Review of notation and properties of sets, Topology 2011

Sets and Subsets

Let A be a set, meaning it is just a collection of objects. The notation $x \in A$ means that x is an element of the set - in other words, x is in A. If we write $x \notin A$, then x is not in A. The symbol $\emptyset$ denotes the empty set, meaning the set that consists of nothing. Given two sets A and B, if we write $B \subset A$, then B is a subset of A, meaning that every element of B is also in A. For any set A it is true that $\emptyset \subset A$. Sometimes one uses the notation $B \subseteq A$ to emphasize that B may possibly be equal to A. Similarly, $B \subsetneq A$ means that B is a subset of A but is not equal to A, and $B \not\subset A$ means that B is not a subset of A. A set B such that $B \subsetneq A$ is called a proper subset of A.

Let X and A be sets, with $A \subset X$. The complement of A in X (often just written "complement of A", if it is understood what X is), is the set of all points that are in X but not in A. It is written

$$A^c = \{x \in X : x \notin A\}.$$

Another way to write this is $X \setminus A$ or $X - A$, which means the set X minus all elements of A.

Given a set X, the collection of all subsets of X is called the power set of X and is denoted $P(X)$. Note that $P(X)$ is a set of sets. So if $O \in P(X)$, then O itself is a set – in particular, a subset of X.

Combining sets

There are at least two key ways to combine sets: intersections and unions. The intersection $A \cap B$ is the collection of elements, or points, that are in both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

Note that $A \cap B \subset A$ and $A \cap B \subset B$. If $A \cap B = \emptyset$, we say that A and B are disjoint, meaning they have no common elements. The union of A and B, written $A \cup B$, is the set of points that are in either A or B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

Thus, $A \subset A \cup B$ and $B \subset A \cup B$. The difference between sets is

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

The following useful facts are referred to as DeMorgan's laws:

$$(A \cap B)^c = A^c \cup B^c, \qquad (A \cup B)^c = A^c \cap B^c.$$

Another way to combine sets is through their Cartesian product. The set $A \times B$ is the set of ordered pairs (a, b) with $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

Indexed families

Let I be a set and suppose that for each $i \in I$ we are given a set A_i. The collection of sets $\{A_i : i \in I\}$ is called a family of sets indexed by I. For example, if we have an infinite family of sets $A_1, A_2, A_3, \ldots$, then this family is indexed by the natural numbers, $I = \mathbb{N} = \{1, 2, 3, \ldots\}$. The indexing set can have finitely or infinitely many elements, and it need not be countable. We could have a family A_α for each $\alpha \in \mathbb{R}$, for example. One can consider unions and intersections, based upon this indexing:

$$\bigcup_{i \in I} A_i = \{x : x \in A_j \text{ for some } j \in I\}, \qquad \bigcap_{i \in I} A_i = \{x : x \in A_j \text{ for all } j \in I\}.$$

Exercises

1. Let $X = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{5, 6, 7, 8\}$. Determine $A \cap B$, $A \cup B$, A^c, and $A \setminus B$. Is $A \subset B$? Is $B \subset A$? What is $X \setminus (A \cup B)$?

2. Prove DeMorgan's laws. [Hint: Given two sets X and Y, if one can show that $X \subset Y$ and $Y \subset X$, then it must be true that $X = Y$.]

3. For each $x \in \mathbb{R}$, let $A_x = \{y \in \mathbb{R} : y = nx \text{ for some } n \in \mathbb{Z}\}$. Given two numbers $x_1, x_2 \in \mathbb{R}$, when is $A_{x_1} \cap A_{x_2} \neq \emptyset$? If $A_{x_1} \cap A_{x_2} \neq \emptyset$, what is $A_{x_1} \setminus A_{x_2}$? What is $\cup_{x \in \mathbb{R}} A_x$? What is $\cap_{x \in \mathbb{R}} A_x$?