F11PE1

Department of Mathematics

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Pure Mathematics E

Duration: 3 Hours

Sample Paper 2011/2012

Attempt all questions

A University approved calculator may be used
for basic computations, but
appropriate working must be shown to obtain full credit.

Question 1

(i) Carefully define what it means for a topological space X to be Hausdorff. **[2 marks]**

(ii) Are the following spaces Hausdorff?

(a) The metric space (X, d) with the associated metric topology, where X contains at least two elements. **[2 marks]**

(b) The set $\mathbb{R}^2$, with the particular point topology where the particular point is chosen to be $(5, -2)$. **[2 marks]**

(c) The circle S^1 with the finite complement topology. **[2 marks]**

Provide brief arguments supporting your answers.

(iii) Let X be Hausdorff. Prove that every subset of the form $\{x\}$ for $x \in X$ is closed. **[6 marks]**

(iv) If $f : X \to Y$ is one-to-one and continuous and Y is Hausdorff, is it necessarily true that X is Hausdorff? If so, provide an proof, if not, provide a counterexample. **[6 marks]**

Question 2

(i) Given a subset $A \subset X$, where X is a topological space, carefully define what it means for p to be a limit point of A. **[2 marks]**

(ii) Consider the set $K = \{1/n : n = 1, 2, \ldots\} \subset \mathbb{R}$. Determine the closure of K when $\mathbb{R}$ is endowed with the following topologies.

(a) The topology determined by the basis $\beta = \{[a, b) : a < b\}$. **[2 marks]**

(b) The particular point topology with the particular point chosen to be 0. **[2 marks]**

(c) The finite complement topology. **[2 marks]**

Provide brief arguments supporting your answers.

(iii) Prove that U is open if and only if $U = \text{int}(U)$. **[6 marks]**

(iv) Consider $X = \{(x, y) : x = 1/2^n, n = 1, 2, \ldots, y \in [0, 1]\} \cup \{(0, 0), (0, 1)\}$ with the subspace topology inherited from $\mathbb{R}^2$. Prove that any subset of X that is both open and closed and that contains $(0, 0)$ must also contain $(0, 1)$. **[6 marks]**

Question 3

(i) Carefully state what it means for a subset of a topological space to be path connected. **[2 marks]**

(ii) Provide arguments supporting your answers to the following questions.

(a) Is the product of two path connected spaces necessarily connected? **[2 marks]**

(b) If $A \subset X$ and A is path connected, is $\bar{A}$ necessarily path connected? **[2 marks]**

(c) If $f : X \to Y$ is continuous and X is path-connected, is $f(X)$ necessarily path connected? **[2 marks]**

(iii) Given continuous functions $f_1, f_2 : X \to Y$ such that f_1 and f_2 are homotopic, and also $g_1, g_2 : Y \to Z$ such that g_1 and g_2 are homotopic, prove that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic. **[6 marks]**

(iv) Let A be a subspace of $\mathbb{R}^n$ and let $h : A \to Y$ be such that $h(x_0) = y_0$. Suppose there is a continuous function $H : \mathbb{R}^n \to Y$ such that $H(x) = h(x)$ for all $x \in A$. Prove that the induced map h_* on the fundamental groups, $h_* : \pi_1(A, x_0) \to \pi_1(Y, y_0)$, is the trivial homomorphism, meaning that it maps everything to the identity element. **[6 marks]**

Question 4

(i) Carefully state the Heine-Borel Theorem. **[2 marks]**

(ii) For the following questions, all sets are considered to be subspaces of $\mathbb{R}^2$, with the usual topology.

(a) Is the set $A = \{(x, y) : x, y \in \mathbb{Z}\}$ closed? Provide a argument supporting your answer. **[2 marks]**

(b) If I remove finitely many points from the set $D = \{(x, y) : x^2 + y^2 \leq 1\}$, is the resulting set compact? Provide a argument supporting your answer. **[2 marks]**

(c) If I remove finitely many points from the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, is the resulting set connected? Provide a argument supporting your answer. **[2 marks]**

(iii) Consider the map $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p_1(x, y) = x$. Let $A = \{(x, y) : x \geq 0 \text{ or } y = 0 \text{ (or both)}\}$, with the subspace topology. Let $q : A \to \mathbb{R}$ be defined by the restriction of p_1 to A: $q = p_1|_A$. Prove that q is an identification map but that it does not necessarily send open sets to open sets. **[6 marks]**

(iv) Let X be the so-called Hawaiian earring, which is defined by $X = \cup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$. So X is the union of the circles with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \ldots$. Let Y be the identification space formed by starting with $\mathbb{R}$ and defining $x \sim y$ if either $x = y$ or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic. (Hint: think about compactness.) **[6 marks]**

Question 5

(i) Carefully define a topological group. **[3 marks]**

(ii) Consider the set $G = GL(n)$, the set of invertible $n \times n$ matrices, with the operation of matrix multiplication. Prove that G is a topological group. **[5 marks]**

(iii) (a) Let U be an open set containing the identity e in a topological group. Let $V^{-1} = i(V)$ for any set V. Prove that $W = UU^{-1}$ is an open set containing e that satisfies $W = W^{-1}$. **[6 marks]**

(b) Use the result of part a) to prove the following: If V is any open set containing e, there is an open set $U \subset V$, such that $e \in U$ and $U = U^{-1}$. **[6 marks]**