Tutorial Sheet 1, Topology 2011 (with solutions)

1. Let Γ be a graph. Prove that $v(\Gamma) - e(\Gamma) \leq 1$. Furthermore, show that $v(\Gamma) - e(\Gamma) = 1$ if and only if Γ is a tree.

   Solution: We'll prove this using induction. Let n be the number of vertices. When $n = 2$, it is clear that $v - e = 1 \leq 1$. Suppose the result is true for $n = k$, and consider the case $n = k + 1$. So, we've added a vertex to the previous graph. Since it is part of the graph, we've added at least one edge, as well, and we could have added more edges. Thus, $v_{k+1} - e_{k+1} \leq v_k - e_k \leq 1$.

   To prove the quantity is equal to one precisely when we have a tree: first consider the case when Γ is a tree. Use the same induction proof as above, noting that whenever we add a vertex we can only add one edge. (Otherwise, we will have created a loop.) Thus, $v - e = 1$. Conversely, if $v - e = 1$, suppose the graph contains a loop. But then we can remove an edge, without removing a vertex, to obtain $v - e = 2$, which is a contradiction, since we have already shown that $v - e \leq 1$.

2. Show that any graph contains a tree which includes all the vertices.

   Solution: Proof by way of contradiction (BWOC): Take the maximal (ie containing the most vertices) tree inside the graph, and suppose there is some vertex of the graph that isn't in the tree. This implies there is some edge connected to this vertex, whose other end is connected to a vertex that is in the tree. But we can add this edge to the tree and obtain a larger one, which contradicts maximality.

3. Let P be a polyhedron satisfying the assumptions of Euler's Theorem. Suppose that each face of P is a regular polygon with p edges and that q faces meet at each vertex. Use Euler's formula to prove that
$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$
   Use this result to conclude that there are only 5 such polyhedra.

   Solution: Let f be the number of faces. Then the total number of edges (counting some more than once) is fp, and the total number of vertices (counting some more than once) is fp. By counting them in this way, we've counted each edge twice, making $e = fp/2$, and we've counted each vertex q times, making $v = fp/q$. By Euler's formula,
$$2 = v - e + f = f\left(\frac{p}{q} - \frac{p}{2} + 1\right) \qquad \Rightarrow \qquad f = \frac{4q}{2p - qp + 2q}.$$
   Hence, $v = 4p/(2p - qp + 2q)$, and so
$$2 = \frac{4p}{2p - qp + 2q} - e + \frac{4q}{2p - qp + 2q} \qquad \Rightarrow \qquad \frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$
   To obtain information about the total number of such polyhedra, notice that $1/e > 0$, so $1/p + 1/q > 1/2$. Rearranging, we have $pq < 2(p + q)$. If we now just start testing possible values of p and q, we see this can only hold if $(p, q) \in \{(3, 3), (3, 4), (4, 3), (5, 3), (3, 5)\}$, which proves the result.

4. Find a continuous function that maps the hyperboloid to the annulus,
$$f : \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 - z^2 = 1\} \to \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 3\}.$$

Suggestion: write the hyperboloid in cylindrical coordinates (r, θ, z) and the annulus in polar coordinates (r, θ). Map the set $\{\theta = 0\}$ in the hyperboloid to the set $\{\theta = 0\}$ in the annulus via

$$g(r, 0, z) = \left(\frac{z}{1 + |z|} + 2, 0 \right).$$

Do this for each value of θ, and check that the resulting function is continuous. Does your function have an inverse? If so, is it continuous?

Solution: In cylindrical coordinates, the hyperboloid is given by $\{(r, \theta, z) \in \mathbb{R}^3 : r^2 - z^2 = 1\}$. This is a surface whose cross sections, parallel to the xy plane, are circles of radius r at heights $z = \pm\sqrt{r^2 - 1}$. Thus, for any fixed value of θ, z can take on any value in $\mathbb{R}$. Note that the function $z \mapsto z/(1 + |z|) + 2$ is a bijection from $\mathbb{R}$ to $(1, 3)$. Furthermore, it is continuous (in the ϵ - δ sense) at each point z_0 for the following reason. If $z_0 \neq 0$, wlog $z_0 > 0$ and we can chose a $\delta > 0$ so that all z with $d(z, z_0) < \delta$ satisfy $z > 0$ also. Hence

$$\left| \frac{z}{1 + |z|} + 2 - \left(\frac{z_0}{1 + |z_0|} + 2 \right) \right| = \frac{|z - z_0|}{(1 + z)(1 + z_0)} \leq |z - z_0|.$$

Thus, given any $\epsilon > 0$ if we take $\delta = \epsilon$, then $d(z, z_0) < \delta$ implies that $d(z/(1+|z|)+2, z_0/(1+|z_0|)+2) < \epsilon$. This implies that the function

$$g(r, \theta, z) = \left(\frac{z}{1 + |z|} + 2, \theta \right).$$

is continuous for each θ. Therefore, this is a continuous bijection between the hyperboloid and the annulus. To define an inverse, set

$$f(y) = \begin{cases} -\frac{(2-y)}{(y-1)} & \text{if} \quad y \in (1, 2) \\ \frac{(y-2)}{(3-y)} & \text{if} \quad y \in (2, 3) \end{cases}.$$

Then $g^{-1}(y, \theta) = (\sqrt{1 + f(y)^2}, \theta, f(y))$, which one can show is continuous by a similar argument.