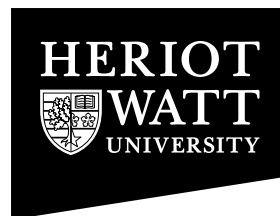


F10PC1



Department of Mathematics

F10PC1

Pure Mathematics C

Duration: 2 Hours

Sample Paper 2011/2012

Attempt three questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

Question 1

- (i) Carefully define what it means for a topological space X to be Hausdorff. [2 marks]
- (ii) Are the following spaces Hausdorff?
 - (a) The metric space (X, d) with the associated metric topology, where X contains at least two elements. [2 marks]
 - (b) The set \mathbb{R}^2 , with the particular point topology where the particular point is chosen to be $(5, -2)$. [2 marks]
 - (c) The circle S^1 with the finite complement topology. [2 marks]

Provide brief arguments supporting your answers.

- (iii) Let X be Hausdorff. Prove that every subset of the form $\{x\}$ for $x \in X$ is closed. [6 marks]
- (iv) If $f : X \rightarrow Y$ is one-to-one and continuous and Y is Hausdorff, is it necessarily true that X is Hausdorff? If so, provide an proof, if not, provide a counterexample. [6 marks]

Question 2

- (i) Given a subset $A \subset X$, where X is a topological space, carefully define what it means for p to be a limit point of A . [2 marks]
- (ii) Consider the set $K = \{1/n : n = 1, 2, \dots\} \subset \mathbb{R}$. Determine the closure of K when \mathbb{R} is endowed with the following topologies.
 - (a) The topology determined by the basis $\beta = \{[a, b) : a < b\}$. [2 marks]
 - (b) The particular point topology with the particular point chosen to be 0. [2 marks]
 - (c) The finite complement topology. [2 marks]

Provide brief arguments supporting your answers.

- (iii) Prove that U is open if and only if $U = \text{int}(U)$. [6 marks]
- (iv) Consider $X = \{(x, y) : x = 1/2^n, n = 1, 2, \dots, y \in [0, 1]\} \cup \{(0, 0), (0, 1)\}$ with the subspace topology inherited from \mathbb{R}^2 . Prove that any subset of X that is both open and closed and that contains $(0, 0)$ must also contain $(0, 1)$. [6 marks]

Question 3

- (i) Carefully state what it means for a subset of a topological space to be path connected. [2 marks]
- (ii) Provide arguments supporting your answers to the following questions.
 - (a) Is the product of two path connected spaces necessarily connected? [2 marks]

- (b) If $A \subset X$ and A is path connected, is \bar{A} necessarily path connected? [**2 marks**]
- (c) If $f : X \rightarrow Y$ is continuous and X is path-connected, is $f(X)$ necessarily path connected? [**2 marks**]
- (iii) Given continuous functions $f_1, f_2 : X \rightarrow Y$ such that f_1 and f_2 are homotopic, and also $g_1, g_2 : Y \rightarrow Z$ such that g_1 and g_2 are homotopic, prove that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic. [**6 marks**]
- (iv) Let A be a subspace of \mathbb{R}^n and let $h : A \rightarrow Y$ be such that $h(x_0) = y_0$. Suppose there is a continuous function $H : \mathbb{R}^n \rightarrow Y$ such that $H(x) = h(x)$ for all $x \in A$. Prove that the induced map h_* on the fundamental groups, $h_* : \pi_1(A, x_0) \rightarrow \pi_1(Y, y_0)$, is the trivial homomorphism, meaning that it maps everything to the identity element. [**6 marks**]

Question 4

- (i) Carefully state the Heine-Borel Theorem. [**2 marks**]
- (ii) For the following questions, all sets are considered to be subspaces of \mathbb{R}^2 , with the usual topology.
- (a) Is the set $A = \{(x, y) : x, y \in \mathbb{Z}\}$ closed? Provide a argument supporting your answer. [**2 marks**]
- (b) If I remove finitely many points from the set $D = \{(x, y) : x^2 + y^2 \leq 1\}$, is the resulting set compact? Provide a argument supporting your answer. [**2 marks**]
- (c) If I remove finitely many points from the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, is the resulting set connected? Provide a argument supporting your answer. [**2 marks**]
- (iii) Consider the map $p_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $p_1(x, y) = x$. Let $A = \{(x, y) : x \geq 0 \text{ or } y = 0 \text{ (or both)}\}$, with the subspace topology. Let $q : A \rightarrow \mathbb{R}$ be defined by the restriction of p_1 to A : $q = p_1|_A$. Prove that q is an identification map but that it does not necessarily send open sets to open sets. [**6 marks**]
- (iv) Let X be the so-called Hawaiian earring, which is defined by $X = \cup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$. So X is the union of the circles with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \dots$. Let Y be the identification space formed by starting with \mathbb{R} and defining $x \sim y$ if either $x = y$ or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic. (Hint: think about compactness.) [**6 marks**]