Practice Test, Topology, Autumn 2011

Instructions: Answer two of the following three questions. Each question is worth 10 marks. This test will be marked, to provide you with feedback regarding how well you are understanding the material, but it will not counts towards your assessment for the course.

Question 1

(i) Carefully define what it means for a set to be open in the finite complement topology. [**1 mark**]

Solution: A set is open if it is either empty or it its complement is finite.

(ii) Prove that the finite complement topology is, in fact, a topology. [**4 marks**]

Solution: Given a set X, the finite complement topology is $\tau = \{U \subset X : X \setminus U \text{ if finite, or } U = \emptyset\}$. By definition, $\emptyset \in \tau$, and $X \in \tau$ because its complement is empty, hence finite. If U and V are open, $X \setminus (U \cap V) = (X \setminus U) \cup (X \setminus V)$, which is finite because each individual complement is finite. Hence, the intersection is open. If $\{U_\alpha\}$ is a collection of open sets, the complement is the intersection of the individual complements: $\cap(X \setminus U_\alpha)$. Since each of these is finite, the intersection is finite, so the union is open.

(iii) Is $\mathbb{R}$, equipped with the finite complement topology, compact? Provide a proof supporting your answer. [**5 marks**]

Solution: This set is compact. Let $\{U_\alpha\}$ be an open cover. Pick any single element of the cover, U_{α_0}. Its complement has finitely many elements, so there are only finitely many elements of $\mathbb{R}$, $x_1, \ldots, x_n$, that are not in this set. For each one, find a U_{α_i} containing x_i. Then $\{U_{\alpha_i}\}_{i=0}^{n}$ is a finite subcover.

Question 2

(i) Give a careful definition of what it means for a set to be closed. [**1 mark**]

Solution: A set is closed if its complement is open.

(ii) (a) Is the set $A = \mathbb{Q} \cap (1, 2)$, considered as a subset of the real line with the usual topology, closed? Provide a brief argument to support your answer. [**2 marks**]

Solution: No, this set is not closed. Consider its complement - the irrational numbers in $(1, 2)$. Given any irrational number $x \in (1, 2)$, there is no open ball containing it that doesn't contain a rational number. This is because the rational numbers are dense. Hence, the complement is not open, so the set is not closed.

(b) Consider the space $X = \mathbb{R}$, with the particular point topology for the particular point zero. What is the closure of $(-1, 1)$, considered as a subset of X? Provide a brief argument to support your answer. [**2 marks**]

Solution: The closure of this set is the entire real line. This is because if $x \in \mathbb{R}$ and U is an open set containing x, then $0 \in U$. Hence, $U \cap [(-1, 1) \setminus x] \neq \emptyset$. Thus, every point (except zero) is a limit point of this set.

(iii) Given a continuous function f from the real line to itself (with the usual topology in both cases), prove that the set $E = \{x : f(x) = x\}$ is closed. [**5 marks**]

Solution: Let p be a limit point of E. Then for each $B_{1/n}(p)$, there is a point $x_n \in E$ such that $x_n \in B_{1/n}(p)$. Since f is continuous, $f(p) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n = p$. Hence, $p \in E$. Therefore, E contains its limit points, which implies that E is closed.

Question 3

(i) Carefully state the definition of a continuous function. [**1 mark**]

Solution: Let $f : X \to Y$, where X and Y are topological spaces. f is continuous if for any set U that is open in Y the inverse image $f^{-1}(U)$ is open in X.

(ii) Consider the function $f : X \to Y$, where X is the real line with the discrete topology and Y is the real line with the usual topology, defined by

$$f(x) = \begin{cases} 1 & \text{if} \quad x \geq 0 \\ 0 & \text{if} \quad x < 0 \end{cases}$$

Is f continuous? Provide a brief argument to support your answer. [**3 marks**]

Solution: Yes, f is continuous. If U is open in Y, then $f^{-1}(U)$ is open in X because any subset is open in the discrete topology. (Note this proof doesn't rely on the definition of f – as discussed in lecture, any function whose domain is discrete is continuous.)

(iii) (a) Suppose $f : X \to Y$ is one-to-one and continuous and Y is Hausdorff. Prove that X is Hausdorff. [**3 marks**]

Solution: Let $x_1, x_2 \in X$ such that $x_1 \neq x_2$. Since f is one-to-one, $f(x_1) \neq f(x_2)$, and since Y is Hausdorff there exist two nonempty disjoint open sets, U_1, U_2, such that $x_1 \in U_1$ and $x_2 \in U_2$. Since f is continuous, $f^{-1}(U_1)$ is open, as is $f^{-1}(U_2)$. They're both nonempty, because they contain x_1 and x_2, respectively. We must show they're disjoint. Suppose $z \in f^{-1}(U_1) \cap f^{-1}(U_2)$. This implies $f(z) \in U_1$ and $f(z) \in U_2$, which can't happen because $U_1 \cap U_2 = \emptyset$. Hence, $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$, and X is Hausdorff.

(b) Use part a) to prove that the property of a space being Hausdorff is a topological invariant. [**3 marks**]

Solution: If $f : X \to Y$ is a homeomorphism, then it is one-to-one and continuous. So if Y is Hausdorff, so is X. If Y is not Hausdorff, then $f^{-1} : Y \to X$ is one-to-one and continuous, so X cannot be Hausdorff (or else the statement of part a) would be contradicted). Therefore, Y is Hausdorff if and only if X is Hausdorff.