Tutorial Sheet 3, Topology 2011

1. Consider the following theorem:

   Theorem 1. *Let β be a nonempty collection of subsets of X. If the intersection of any finite number of elements of β is always in β, and if*
   $$\bigcup_{B\in\beta} B = X,$$
   then β is a basis for a topology on X.

   Use this theorem to do the following. Let X be the real line and let $\beta = \{[a,b) : a < b\}$. Prove that β is a base for a topology and that in this topology each member of β is both open and closed. (This topology is called the half-open interval topology.)

   Remark: Apologies if this question was in any way unclear. I should perhaps have said explicitly that the corresponding topology is $\tau = \{U : U = \emptyset \text{ or } U = \cup_\alpha [a_\alpha, b_\alpha)\}$. Therefore, you only really need to check that any nonempty intersection of sets in β is in β. [Or, I could have allowed for $a = b$ in the definition of β, with the convention that $[a,a) = \emptyset$.]

   Solution: To check that it is a base, we need to check that its union is the entire real line (this is clear – take $a = n, b = n+1$ for all n) and that it contains all finite intersections (if the intersection is nonempty, take the largest a and smallest b – the intersection is then $[a,b)$). Hence, it is a base. Note that the associated topology, by the definition of basis, is defined as follows: a set is open if it can be written as the union of basis elements. (Or if it is the empty set.)

   Note that each member of β is open by definition. The complement is $(-\infty, a) \cup [b, \infty)$, which can be written as the union of sets of the form $[-n, a)$ and $[b, n)$. Hence this complement is open, and so the set is closed.

2. Find a countable basis for the usual topology on $\mathbb{R}$.

   [Some remarks on terminology: A topological space with a countable basis is called *second countable*. An example of a space that is not second countable is the real line with the half-open interval topology, defined above (you don't need to prove this). A related concept is as follows. A space that has a countable dense subset is called *separable*. We've already seen an example of this - the real line with the usual topology, which has the rationals as a countable dense set.]

   Solution: A countable basis for the real line with the usual topology is the collection of all open intervals whose endpoints are rational.

3. Verify the following for arbitrary subsets A and B of a topological space X: $\overline{A\cap B} \subseteq \bar{A} \cap \bar{B}$. Show that equality need not hold.

   Solution: First note that if $x \in A \cap B$, then clearly $x \in \bar{A} \cap \bar{B}$. So assume $x \in \overline{A\cap B} \setminus (A \cap B)$, which means that x is a limit point of $A \cap B$. Then for any open set O containing x, $O \cap ((A\cap B) \setminus \{x\}) \neq \emptyset$. BWOC – without loss of generality, suppose x is not a limit point for A. Then we can find an open set U containing x such that $U \cap (A \setminus \{x\}) = \emptyset$. But this implies $U \cap ((A \cap B) \setminus \{x\}) = \emptyset$, which is a contradiction. Thus, $x \in \bar{A} \cap \bar{B}$.

A simple example where equality doesn't hold is $A = (0,1)$ and $B = (1,2)$. Another one is

$$A = \{p/q \in \mathbb{Q} : q = 2^n, n \in \mathbb{N}\}, \qquad B = \{p/q \in \mathbb{Q} : q = 3^n, n \in \mathbb{N}\}.$$

We have $\overline{A \cap B} = \overline{\emptyset} = \emptyset$, but $\bar{A} \cap \bar{B} = \mathbb{R} \cap \mathbb{R} = \mathbb{R}$.

4. Determine the interior, closure, and frontier of each of the following sets.

   (a) The plane with both axes removed.

   (b) $\mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$

   Solution: a) The closure is the entire plane, the interior is the set itself, and the frontier is the axes. b) Denote $A = \mathbb{R}^2 \setminus \{(x, \sin(1/x)) : x > 0\}$. Then $\text{int}(A) = A \setminus \{(0, y) : -1 \leq y \leq 1\}$, $\text{cl}(A) = \mathbb{R}^2$, $\text{front}(A) = \mathbb{R}^2 \setminus A$.

5. Let X be the real line equipped with the finite complement topology. Prove that if A is an infinite set, then every point is a limit point of A. In addition, prove that if A is a finite set, then it has no limit points.

   Solution: In the first case, let U be open and contain x. Then the complement of U is finite. As a result, there must be some point in A (other than x) that's also in U. Hence, x is a limit point. Conversely, if A is finite, then consider the open set $U = X \setminus A$. (Put x back in if $x \in A$). This is an open set containing x that doesn't intersect A, so x cannot be a limit point.

6. Prove that $f : X \to Y$ is continuous if and only if C begin closed implies $f^{-1}(C)$ is also closed.

   Solution: Assume that f is continuous. Then if C is closed, $Y \setminus C$ is open, so $f^{-1}(Y \setminus C)$ is open, and so $X \setminus f^{-1}(Y \setminus C) = f^{-1}(C)$ is closed. Suppose now that inverse images of closed sets are closed. Let O be open, so $Y \setminus O$ is closed. The proof now follows as before.

7. Prove that any two open intervals in the real line (with the usual topology) are homeomorphic.

   Solution: Suppose the intervals are given by (a, b) and (c, d). Use the linear homeomorphism $f(x) = (d - c)(x - a)/(b - a) + c$.

8. Prove that the function defined in lecture is really a homeomorphism between the square and the disk.

   Solution: The boundary of the square consists of points of the form $(-1/2, y^*)$, $(1/2, y^*)$, $(x^*, -1/2)$, and $(x^*, 1/2)$. In the first case, the line containing the point and the origin is $\{y = -2y^* x\}$. The corresponding point on the circle is $x = -\sqrt{1/(1 + 4(y^*)^2)}$, $y = 2y^* \sqrt{1/(1 + 4(y^*)^2)}$. Thus, on that part of the boundary, the function is defined by

   $$f(-1/2, y) = \left(-\sqrt{\frac{1}{1 + 4y^2}}, 2y\sqrt{\frac{1}{1 + 4y^2}} \right).$$

   You can now prove directly that this function is continuous, compute its inverse explicitly, and check that it is continuous. For example, the inverse is

   $$f^{-1}(a, b) = \left(-\frac{1}{2}, \text{sgn}(b)\sqrt{\frac{1 - a^2}{4a^2}} \right), \quad \text{if} \quad a^2 + b^2 = 1, a \in [-1, -1/\sqrt{2}].$$

   You can then do a similar calculation for each piece of the boundary.

9. Let D and E be disks with boundaries ∂D and ∂E. Prove that any homeomorphism $h : \partial D \to \partial E$ extends to a homeomorphism from D to E. This means there exists a homeomorphism $\tilde{h} : D \to E$ such that $\tilde{h}|_{\partial D} = h$. (You may assume that any homeomorphism from one disk to another maps the boundary of one disk to the boundary of the other.)

   Solution: Let D_1 be the unit disk. Since they are both 2-disks, there exist homeomorphisms taking them to D_1, call them h_1 and h_2. Since h_i must map the bondary of D_i to the boundary of the unit disk, $g := h_2 \circ h \circ h_1^{-1}$ is a homeomorphism from the boundary of D_1 to itself. Given any $x \neq 0$ in the interior of D_1, define $\tilde{g}(x) = |x|g(x/|x|)$ and $\tilde{g}(0) = 0$, which extends g to the entire disk. Then, we take $\tilde{h} = h_2^{-1} \circ \tilde{g} \circ h_1$.