Sample Exam, F11PE Solutions, Topology, Autumn 2011

Question 1

(i) Carefully define what it means for a topological space X to be Hausdorff.

Solution: A space X is Hausdorff if, given any two points $x, y \in X$ such that $x \neq y$, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$. [**2 marks**]

(ii) Are the following spaces Hausdorff?

(a) The metric space (X, d) with the associated metric topology, where X contains at least two elements.

Solution: Yes, such a space is always Hausdorff. Let $x, y \in X$ such that $x \neq y$. If $d(x, y) = \epsilon$, then $B_{\epsilon/2}(x)$ and $B_{\epsilon/2}(y)$ are disjoint open sets containing x and y, respectively. [**2 marks**]

(b) The set $\mathbb{R}^2$, with the particular point topology where the particular point is chosen to be $(5, -2)$.

Solution: No, this space is not Hausdorff. No two nonempty open sets are disjoint, because they must both contain $(5, -2)$. [**2 marks**]

(c) The circle S^1 with the finite complement topology.

Solution: This set is not Hausdorff, because no two (nonempty) open sets are disjoint. If U is open, then its complement is finite. Thus, if V is disjoint from U, V must be finite, which means it cannot be open. [**2 marks**]

Provide brief arguments supporting your answers.

(iii) Let X be Hausdorff. Prove that every subset of the form $\{x\}$ for $x \in X$ is closed.

Solution: To show that $\{x\}$ is closed we must show that $X \setminus \{x\}$ is open. This set will be open if, given any point y in the set, we can find a neighborhood N of y such that $N \subset X \setminus \{x\}$. Since $y \in X \setminus \{x\}$, $x \neq y$, so let U and V be disjoint open sets containing the two points. But then $U \subset X \setminus \{x\}$, and U is a neighborhood of y. Hence, the set is open, so $\{x\}$ is closed. [**6 marks**]

(iv) If $f : X \to Y$ is one-to-one and continuous and Y is Hausdorff, is it necessarily true that X is Hausdorff? If so, provide an proof, if not, provide a counterexample.

Solution: Yes, this is true. Let $x_1, x_2 \in X$ be distinct points, so that $f(x_1) \neq f(x_2)$ because f is one-to-one. Let U and V be disjoint open sets in Y containing these points. Then $x_1 \in f^{-1}(U)$ and $x_2 \in f^{-1}(V)$, and both $f^{-1}(U)$ and $f^{-1}(V)$ are open since f is continuous. Also, they must be disjoint, since U and V are. Thus, X is Hausdorff. [**6 marks**]

Question 2

(i) Given a subset $A \subset X$, where X is a topological space, carefully define what it means for p to be a limit point of A.

Solution: The point p is a limit point of A if, given any open set U containing p, $U \cap (A \setminus \{p\}) \neq \emptyset$. **[2 marks]**

(ii) Consider the set $K = \{1/n : n = 1, 2, \dots\} \subset \mathbb{R}$. Determine the closure of K when $\mathbb{R}$ is endowed with the following topologies.

(a) The topology determined by the basis $\beta = \{[a, b) : a < b\}$.

Solution: In this case, $\bar{K} = K \cup \{0\}$. For any other point x, we can consider three cases. If $x < 0$, then for some ϵ the set $[x, -\epsilon)$ is an open set that doesn't intersect K. if $x > 1$, then $[x, x+1)$ is such an open set. If $x \in (0, 1)$ and $x \notin K$, just find the smallest n such that $1/n < x$, and then $[1/n, 1/(n-1))$ is such an open set. The point 0 is a limit point because $[0, \epsilon)$ intersects K for any ϵ, because there is always an n large enough so that $1/n < \epsilon$. **[2 marks]**

(b) The particular point topology with the particular point chosen to be 0.

Solution: In this case, $\bar{K} = K$. For any other point $x \notin K$, the set $\{0, x\}$ (or just $\{0\}$, if $x = 0$) is an open set that doesn't intersect K. Thus, K has no limit points. **[2 marks]**

(c) The finite complement topology.

Solution: In this case, $\bar{K} = \mathbb{R}$. To see this, pick any $x \in \mathbb{R}$ and let U be an open set containing x. Since its complement must be finite, and K has infinitely many points, $U \cap K \neq \emptyset$ - in fact there are infinitely many points in the intersection. Hence, x is a limit point. **[2 marks]**

Provide brief arguments supporting your answers.

(iii) Prove that U is open if and only if $U = \text{int}(U)$.

Solution: Let U be open. Since $\text{int}(U)$ is the union of all open sets contained in U, by definition $\text{int}(U) \subset U$. Also, $U \subset \text{int}(U)$, since U is open and contained in itself. Thus, $U = \text{int}(U)$. Conversely, since the interior is the union of a collection of open sets, it is open. Thus, if $U = \text{int}(U)$ then U is open. **[6 marks]**

(iv) Consider $X = \{(x, y) : x = 1/2^n, n = 1, 2, \dots, y \in [0, 1]\} \cup \{(0, 0), (0, 1)\}$ with the subspace topology inherited from $\mathbb{R}^2$. Prove that any subset of X that is both open and closed and that contains $(0, 0)$ must also contain $(0, 1)$.

Solution: Let O be open and closed and contain the origin. Thus, $O = U \cap X$, where U is open in $\mathbb{R}^2$ and contains the origin. Since it is closed, $X \setminus O = V \cap X$, where V is open in $\mathbb{R}^2$. Suppose that $(0, 1) \notin O$. Then $(0, 1) \in X \setminus O$ so $(0, 1) \in V$. Since V is open, for n sufficiently large we have $L_v := V \cap \{(1/2^n, y) : y \in [0, 1]\} \neq \emptyset$, and similarly $L_u := U \cap \{(1/2^n, y) : y \in [0, 1]\} \neq \emptyset$. Also, by construction $L_u \cup L_v = \{(1/2^n, y) : y \in [0, 1]\}$, which is homeomorphic to the interval $[0, 1]$. Thus, we have found disjoint, nonempty open sets, U and V, that disconnect a set that is homeomorphic to an interval. This is a contradiction, since the interval is connected. **[6 marks]**

Question 3

(i) Carefully state what it means for a subset of a topological space to be path connected.

Solution: A topological space X is path connected if for all $x, y \in X$ there exists a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. [**2 marks**]

(ii) Provide arguments supporting your answers to the following questions.

(a) Is the product of two path connected spaces necessarily connected?

Solution: Yes. Let X and Y be path connected, and let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Let α be a path in X from x_1 to x_2, and let β be a path in Y from y_1 to y_2. Define $\gamma : [0,1] \to X \times Y$ to be $\gamma(t) = (\alpha(t), \beta(t))$. Since $p_1 \circ \gamma = \alpha$ and $p_2 \circ \gamma = \beta$ are continuous, γ is continuous and it is the required path. [**3 marks**]

(b) If $A \subset X$ and A is path connected, is $\bar{A}$ necessarily path connected?

Solution: No. The topologists sine curve is a counterexample: $S = \{(x, \sin(1/x)) : x \in (0,1)\}$. As discussed in lecture, it is path connected, but its closure isn't. [**3 marks**]

(c) If $f : X \to Y$ is continuous and X is path-connected, is $f(X)$ necessarily path connected?

Solution: Yes. Let $y_{1,2} \in f(X)$, so there exist $x_{1,2} \in X$ such that $f(x_i) = y_i$, $i = 1, 2$. Let γ be a path between x_1 and x_2. Then $f \circ \gamma$ is continuous, because it is the composition of continuous functions, and it is a path from y_1 to y_2 in $f(X)$. [**3 marks**]

(iii) Given continuous functions $f_1, f_2 : X \to Y$ such that f_1 and f_2 are homotopic, and also $g_1, g_2 : Y \to Z$ such that g_1 and g_2 are homotopic, prove that $g_1 \circ f_1$ and $g_2 \circ f_2$ are homotopic.

Solution: Let F and G be the homotopies between the respective maps. Consider $H(t, x) = G(t, F(t, x))$. Note that $H(0, x) = G(0, F(0, x)) = G(0, f_1(x)) = g_1(f_1(x))$ and $H(1, x) = G(1, F(1, x)) = G(1, f_2(x)) = g_2(f_2(x))$. Also, H is continuous because it is the composition of continuous functions. [**3 marks**]

(iv) Let A be a subspace of $\mathbb{R}^n$ and let $h : A \to Y$ be such that $h(x_0) = y_0$. Suppose there is a continuous function $H : \mathbb{R}^n \to Y$ such that $H(x) = h(x)$ for all $x \in A$. Prove that the induced map h_* on the fundamental groups, $h_* : \pi_1(A, x_0) \to \pi_1(Y, y_0)$, is the trivial homomorphism, meaning that it maps everything to the identity element.

Solution: Recall that, by definition, $h_*(\langle\alpha\rangle) = \langle h \circ \alpha\rangle$. We must show that, for any α, $h \circ \alpha$ is homotopic in Y to a constant loop. Since α is a loop in A, $h \circ \alpha = H \circ \alpha$. Since $\mathbb{R}^n$ is simply connected, there exists a homotopy $F(t, x)$ from α to the loop $\gamma(t) = x_0$ for all $t \in [0,1]$: $F(t, x) : [0,1] \times \mathbb{R}^n \to \mathbb{R}^n$, $F(0, x) = \alpha(x)$, $F(1, x) = x_0$. But then $H(F(t, x))$ is a homotopy in Y from $h \circ \alpha$ to $h \circ \gamma$, where $h \circ \gamma$ is constant and equal to y_0. Thus, $h_*(\langle\alpha\rangle) = \langle y_0\rangle$ for all α. [**6 marks**]

Question 4

(i) Carefully state the Heine-Borel Theorem.

Solution: A subset of $\mathbb{R}^n$ (with the usual topology) is compact if and only if it is closed and bounded. [**2 marks**]

(ii) For the following questions, all sets are considered to be subspaces of $\mathbb{R}^2$, with the usual topology.

(a) Is the set $A = \{(x, y) : x, y \in \mathbb{Z}\}$ closed? Provide a argument supporting your answer.

Solution: Yes. The set A has no limit points: for any point (x, y), it is possibly to take $B_\epsilon(x, y)$, for ϵ sufficiently small, so that $B_\epsilon(x, y) \cap A = \emptyset$ or $\{(x, y)\}$, if $(x, y) \in A$. Thus, $A = \bar{A}$, and it is closed. [**2 marks**]

(b) If I remove finitely many points from the set $D = \{(x, y) : x^2 + y^2 \leq 1\}$, is the resulting set compact? Provide a argument supporting your answer.

Solution: No, because it will not be closed - each one of the removed points is a limit point. Hence, by Heine-Borel, it cannot be compact. [**2 marks**]

(c) If I remove finitely many points from the set $S^1 = \{(x, y) : x^2 + y^2 = 1\}$, is the resulting set connected? Provide a argument supporting your answer.

Solution: Yes, if you remove only one point, no otherwise. If you remove only one point, the set is homeomorphic to an open interval, hence connected. If you remove more than one, it is homeomorphic to a finite, disjoint union of open intervals, which is not connected. [**2 marks**]

(iii) Consider the map $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $p_1(x, y) = x$. Let $A = \{(x, y) : x \geq 0 \text{ or } y = 0 \text{ (or both)}\}$, with the subspace topology. Let $q : A \to \mathbb{R}$ be defined by the restriction of p_1 to A: $q = p_1|_A$. Prove that q is an identification map but that it does not necessarily send open sets to open sets.

Solution: We must show q is continuous, onto, and that U is open in $\mathbb{R}$ if and only if $p_1^{-1}(U)$ is open in A. Note that it is continuous because it is the restriction of a continuous function. It is onto because $p_1(x, 0) = x$ and $(x, 0) \in A$ for all $x \in \mathbb{R}$. Let U be any subset of $\mathbb{R}$. Then $q^{-1}(U) = (U \times \mathbb{R}) \cap A$, which is open if and only if U is open: If U is open, then $(U \times \mathbb{R}) \cap A$ is open by the definition of the subspace topology. Conversely, if $(U \times \mathbb{R}) \cap A$ is open, then for any $(x, y) \in (U \times \mathbb{R}) \cap A$ there is a $B_\epsilon(x) \times B_\delta(y) \subset (U \times \mathbb{R}) \cap A$, which implies $B_\epsilon(x) \subset U$, so U is open. Thus, q is an identification map. However, $B = ((-1, 1) \times (1, 2)) \cap A$ is open in A, but $q(B) = [0, 1)$, which is not open in $\mathbb{R}$. [**6 marks**]

(iv) Let X be the so-called Hawaiian earring, which is defined by $X = \cup_{n=1}^{\infty} C_n$, where $C_n = \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}$. So X is the union of the circles with center $(1/n, 0)$ and radius $1/n$ for $n = 1, 2, 3, \ldots$. Let Y be the identification space formed by starting with $\mathbb{R}$ and defining $x \sim y$ if either $x = y$ or if $x, y \in \mathbb{Z}$. Prove that X and Y are not homeomorphic. (Hint: think about compactness.)

Solution: Suppose they were homeomorphic. I claim that X is compact, but Y isn't, which is then a contradiction. To see that Y is not compact, consider the open cover $O_j = (j, j + 1)$ for all $j \in \mathbb{Z}$ together with the image of the set $(-1/4, 1/4)$ under the identification map. This set has no finite subcover, because if any set is removed, either $j + 1/2$ for some j or all the integers are not contained in the remaining subcover. To see that X is compact, note first that it is bounded. To see that it is also closed, hence compact, take $p \notin X$. We'll show that p cannot be a limit point. First note that $p \neq 0$, because $0 \in X$. If p is outside all of the circles, then $B_\epsilon(p)$, where ϵ is less than the distance to the outermost circle, is an open set that is disjoint from X. If p is inside any circle, it must be outside some smaller circle – so it lies "between" two circles. To make this precise, let D_n be the open region enclosed by the circle C_n. One can find an n such that $p \in D_n \setminus \bar{D}_{n+1}$, which is an open set that is disjoint from X. Hence, p is not a limit point. Thus, X is closed and bounded, so compact, and it cannot be homeomorphic to Y. [**6 marks**]

Question 5

(i) Carefully define a topological group.

Solution: A topological group is a set G that is both a Hausdorff topological space and a group. In addition, the two functions $m : G \times G \to G$, $m(x, y) = x \cdot y$, where $\cdot$ is the group operation, and $i : G \to G$, $i(x) = x^{-1}$, must be continuous. [**3 marks**]

(ii) Consider the set $G = GL(n)$, the set of invertible $n \times n$ matrices, with the operation of matrix multiplication. Prove that G is a topological group.

Solution: The set G is a Hausdorff topological space because we can view it as a subset of $\mathbb{R}^{n^2}$, with the subspace topology, and $\mathbb{R}^{n^2}$ is Hausdorff - any metric space with at least two elements is Hausdorff. The space G is a group because inverses exist - any matrix with nonzero determinant has an inverse that also has nonzero determinant, the identity exists since the identity matrix has a nonzero determinant of one, and the operation is associative because matrix multiplication is associative. The inverse function $i(A) = A^{-1}$ is continuous because the cofactor formula for a matrix inverse is just a polynomial function in each entry, hence continuous. The multiplication function is continuous for the same reason - it is also a polynomial function in its components. [**5 marks**]

(iii) (a) Let U be an open set containing the identity e in a topological group. Let $V^{-1} = i(V)$ for any set V. Prove that $W = UU^{-1}$ is an open set containing e that satisfies $W = W^{-1}$.

Solution: Consider $W = UU^{-1}$. Since $e \in U$ and $e \in U^{-1}$, $e = ee^{-1} \in W$. If $h \in W$, then $h = g_1 g_2^{-1}$ for some $g_1, g_2 \in U$, so $h^{-1} = g_2 g_1^{-1} \in W$. Thus, $W^{-1} \subset W$. If $h \in W^{-1}$, then there exists a $\tilde{h} = g_1 g_2^{-1} \in UU^{-1}$ such that $h = \tilde{h}^{-1}$. But then $h = \tilde{h}^{-1} = g_2 g_1^{-1} \in UU^{-1}$, and so $W^{-1} \subset W$. Hence, $W = W^{-1}$. Finally, since both U and U^{-1} are open, and also gU^{-1} is open for all $g \in U$ (since translation is a homeomorphism), we find $W = \cup_{g \in U} gU^{-1}$ is open. [**6 marks**]

(b) Use the result of part a) to prove the following: If V is any open set containing e, there is an open set $U \subset V$, such that $e \in U$ and $U = U^{-1}$.

Solution: We cannot just take $U = VV^{-1}$, because this is not necessarily in V. Since m is continuous, $m^{-1}(V)$ is open. Therefore, $(V \times V^{-1}) \cap m^{-1}(V)$ is open, and there is a basis element of the form $U_1 \times U_2 \subset (V \times V^{-1}) \cap m^{-1}(V)$ that contains (e, e). Define $\tilde{U} = U_1 \cap U_2 \cap U_1^{-1} \cap U_2^{-1}$, which is open and nonempty (it contains e), and define $U = \tilde{U}\tilde{U}^{-1}$. Then $e \in U$, U is open, and $U^{-1} = U$ by the above argument. Also, by construction, $U \subset V$. [**6 marks**]