Solutions to Exercises on Topological Groups, Topology 2011

1. (4.3 #13) Let G_1 and G_2 be topological groups. Since they are both Hausdorff, $G_1 \times G_2$ is a Hausdorff topological space under the product topology. Also, $i : G_1 \times G_2 \to G_1 \times G_2$ is continuous because $p_{1,2} \circ i$ is continuous for each projection function. Similarly, $m : (G_1 \times G_2) \times (G_1 \times G_2) \to G_1 \times G_2$ because its composition with the projection functions are continuous.

2. (4.3 #14) We must check that $\bar{H}$ is a group under $\cdot$, the operation of G. First, note that $e \in H$ so $e \in \bar{H}$. If $h \in \bar{H}$, is $h^{-1} \in \bar{H}$? If $h \in H$, then yes, clearly. So assume $h \in \bar{H} \setminus H$. bwoc, assume $h^{-1} \notin \bar{H}$. Then $\exists$ an open set U containing h^{-1} such that $U \cap H = \emptyset$. Since i, the inverse function, is continuous, $i^{-1}(U)$ is open in G, and $h \in i^{-1}(U)$. Since h is a limit point of H, there is a $g \in i^{-1}(U) \cap H$, $g \neq h$. But then $g^{-1} \in U \cap H$, which is a contradiction. So $h^{-1} \in \bar{H}$. Finally, if $h_1, h_2 \in \bar{H}$, we need to show that $h_1 \cdot h_2 \in \bar{H}$. If they're both in H, this is clear. Assume $h_1 \in \bar{H} \setminus H$, $h_2 \in H$, and bwoc assume $h_1 \cdot h_2 \notin \bar{H}$. So there is an open U such that $h_1 \cdot h_2 \in U$ and $U \cap H = \emptyset$. Since m is continuous, $m^{-1}(U)$ is open in $G \times G$, so there is an open set of the form $U_1 \times U_2 \subset m^{-1}(U)$, and $h_1 \in U_1$, $h_2 \in U_2$. Thus, there is a $g \in U_1 \cap H$, which implies $(g, h_2) \in U_1 \times U_2 \subset m^{-1}(U)$. But this is a contradiction, since $g \cdot h_2 \in U \cap H$. The case where $h_1, h_2 \in \bar{H} \setminus H$ is similar.

   To check that $\bar{H}$ is normal if H is, we must show that $g\bar{H}g^{-1} \subset \bar{H}$ for all $g \in G$. Since $ghg^{-1} \in H$ for all $h \in H$, suppose $h \in \bar{H} \setminus H$ and $ghg^{-1} \notin \bar{H}$. The there is an open U containing ghg^{-1} such that $U \cap H = \emptyset$. Note that $(g, hg^{-1}) \in m^{-1}(U)$, which is open, so there is an open set $U_1 \times V_1 \subset m^{-1}(U)$ such that $(g, hg^{-1}) \in U_1 \times V_1$. Similarly, we can find $\tilde{U}_1 \times \tilde{V}_1$ such that $(h, g^{-1}) \in \tilde{U}_1 \times \tilde{V}_1 \subset m^{-1}(V_1)$. Since $h \in \bar{H} \setminus H$, there is a point $\hat{h} \in \tilde{U}_1 \cap H$, and so

   $$m(\hat{h}, g^{-1}) \in V_1 \quad m(g, \hat{h}g^{-1}) \in U, \quad g\hat{h}g^{-1} \in U \cap H,$$

   since $\hat{h} \in H$ and H is normal. This is a contradiction.

3. (4.3 #15) Suppose $m : G \times G \to G$ is continuous. Let C be closed. We must show that $i^{-1}(C)$ is closed. Since G is compact and Hausdorff, this will follow if we can show that $i^{-1}(C)$ is compact. Let $\{V_\alpha\}$ be an open cover of $i^{-1}(C)$. Note that, since G is Hausdorff, the set $\{e\}$ is closed, and hence also compact. [It is closed because, if $g \in G \setminus e$, since $g \neq e$ there exists disjoint open sets U_g and V_e containing those points. But then $U_g \subset (G \setminus e)$, so $G \setminus e$ is open.] Thus, $m^{-1}(e)$ is closed, and hence compact. Note that $G \times C$ is closed and compact, because it is the product of compact spaces, and so

   $$K := m^{-1}(e) \cap (G \times C) = \{(g^{-1}, g) : g^{-1} \in C\}$$

   is the intersection of two closed sets, so it is closed, and hence compact. Note that $\{G \times V_\alpha\}$ is an open cover of K, and so there is a finite subcover $\{G \times V_{\alpha_i}\}_{i=1}^n$. But then $\{V_{\alpha_i}\}_{i=1}^n$ is a finite subcover of $i^{-1}(C)$.

4. (4.3 #16) Let $I_- \in O(n)$ be any fixed matrix with determinant -1. For example,

   $$I_- := \begin{pmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Consider the function $h : SO(n) \times \mathbb{Z}_2 \to O(n)$, $\mathbb{Z}_2 = \{1, -1\}$,

$$h(M, g) = \begin{cases} M & \text{if} \quad g = 1 \\ I_- M & \text{if} \quad g = -1 \end{cases}$$

This is onto, because if $N \in O(n)$ with $\det(N) = 1$, then $h(N, 1) = N$, and if $N \in O(n)$ with $\det(N) = -1$, then $h(I_-^{-1}N, -1) = N$. A similar calculation shows it is one-to-one. We need to check it is continuous with continuous inverse. Note that the function h is piecewise linear, and $O(n)$ consists of two connected components - in fact it can be written as the union of two disjoint, open sets (matrices with determinant plus one and minus one) - and it is exactly on these two open sets that the function h is linear. Thus, it is a homeomorphism.

To determine if it is an isomorphism, note that, if it were, either $SO(n) \times \mathbb{Z}_2$ and $O(n)$ would both be abelian, or neither would be. For $n = 2$, $SO(n)$ is abelian, because it is just the set of all rotations:

$$SO(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in [0, 2\pi) \right\} \cong S^1.$$

However, $O(2)$ is not abelian, because the inversion I_- doesn't necessarily commute with a rotation. Let R be a rotation that doesn't commute with I_-. If the groups were isomorphic, we would have

$$R = h(R, 1) = h(R \cdot I, -1 \cdot -1) = h(R, -1) \cdot h(I, -1) = I_- R I_-,$$

which could imply that $I_- R = I_-^{-1} R = R I_-$, which is a contradiction.

5. (4.3 #17) Since $m : G \times G \to G$ is continuous and $A \times B$ is compact, $m(A \times B) = AB$ is compact. (Continuous images of compact sets are compact.)

6. (4.3 #18) Since $e \in U$ and U is a neighborhood, there is an open set $\tilde{V}$ such that $e \in \tilde{V} \subset U$. This implies that $\tilde{V}^{-1} = i^{-1}(\tilde{V})$ is open. Also, $m^{-1}(\tilde{V})$ is open and contains (e, e), so $m^{-1}(\tilde{V}) \cap (\tilde{V} \times \tilde{V}^{-1})$ is open and nonempty. This implies it contains a basis element $(e, e) \in (V_1 \times V_2)$, and so we can set $V := i^{-1}(V_1) \cap i^{-1}(V_2) \cap V_1 \cap V_2$, which is open.

7. (4.3 #19) Since H is a discrete subgroup, $e \in H$ and $\{e\}$ is open in the subspace topology, and hence there must be an open set U in G such that $U \cap H = \{e\}$. This set U is a neighborhood of e in G. I claim the translates hU are all disjoint. Note that $h \in hU \cap H$, since $he = h$. Suppose $\exists g \in H$ such that $g \in hU$, $g \neq h$. This implies $e \in g^{-1}(hU) = (g^{-1}h)U$, which implies that $(g^{-1}h)^{-1} = h^{-1}g \in U$. Since $h^{-1}g \in H$ and $h^{-1}g \neq e$, this is a contradiction.

8. (4.3 #20) BWOC, if $C \cap H$ is not finite, then by Bolzano-Weierstrass it has a limit point $p \in C$. First, note that $p \notin H$, which follows from the following argument. Since any open set U with $p \in U$ satisfies $U \cap (H \setminus p) \neq \emptyset$, there is an $h \in H$ such that $h \neq p$, $h \in U$. But this would contradict the fact that H is discrete – it would imply that $\{p\} \subset H$ is not open. Using the result of problem 19, let U_e be an open set containing the identity whose translates $U_h = hU_e$ are disjoint. Using the result of problem 18, find an open set $V \subset U_e$ such that $VV^{-1} \subset U_e$. Note that $W = VV^{-1} = \cup_{g \in V}\{gV^{-1}\}$ is open, because the translate gV^{-1} is open since V^{-1} is, and so VV^{-1} is a union of open sets. Also, W contains e, and $W^{-1} = W$. Thus, pW is an open set containing p, so there is a $h \in H$ such that

$h \in pW$. So, $h = pg$ for some $g \in W$. Since $g^{-1} \in W \subset U_e$ and $p = hg^{-1}$, $p \in hU_e$. But then hU_e is an open set that contains p and that contains only one point $h \in H$. This contradicts the fact that p is a limit point: since the space is Hausdorff, we can now find an open set O containing p that doesn't contain h. Then $O \cap hU_e$ is an open set containing p that contains no element of H.