Tutorial Sheet 3, Topology 2011

1. Consider the following theorem:

   Theorem 1. *Let β be a nonempty collection of subsets of X. If the intersection of any finite number of elements of β is always in β, and if*
   $$\bigcup_{B \in \beta} B = X,$$
   then β is a basis for a topology on X.

   Use this theorem to do the following. Let X be the real line and let $\beta = \{[a, b) : a < b\}$. Prove that β is a base for a topology and that in this topology each member of β is both open and closed. (This topology is called the half-open interval topology.)

2. Find a countable basis for the usual topology on $\mathbb{R}$.

   [Some remarks on terminology: A topological space with a countable basis is called *second countable*. An example of a space that is not second countable is the real line with the half-open interval topology, defined above (you don't need to prove this). A related concept is as follows. A space that has a countable dense subset is called *separable*. We've already seen an example of this - the real line with the usual topology, which has the rationals as a countable dense set.]

3. Verify the following for arbitrary subsets A and B of a topological space X: $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$. Show that equality need not hold.

4. Determine the interior, closure, and frontier of each of the following sets.

   (a) The plane with both axes removed.

   (b) $\mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$

5. Let X be the real line equipped with the finite complement topology. Prove that if A is an infinite set, then every point is a limit point of A. In addition, prove that if A is a finite set, then it has no limit points.

6. Prove that $f : X \to Y$ is continuous if and only if C begin closed implies $f^{-1}(C)$ is also closed.

7. Prove that any two open intervals in the real line (with the usual topology) are homeomorphic.

8. Prove that the function defined in lecture is really a homeomorphism between the square and the disk.

9. Let D and E be disks with boundaries ∂D and ∂E. Prove that any homeomorphism $h : \partial D \to \partial E$ extends to a homeomorphism from D to E. This means there exists a homeomorphism $\tilde{h} : D \to E$ such that $\tilde{h}|_{\partial D} = h$. (You may assume that any homeomorphism from one disk to another maps the boundary of one disk to the boundary of the other.)