Boston University
MA 581
Probability, Discussion 4
Summer 2, 2013

1 Expectation

1. Consider n cards, labeled with $1, 2, \ldots, n$, draw k cards with replacement. What is the expectation of sum of numbers of this k cards.

2. n balls numbered by $1, \ldots, n$ are to be placed in n boxes numbered by $1, \ldots, n$, each box with one ball. We call it a match if the ith ball is place in the ith box. Let X be the number of matches. Find $E(X)$.

3. Consider n keys that look identical where only one of them can be used to open a lock. Luis selected the keys randomly and then tried it on the lock. If the key could not open the lock, then it was removed from the key set. Let X denote the number of times Luis tried until the lock was opened.

   (a) Find the PMF of X

   (b) Find $E(X)$

4. In an ordinary blood test, a medical center collected N blood samples from N people to test for a specific disease. It is known that $p \times 100\%$ of the population has the disease. There are two ways to carry out the blood test (1) one test on each blood sample, so that N tests are needed. (2) one test on k samples together. If the test result is negative, there is evidence that those k people do not have the disease. While if the result is positive, then k more tests will be performed on the k blood samples separately. Which test method requires fewer tests on average when (1) $k = 4$ and $p = 0.1$; (2) $k = 4$ and $p = 0.8$?

5. The following problem was posed and solved by Daniel Bernoulli in the 18th century. Suppose a jar contains $2n$ cards, two of them marked 1, two of them marked 2, and so on. Draw out m cards at random. What is the expected number of pairs remained in the jar.

6. Suppose there are N different types of coupons and each time a person obtains a coupon it is equally likely to be any one of the N types.

   (a) Find the expected number of different types of coupons that are contained in a set of n coupons.

   (b) Find the expected number of coupons that one needs to amass before obtaining a complete set of at least one of each.

2 Variance, Covariance

1. Let I_A and I_B be the indicator random variables of event A and B, respectively.
   (a) Find $Cov(I_A, I_B)$
   (b) Based on (a), determine when is A and B positively correlated, negatively correlated and uncorrelated.
2. Let X have a discrete uniform distribution on $S = \{-1, 0, 1\}$ and let $Y = X^2$.
   (a) Are they correlated?
   (b) Are they independent?
3. Random variables $X_1, \ldots, X_n$ are i.i.d. $E(X_i) = \mu, Var(X_i) = \sigma^2$.
   (a) Find $E(\overline{X})$, $Var(\overline{X})$ and $E(S^2)$ where $S^2 = \sum_{i=1}^{n}(X_i - \overline{X})^2$.
   (b) Show that $X_i - \overline{X}$ and $\overline{X}$ are uncorrelated.
4. Suppose that $X \sim Pois(\lambda)$.
   (a) For $n \in \mathcal{N}$, determine $E\big(X(X-1)\cdots(X-n+1)\big)$, called the nth factorial moment.
   (b) Use part (a) to find $E(X)$ and $Var(X)$
5. $E(X) = 1, E(Y) = 1, Var(X) = 1, Var(Y) = 4, \rho(X,Y) = -0.5$. Find
   (a) $Cov(X,Y)$
   (b) $Cov(X+Y, X-Y)$
   (c) $Var(X+Y)$
   (d) $Var(X-Y)$
   (e) $Var(3X+2Y)$
   (f) $Cov(2X+Y, -3X+2Y)$

3 Conditional Expectation, Variance

1. Let X and Y be i.i.d. Binomial random variables with parameters n and p. Identify the PMF of $X|(X+Y)$ and find the conditional expectation and variance of X given that $X+Y=m$.
2. An urn contains a white balls and b black balls. One ball at a time is withdrawn randomly until the first white ball is withdrawn. Find the expected number of black balls that are withdrawn.
3. Let X denote the number of Bernoulli trials until the first success. Next, X more trials are performed. Let Y denote the number of successes in the second group of Bernoulli trials. Find $E(X)$ and $Var(X)$.