Rapid convergence to quasi-stationary states in the 2D Navier-Stokes equation

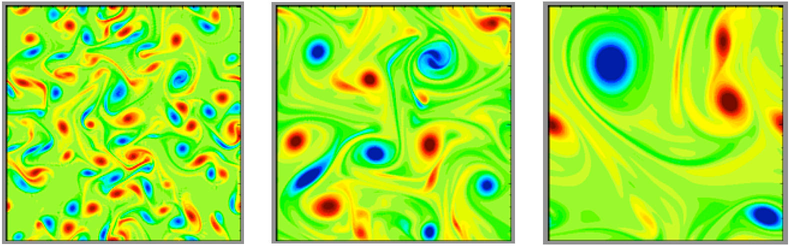
Margaret Beck
Heriot-Watt University and Boston University

joint work with C. Eugene Wayne (Boston University)

IMA, Sept 24, 2012

Observed dynamics

2D incompressible Navier-Stokes on the torus with small viscosity:



[Fluid dynamics laboratory, Eindhoven]

- Vorticity evolves from small scale to large scale structures
- Localized vortices persist and organize the dynamics
- Separation of time scales
    - Rapid convergence to localized vortices
    - Slow motion and merger of vortices

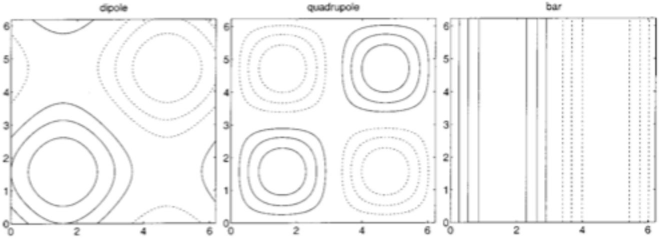
2D Navier-Stokes: decaying turbulence

Some questions:

- How to characterize the quasi-stationary states? [Y, M, C 2003]
- What causes the separation in time scales? [This talk]

Determine quasi-stationary states via statistical mechanics:

- Stationary solutions of inviscid Euler equations seem to play a role
- Such states with maximum entropy are good candidates



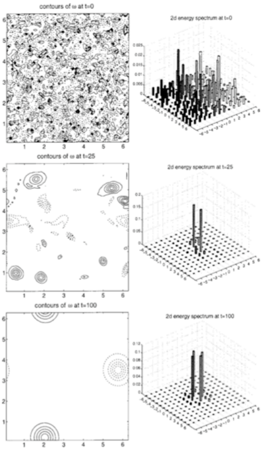
(a) (b) (c)

[Yin, Montgomery, Clercx 2003]

Quasi-stationary states

Yin, Montgomery, Clercx 2003:

- Euler: formal calculations and numerical analysis determined these states
- Navier-Stokes: dynamic calculations confirmed predictions ($\nu = 1/5000$)

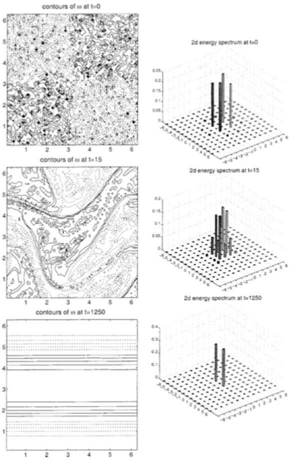


$t = 0$

$t = 25$

$t = 100$

Dipole



$t = 0$

$t = 15$

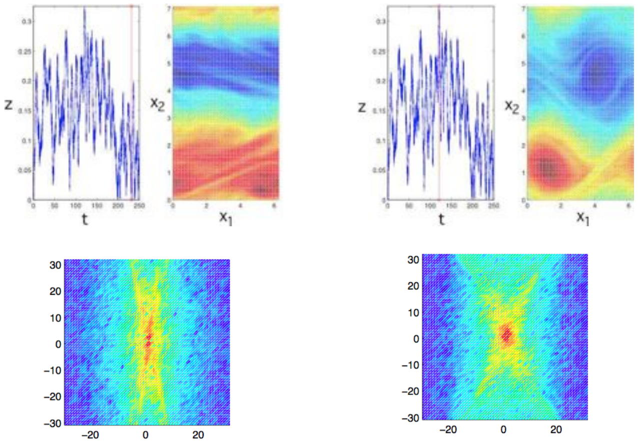
$t = 1250$

Bar

Related work for stochastically forced Navier-Stokes equation

Statistical equilibrium consists of bars and dipoles [Bouchet, Simonnet 09]:

- Square torus: dipole dominates
- Asymmetric (rectangular) torus: bar dominates



Figures produced by Gabriel Lord (Heriot-Watt)

Related work for Burgers equation

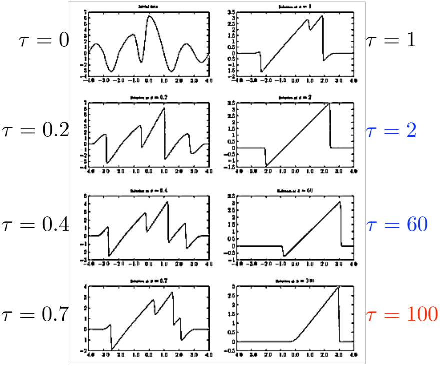
1D Burgers equation; figure is for similarity variables:

$$u_t = \nu u_{xx} - u u_x$$

$$0 < \nu \ll 1$$

Zero viscosity:
N-waves stable

Nonzero viscosity:
Diffusion waves stable



[Kim & Tzavaras, SIAM J. Math. Anal., 01]

Results from [Kim & Tzavaras 01]:

- Observed numerically
- Explained formally using asymptotic expansions

Related work for Burgers equation

Burgers Equation:

$$\begin{aligned} u_t &= \nu u_{xx} - u u_x, && x \in \mathbb{R}, \quad t > 0, \quad u \in \mathbb{R} \\ u(x,0) &= u_0(x), && 0 < \mu \ll 1 \end{aligned}$$

Scaling variables - deal with continuous spectrum:

$$u(x,t) = \frac{1}{\sqrt{t+1}} w\left(\frac{x}{\sqrt{t+1}}, \log(t+1)\right)$$

$$\xi = \frac{x}{\sqrt{t+1}}, \quad \tau = \log(t+1)$$

Scaled Burgers equation:

$$\begin{aligned} w_\tau &= \nu w_{\xi\xi} + \frac{1}{2}\xi w_\xi + \frac{1}{2} w - w w_\xi \\ \mathcal{L}_\nu w &= \partial_\xi^2 w + \frac{1}{2}\partial_\xi(\xi w) \end{aligned}$$

Related work for Burgers equation

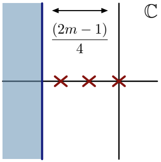
$$w_\tau = \mathcal{L}_\nu w - w w_\xi$$

In the space

$$L^2(m) := \left\{ w \in L^2(\mathbb{R}) : \int_{\mathbb{R}} (1+\xi^2)^m w^2(\xi) d\xi < \infty \right\}$$

the spectrum of $\mathcal{L}$ is [Gallay & Wayne 02]

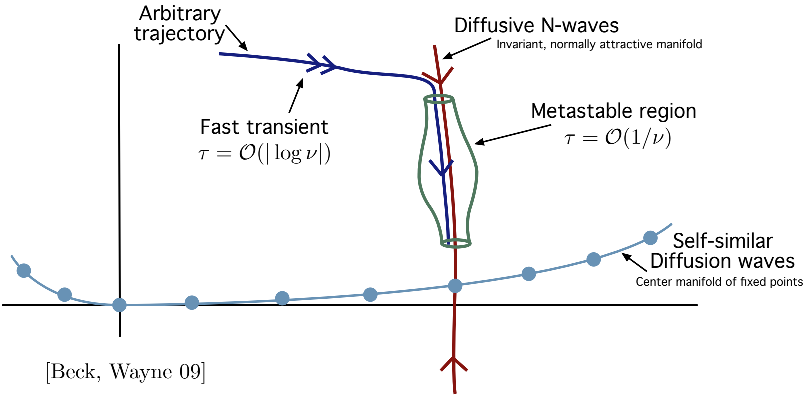
$$\sigma(\mathcal{L}) = \left\{ -\frac{n}{2} : n \in \mathbb{N} \right\} \cup \left\{ \lambda \in \mathbb{C} : \mathrm{Re}\lambda \le \frac{1-2m}{4} \right\}$$



Cole-Hopf still applies:

$$W(\xi,\tau) = w(\xi,\tau) e^{-\frac{1}{2\nu}\int_{-\infty}^{\xi} w(y,\tau)dy} \quad \Rightarrow \quad W_\tau = \mathcal{L}_\nu W.$$

Related work for Burgers equation



[Beck, Wayne 09]

Reason for timescales:

- Spectrum independent of ν
- Large coefficients in eigenfunction expansion: $w(\tau) = c_0\phi_0 + c_1\phi_1 e^{-\frac{1}{2}\tau} \ldots$
- Due to pseudospectrum or Cole-Hopf?

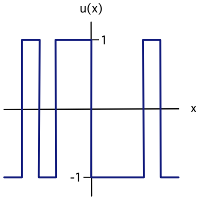
Related results in reaction-diffusion equations

Metastability in gradient systems:

$$u_t = \epsilon^2 u_{xx} - u(u^2 - 1), \qquad x \in (0,1)$$

Eg: [Carr & Pego 89], [Fusco & Hale 89], [Chen 04], [Otto & Reznikoff 07]

- Stable states: $u \equiv \pm 1$
- Metastable states: step functions connecting ± 1 numerous times



Different mechanisms:

- Gradient: utilize energy functional

$$E[u](t) = \int_0^1 \left[\frac{\epsilon^2}{2} u_x^2 + \frac{1}{4}(u^2 - 1)^2 \right] dx$$

- Burgers: spectrum independent of viscosity.
- Navier-Stokes: "spectrum" depends strongly on viscosity.
- Timescale differences

2D Navier-Stokes on the torus

$$\partial_t \mathbf{u} = \nu \Delta \mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla p, \qquad \nabla \cdot \mathbf{u} = 0, \qquad (x, y) \in \mathbb{T}^2$$

Assume viscosity is small

$$0 < \nu \ll 1, \qquad \text{physical range} = \mathcal{O}(10^{-3}).$$

Vorticity formulation: $\omega = \nabla \times \mathbf{u}$

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \int_{\mathbb{T}^2} \omega = 0, \qquad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}.$$

Decay of energy due to diffusion

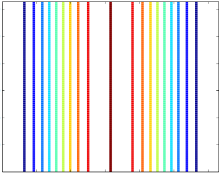
$$\frac{d}{dt} \frac{1}{2} \int_{\mathbb{T}^2} \omega^2(x, y) dxdy = -\nu \int_{\mathbb{T}^2} |\nabla \omega(x, y)|^2 dxdy \leq -\nu \int_{\mathbb{T}^2} \omega^2(x, y) dxdy$$

is very slow

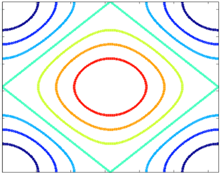
$$\|\omega(t)\|_{L^2} = \mathcal{O}(e^{-\nu t}).$$

Explicit families of metastable states

$$\omega^{bar}(x,y,t) = e^{-\nu t}\cos(x), \qquad \omega^{dipole}(x,y,t) = e^{-\nu t}[\cos(x)+\cos(y)]$$



Bar



Dipole

These solutions:

- Are quasi-stationary if $0 < \nu \ll 1$.
- Match observations of [Yin et al 03] and [Bouchet and Simonnet 09].
- Are stationary solutions of the Euler equations when $\nu = 0$.
- Should attract (some) nearby solutions faster that $\mathcal{O}(e^{-\nu t})$.
- Are part of an infinite family:

$$\omega^{slow}(x,y,t) = e^{-\nu m^2 t}[a_1\cos(mx) + a_2\cos(my) + a_3\sin(mx) + a_4\sin(my)]$$

Linearization about a bar state

$$\partial_t \omega = \nu \Delta \omega - \mathbf{u} \cdot \nabla \omega, \qquad \mathbf{u} = \begin{pmatrix} -\partial_y \Delta^{-1} \omega \\ \partial_x \Delta^{-1} \omega \end{pmatrix}.$$

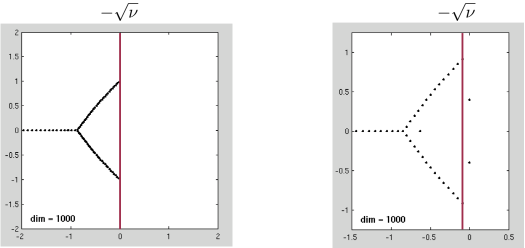
Ansatz: $\omega = \omega^{bar} + v$

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v - \mathbf{u}^v \cdot \nabla v.$$

Approximate linearization similar to advection of passive scalar by a shear flow:

$$\partial_t v = \nu \Delta v - \sin x \partial_y v$$

- Asymptotic of eigenvalues in [Vanneste, Byatt-Smith 07]: $\mathcal{O}(e^{-\sqrt{\nu} t})$
- Compute spectrum (Eigtool; Fourier approximation, $(k, l) = (k, 1)$):



Approximate operator $\nu = 0.001$ Full operator

What causes the fast decay?

$$u_t = Lu$$

Villani, 2009, considers operators of the form

$$L = A^*A + B, \qquad B^* = -B$$

- $\underline{AB = BA}$: antisymmetry of B implies $\|e^{Bt}u\| = \|u\|$, and so

$$\|e^{Lt}\| = \|e^{A^*At}e^{Bt}\| = \|e^{A^*At}\|,$$

so B cannot increase the decay rate of the semigroup.

- $\underline{AB \neq BA}$: rapid decay possible via hypoceorcivity

Define commutator $C = [A, B] = AB - BA$ and an inner product

$$((u, u)) = (u, u) + \alpha(Au, Au) - 2\beta\mathrm{Re}(Au, Cu) + \gamma(Cu, Cu)$$

Careful choice of α, β, and γ can show faster than expected decay.

Back to our problem...

$$\partial_t v = \nu\Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v =: \mathcal{L}(t) v$$

Slow modes: Cannot expect rapid decay on all of L^2

$$\lambda v_{slow} = \partial_t v_{slow} = \mathcal{L}(t) v_{slow}, \qquad \lambda = \mathcal{O}(\nu)$$

$$v_{slow} \in \{e^{-\nu m^2 t + imx}, e^{-\nu t \pm iy} : m \in \mathbb{Z}_0\}.$$

Like an infinite-dimensional eigenspace – need to “project” off it.

Intuitively:

- Expect something like a center manifold with slow decay $\mathcal{O}(e^{-\nu t})$
- and something like a stable manifold with rapid decay $\mathcal{O}(e^{-\sqrt{\nu} t})$
- Use hypocoercivity to get rapid decay rate in stable manifold.
- But operator is time-dependent.
- Can't use spectral projections to obtain manifolds.

Invariant subspaces:

- Need an alternative way to construct them
- Should be related to movement of energy between Fourier modes.

Construct invariant subspaces

$$v(x,y) = \sum_{k,l\in\mathbb{Z},(k,l)\neq(0,0)} \hat{v}(k,l)e^{i(kx+ly)}$$

Goal: don't excite the slow modes

$$\{e^{-\nu m^2 t+imx}, e^{-\nu t\pm iy}\} \quad \Rightarrow \quad (k,l)\in\{(0,\pm1),(m,0)\}$$

In Fourier space, $v_t = \nu\Delta v - e^{-\nu t}\partial_y \sin x(1+\Delta^{-1})v$ becomes

$$\begin{aligned}\partial_t\hat{v}(k,l) &= -\nu(k^2+l^2)\hat{v}(k,l)\\ &\quad -\frac{l}{2}e^{-\nu t}\left[\left(1-\frac{1}{(k-1)^2+l^2}\right)\hat{v}(k-1,l)-\left(1-\frac{1}{(k+1)^2+l^2}\right)\hat{v}(k+1,l)\right]\end{aligned}$$

Try $\mathcal{M}_x = \{v\in L^2(\mathbb{T}^2) : \hat{v}(m,0)=0,\ m\in\mathbb{Z}\}$

$$\partial_t\hat{v}(m,0) = -\nu m^2\hat{v}(m,0) \qquad \textbf{invariant}$$

Try: $\tilde{\mathcal{M}}_y = \{v\in L^2(\mathbb{T}^2) : \hat{v}(0,\pm1)=0\}$

$$\partial_t\hat{v}(0,\pm1) = -\nu\hat{v}(0,\pm1)\mp\frac{1}{4}e^{-\nu t}\left[\hat{v}(-1,\pm1)-\hat{v}(1,\pm1)\right] \qquad \textbf{not invariant}$$

Construct invariant subspaces

Recall: we don't want to excite the modes $e^{\pm imx}$ and $e^{\pm iy}$

- x-modes: $\mathcal{M}_x = \{w \in L^2(\mathbb{T}^2) : \hat{w}(m,0) = 0\}$
- y-modes: Formal calculations with Fourier equation lead to...

Define

$$p_j^{\pm} := \hat{w}(2j, \pm 1) + \hat{w}(-2j, \pm 1), \qquad q_j^{\pm} := \hat{w}(2j+1, \pm 1) - \hat{w}(-2j-1, \pm 1)$$

One can show:

$$\begin{pmatrix} p^{\pm} \\ q^{\pm} \end{pmatrix} = A^{\pm}(t) \begin{pmatrix} p^{\pm} \\ q^{\pm} \end{pmatrix}$$

Propositon A solution of $w_t = \mathcal{L}(t)$ satisfies $\hat{w}(0,\pm 1)(t) = 0$ for all $t \geq 0$ if and only if $w(0) \in \mathcal{M}_y$, where

$$\mathcal{M}_y = \{w \in L^2 : p_j^{\pm} = q_j^{\pm} = 0 \ \forall j\}.$$

Recall: In [YCM '03], only special initial data converge rapidly to bar states.

Rapid decay in this subspace

From now on, we only work in $\mathcal{M}_x \cap \mathcal{M}_y$.

$$\partial_t v = \nu \Delta v - e^{-\nu t}[\sin x \partial_y (1 + \Delta^{-1})] v.$$

Since there is no y-dependence in the bar state: $v(x, y) = \sum_{l \in \mathbb{Z}} \hat{v}_l(x) e^{ily}$

$$\partial_t \hat{v}_l = \nu \Delta_l \hat{v}_l - ile^{-\nu t}[\sin x (1 + \Delta_l^{-1})] \hat{v}_l, \qquad \Delta_l = \partial_x^2 - l^2.$$

Recall: want $L = A^* A + B$, with $B^* = -B$

- $A = \partial_x$, $A^* = -\partial_x$, so that $\nu \partial_x^2 = -\nu A^* A$
- But the second term is not anti-symmetric! Change variables...

Motivated by Wilkinson's book "The algebraic eigenvalue problem":

$$u := \sqrt{1 + \Delta_l^{-1}} \hat{v}_l$$

$$\widehat{1 + \Delta_l^{-1}} = 1 - \frac{1}{k^2 + l^2} \qquad \Leftrightarrow \qquad |l| + |k| > 1.$$

Invertible transformation in our subspace.

Transformed equation

$$\partial_t u = \nu \Delta_l u - ile^{-\nu t}\left[\sqrt{1+\Delta_l^{-1}}\sin x\sqrt{1+\Delta_l^{-1}}\right]u.$$

We have

- $A := \partial_x$
- $B := -ile^{-\nu t}\left[\sqrt{1+\Delta_l^{-1}}\sin x\sqrt{1+\Delta_l^{-1}}\right],\ B^* = -B$
- $C := [\partial_x, B] = -ile^{-\nu t}\left[\sqrt{1+\Delta_l^{-1}}\cos x\sqrt{1+\Delta_l^{-1}}\right],\ C^* = -C.$

Problem: $[B, C] \neq 0$; will lead to difficult terms in Villani's framework.
Partial solution: first consider only the approximate equation

$$\partial_t u = \nu \Delta_l u - ile^{-\nu t}\sin xu := \mathcal{L}_{approx}(t)u.$$

- $A := \partial_x$
- $B := -ile^{-\nu t}\sin x,\ B^* = -B$
- $C := [\partial_x, B] = -ile^{-\nu t}\cos x,\ C^* = -C.$
- $[B, C] = 0$

Why is this new inner product useful?

Motivated by work of Gallagher, Gallay, and Nier 2009, we rescale time:

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu.$$

Define, for $(u, u) = \|u\|_{L^2}^2$, $\alpha, \beta, \gamma > 0$,

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

If $\beta^2 < \alpha\gamma/4$, Young's inequality implies

$$\|u\|^2 + \frac{\alpha}{2}\|u_x\|^2 + \frac{\gamma}{2}\|Cu\|^2 < \Phi < \|u\|^2 + \frac{3\alpha}{2}\|u_x\|^2 + \frac{3\gamma}{2}\|Cu\|^2.$$

Therefore, by controlling the dynamics of Φ, we can control the above norm.

Strategy:

- Compute $d\Phi/dt$
- Chose α, β, γ to obtain a decay estimate
- Show this implies rapid convergence of solutions to the bar states

Main result

Function space: $C = C(l) = -ile^{-\nu t}\cos x$

$$X = \left\{ u : \hat{u}_0 = 0, \sum_{l\neq 0} [\|\hat{u}_l\|^2 + \sqrt{\frac{\nu}{|l|}}\|\partial_x \hat{u}_l\|^2 + \frac{1}{\sqrt{\nu}|l^{3/2}}\|C(l)\hat{u}_l\|^2] < \infty \right\}$$

Theorem Pick $T \in [0, 1/\nu]$. There exist constants K and M, $\mathcal{O}(1)$ with respect to ν, such that the following holds. If ν is sufficiently small, then the solution to $u_t = \mathcal{L}_{approx}(t)u$ with initial condition $u^0 \in X$ satisfies

$$\|u(t)\|_X^2 \leq Ke^{-M\sqrt{\nu}t}\|u^0\|_X^2$$

for all $t \in [0, T]$.

Implies rapid decay of solutions:

- Decay $e^{-M\sqrt{\nu}t}$ much faster than the viscous time scale $e^{-\nu t}$
- If $T = 1/\nu$, then

$$e^{-M\sqrt{\nu}T} = e^{-\frac{M}{\sqrt{\nu}}} \ll 1, \qquad e^{-\nu T} = e^{-1}$$

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu$$

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

Differentiate:

$$\begin{aligned}\frac{d}{dt}\Phi(t) = {} & [(u_t, u) + (u, u_t)] + \alpha[(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t)] \\ & -2\beta \mathrm{Re}[(\partial_x u_t, Cu) + (\partial_x u, Cu_t)] + \gamma[(Cu_t, Cu) + (Cu, Cu_t)] \\ & +\gamma[(C_t u, Cu) + (Cu, C_t u)].\end{aligned}$$

The first term gives

$$\begin{aligned}(u_t, u) + (u, u_t) &= ((-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u) + (u, (-l^2 + \partial_x^2 + \frac{1}{\nu}B)u) \\ &= -2l^2\|u\|^2 - 2\|u_x\|^2 + \frac{1}{\nu}\underbrace{[(Bu, u) + (u, Bu)]}_{=0}\end{aligned}$$

by the anti-symmetry of B.

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu}Bu$$
$$\Phi(t) := (u,u) + \alpha(\partial_x u, \partial_x u) - 2\beta\mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

The α term gives

$$\begin{aligned}
(\partial_x u_t, \partial_x u) + (\partial_x u, \partial_x u_t) &= (\partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u, u_x) \\
&\quad +(u_x, \partial_x(-l^2 + \partial_x^2 + \frac{1}{\nu}B)u) \\
&= -2l^2\|u_x\|^2 - 2\|u_{xx}\|^2 \\
&\quad +\frac{1}{\nu}[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))]
\end{aligned}$$

We can bound

$$\begin{aligned}
[(\partial_x(Bu), u_x) + (u_x, \partial_x(Bu))] &= (Bu_x, u_x) + (\overbrace{[\partial_x, B]}^{=C}u, u_x) \\
&\quad +(u_x, Bu_x) + (u_x, [\partial_x, B]u) \\
&= 2\mathrm{Re}(u_x, Cu) \\
&\leq 2\|u_x\|\|Cu\|.
\end{aligned}$$

Proof of Theorem

$$\partial_t u = (\partial_x^2 - l^2)u + \frac{1}{\nu} Bu$$

$$\Phi(t) := (u, u) + \alpha(\partial_x u, \partial_x u) - 2\beta \mathrm{Re}(\partial_x u, Cu) + \gamma(Cu, Cu)$$

The β term gives

$$\begin{aligned}(\partial_x u_t, Cu) + (\partial_x u, Cu_t) \quad &= \quad -2l^2 \mathrm{Re}(\partial_x u, Cu) + [(u_{xxx}, Cu) + (u_x, Cu_{xx})] \\ &\quad + \frac{1}{\nu}[(\partial_x(Bu), Cu) + (u_x, C(Bu))]\end{aligned}$$

One can show

$$(\partial_x(Bu), Cu) + (u_x, C(Bu)) = \|Cu\|^2 + (u_x, [C, B]u) = \|Cu\|^2$$

Important term: $-(2\beta/\nu)\|Cu\|^2$

The γ and C_t terms are similar.

Proof of Theorem

Collecting these estimates, we have shown

$$\begin{aligned}\frac{d}{dt}\Phi(t) \quad &\leq \quad -2l^2\|u\|^2 - [2+2\alpha l^2]\|u_x\|^2 - 2\alpha\|u_{xx}\|^2 \\ &\quad + \left(\frac{2\alpha}{\nu} + 2\beta(2l^2+1+\nu)\right)\|u_x\|\|Cu\| + 4\beta\|u_{xx}\|\|Cu_x\| \\ &\quad - \left((2l^2+2)\gamma + \frac{2\beta}{\nu} - 2\gamma\nu\right)\|Cu\|^2 - 2\gamma\|Cu_x\|^2 + 2\gamma\|Bu\|^2.\end{aligned}$$

We now use the fact that $2ab \leq a^2 + b^2$ and scale the parameters as

$$\alpha = \sqrt{\nu}\alpha_0, \qquad \beta = \beta_0, \qquad \gamma = \frac{1}{\sqrt{\nu}}\gamma_0$$

With appropriate conditions on $\alpha_0, \beta_0, \gamma_0$, this gives

$$\frac{d}{dt}\Phi(t) \leq -2\|u\|^2 + 2\frac{\gamma_0}{\sqrt{\nu}}\|Bu\|^2 - \frac{1}{4}\|u_x\|^2 - \frac{3\beta_0}{2\nu}\|Cu\|^2$$

Goal: Show $\Phi' \leq -(M/\sqrt{\nu})\Phi$

Proof of Theorem

$$\|u\|^2 + \frac{\alpha_0\sqrt{\nu}}{2}\|u_x\|^2 + \frac{\gamma_0}{2\sqrt{\nu}}\|Cu\|^2 < \Phi < \|u\|^2 + \frac{3\alpha_0\sqrt{\nu}}{2}\|u_x\|^2 + \frac{3\gamma_0}{2\sqrt{\nu}}\|Cu\|^2$$

$$\frac{d}{dt}\Phi(t) \leq -2\|u\|^2 + 2\frac{\gamma_0}{\sqrt{\nu}}\|Bu\|^2 - \frac{1}{4}\|u_x\|^2 - \frac{3\beta_0}{2\nu}\|Cu\|^2$$

Proposition If $|l| > 1$, then there exists a constant M_0 such that, for all $0 < t < T$,

$$\frac{1}{8}\|u_x\|^2 + \frac{\beta_0}{2\nu}\|Cu\|^2 \geq \frac{M_0|l|\sqrt{\beta_0}}{\sqrt{\nu}}\|u\|^2 \ .$$

Proof: Follows like a similar result in [Gallagher, Gallay, & Nier '09]. Essentially due to connection with harmonic oscillator:

$$H = a\partial_{xx} + bx^2 \quad \Rightarrow \quad (Hu, u)_{L^2(\mathbb{R})} \geq \sqrt{ab}(u, u)_{L^2(\mathbb{R})}$$

Need to be careful about the role of $|l|$. Also, $M_0 = \mathcal{O}(e^{-\nu t})$. □

This implies (after choosing $\alpha_0, \beta_0, \gamma_0$)

$$\Phi'(t) \leq -\frac{M}{\sqrt{\nu}}\Phi(t)$$

Summary and future directions

We have shown:

- Rapid decay for approximate operator: $\mathcal{O}(e^{-\sqrt{\nu}t}) \ll \mathcal{O}(e^{-\nu t})$
- Proof based on Villani's treatment of $L = A^*A + B$, $[A, B] \neq 0$.

To extend to full linear operator:

- Existence of invariant subspaces (and projections) for full operator.
- Use transformation $u = \sqrt{1 + \Delta^{-1}}v$ to make B antisymmetric.

Nonlinear equation; metastability of bar states:

- Use projection operators
- Use estimates similar to invariant manifold existence proofs

Dipoles:

$$\omega^d(x, y, t) = e^{-\nu t}[\cos(x) + \cos(y)],$$

- Much of the proof could be similar
- Need to understand slow modes and invariant subspaces