# Tutorial Sheet 2, Topology 2011 (with Solutions)

1. Let $X$ be any set and $p \in X$ be some point in $X$. Define $\tau$ to be the collection of all subsets of $X$ that do not contain $p$, plus $X$ itself. Prove that $\tau$ is a topology on $X$. (It is called the "excluded point topology.")

   **Solution:** Just check this satisfies the definition of topology: contains the empty set, entire space, unions, and finite complements.

2. Consider the following metrics on $\mathbb{R}^2$ (which are not the usual metric): $d_1(x, y) = \max_{i=1,2} |x_i - y_i|$ and $d_2(x, y) = |x_1 - y_1| + |x_2 - y_2|$. Describe the open sets induced by these metrics. (What does an open ball look like?)

   **Solution:** For the first metric, the open ball of radius $\epsilon$ at the origin is a square with sides length $2\epsilon$, not including its edges, which are parallel to the axes, and centered at the origin. For the second metric, we also get a square but it has been rotated so that its vertices now lie on the axes, at points $(\pm\epsilon, 0)$ and $(0, \pm\epsilon)$. One can actually check, though, that the topologies induced by these metrics are both the same as the usual topology. (Although that wasn't really part of the question.)

3. Let $(X, d)$ be a metric space containing at least two points. Prove that the metric topology cannot be the trivial topology.

   **Solution:** We just need to find an open set other than $\emptyset$ or $X$, which implies the topology cannot be trivial. Take $x, y \in X$ such that $x \neq y$. Then $d(x, y) = \epsilon > \delta > 0$ and so $B_\delta(x)$ is the desired open set, since $x \in B_\delta(x)$ but $y \notin B_\delta(x)$.

4. Prove that, in the real line with the usual topology, every point is a limit point of the rationals.

   **Solution:** Take any $x \in \mathbb{R}$ and any open set $O$ containing $x$. Pick $B_\epsilon(x) \subset O$. Define $N$ so that $1/N < \epsilon$. Then the points of the set $A = \{p/N : p \in \mathbb{Z}\} \subset \mathbb{Q}$ divide the real line up into subintervals of length strictly less than $\epsilon$. Hence, $B_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$, and so $B_\epsilon(x) \cap (\mathbb{Q} \setminus \{x\}) \neq \emptyset$.

5. Find all the limit points of the following subsets of the real line:

   (a) $\{(1/m) + (1/n) : n, m = 1, 2, 3, \ldots\}$

   (b) $\{(1/n) \sin n : n = 1, 2, 3, \ldots\}$

   **Solution:** a) Notice that $1 + (1/n)$ limis to 1 only, $1/2 + (1/n)$ limits to $1/2$ only, etc. Hence, the limit points are the set $\{1/n\}$, as well as zero. b) The only limit point is zero.

6. Let $X$ be the real line equipped with the finite complement topology. Prove that if $A$ is an infinite set, then every point is a limit point of $A$. In addition, prove that if $A$ is a finite set, then it has no limit points.

   **Solution:** In the first case, let $U$ be open and contain $x$. Then the complement of $U$ is finite. As a result, there must be some point in $A$ (other than $x$) that's also in $U$. Hence, $x$ is a limit point. Conversely, if $A$ is finite, then consider the open set $U = X \setminus A$. (Put $x$ back in if $x \in A$). This is an open set containing $x$ that doesn't intersect $A$, so $x$ cannot be a limit point.

7. Find a family of closed subsets of the real line whose union is not closed.

**Solution:** $C_n = [1/n, 1]$, so that $\cup_n C_n = (0, 1]$. (Alternatively, take any non-closed set, for example $(0, 1)$. Let $C_x = \{x\}$ for each $x$ in your set. This also works.)