Boston University
MA 581
Probability, Discussion 5
Summer 2, 2013

1 Continuous Random Variables

1. Let $X_1, \ldots, X_n$ be independent random variables, each having the same distribution as a random variable X. Determine the CDF and PDF of $Y = \min\{X_1, \ldots, X_n\}$ and $Z = \max\{X_1, \ldots, X_n\}$ in terms of the CDF and PDF of X.

2. The lifetime in hours of a certain kind of radio tube is a random variable having a pdf $f_X(x) = 100/x^2$ for $x > 100$ and 0 otherwise. What is the probability that exactly 2 of 5 such tubes will have to be replaced within the first 150 hours of operation. Assume that the tubes work independently.

3. A stick of length 1 is split at a point U that is uniformly distributed over $(0, 1)$. Determine the expected length of the piece that contains point p, $0 < p < 1$.

4. Suppose that if you are t minutes early for an appointment, then you incur the cost ct, and if you are t minutes late for an appointment, then you incur the cost kt. Suppose that the travel time from where you presently are to the location of the appointment is a continuous random variable with pdf f. Determine the time at which you should depart to minimize your expected cost.

5. An examination is considered as being good if the test scores of those taking the exam can be approximated be a normal density $N(\mu, \sigma^2)$. The instructor then assigns letter A to those whose test score is above $\mu + \sigma$, B to those whose test score is between μ and $\mu + \sigma$, C to those whose test score is between $\mu - \sigma$ and μ, D to those whose test score is between $\mu - 2\sigma$ and $\mu - \sigma$ and F to those whose test score is below $\mu - 2\sigma$. Find the percent of class receiving A, B, C, D and F, respectively.

6. Suppose that a binary message, either 0 or 1, must be transmitted by wire from location A to location B. However, the data sent over the wire are subject to a channel noise disturbance, so to reduce the possibility of error, the value 2 is sent over the wire when the message is 1 and the value -2 is sent when the message is 0. If x, $x = \pm 2$, is the value sent at location A, then R, the value received at location B, is given by $R = x + N$, where N is the channel noise disturbance. When the message is received at location B, the receiver decodes it according to the following rule: If $R \geq .5$, then 1 is concluded, otherwise 0 is concluded. Suppose that the channel noise disturbance N is a standard normal random variable. There are two types of error that can occur: one is that the message 1 is incorrectly concluded as 0, the other one is that 0 is concluded to be 1. Find the probabilities of the two types of error.

7. Consider a post office that is staffed by two clerks. Suppose that when Wes enters the system, he discovers that Ian is served by one og the clerks and Hunter by the other one. Wes is told that his service will begin as soon as either Ian or Hunter leaves. If the amount of time that a clerk spent with a customer is exponentially distributed with parameter λ, what is the probability that, of the three customers, Wes is the last to leave the post office?

8. Suppose that the miles that a car can ran before its battery wears out is exponentially distributed with an average value of 10,000 miles. If Yuki decides to take a 5,000-mile trip, what is the probability that he will complete the trip without having to replace the car battery?

2 Joint Continuous Random Variables

1. Consider a circle with radius R and a point is randomly chosen such that the point is uniformly distributed within the circle. Let X and Y be the coordinates of the points. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = c \quad \text{if } x^2 + y^2 \leq R^2$$

for some value c and 0 otherwise.

   (a) Determine c.
   (b) Find $f_X(x)$ and $f_Y(y)$.
   (c) Compute the probability that Z, the distance of the point from the origin, is less than of equal to z.
   (d) Find $E(Z)$.

2. Let X and Y be independent exponential random exponential variables with parameter λ. Find $f_{Y/X}(z)$.

3. Suppose that the joint pdf of X and Y is given by

$$f_{X,Y}(x,y) = \frac{e^{-x/y}e^{-y}}{y} \quad x > 0, y > 0$$

and 0 otherwise. Find $P(X > 1|Y = y)$

4. Consider $m + n$ trials having a common probability of success. However, this success probability is not fixed in advance but a random $U(0,1)$ variable. What is the conditional success probability given that the $m + n$ trials result in n successes. What if the success probability is a $Beta(\alpha, \beta)$ random variable?

3 Expectation

1. An accident occurs at a point X that is uniformly distributed along a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on the road. Assume that X and Y are independent. Find the expected distance between the point of accident and the ambulance.

2. (A random walk in the plane) Consider a particle initially located at the origin in the plane and it undergoes a sequence of steps of length 1 but in a completely random direction. Suppose that the angle of orientation from the precious position is uniformly distributed over $(0, 2\pi)$. Compute the expected square of distance from the origin after n steps.

3. A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel, the second door leads to a tunnel that will return him to the mine after 5 hours of travel and the third door leads to a tunnel that will return him to the mine after 7 hours of travel. Assume that the miner is at all times equally likely to choose anyone of the doors, what is the expected length of time until he reaches safety.

4. Consider n points that are independently and uniformly distributed over interval $(0, 1)$. Say that any of the points are "isolated" if there are no other points within distance d of it, where d is a constant that $0 < d < 1/2$. Compute the expected number of isolated points out of the n points.

5. Let $U_1, U_2, \ldots$ be a sequence of independent uniform (0,1) random variables. Find $E(N)$ where $N = \min\{n : \sum_{i=1}^{n} U_i > x\}$. What is $E(N)$ if $x = 1$?

6. Suppose that by any time t, the number of people that have arrived at BU Central T stop is a Poisson random variable with mean λt. It the initial T arrives at BU Central at a time that is uniformly distributed over $(0, T)$. What is the mean and variance of the number of passengers that enter the T?