the interval of convergence:

(a) $f(x) = \frac{1}{1+x^2}$
Answer: $\sum_{k=0}^{\infty}(-1)^k x^{2k}$, converges for $|x| < 1$

(b) $f(x) = \frac{1}{1-16x^4}$
Answer: $\sum_{k=0}^{\infty} 16^k x^{4k}$, converges for $|x| < 1/2$

(c) $f(x) = \ln\sqrt{1-x^2}$
Answer: $-\frac{1}{2}\sum_{k=1}^{\infty} \frac{x^{2k}}{k}$, converges for $|x| < 1$

6. Taylor series centered ar $a \neq 0$

(a) $f(x) = 1/x$, $a = 1$
Answer: $\sum_{k=0}^{\infty}(-1)^k(x-1)^k$

(b) $f(x) = 1/x$, $a = 2$
Answer: $\sum_{k=0}^{\infty}(-1)^k\frac{1}{2^{k+1}}(x-2)^k$

(c) $f(x) = \ln x$, $a = 3$
Answer: $\ln 3 + \sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k3^k}(x-3)^k$

(d) $f(x) = e^x$, $a = \ln 2$
Answer: $\sum_{k=0}^{\infty} \frac{2}{k!}(x-\ln 2)^k$