1. Find an open cover of \( \mathbb{R}^1 \) that does not contain a finite subcover. Do the same for \((0,1)\).

2. Let \( X \) be an infinite set with the finite complement topology.
   
   (a) Prove that \( X \) is not Hausdorff.
   
   (b) Prove that every subset of \( X \) is compact.
   
   (c) Find an example of a subset of \( X \) that is not closed, which is therefore an example of a compact set that is not closed.

3. Are either of the following sets compact?
   
   (a) The rational numbers, considered as a subset of the real line.
   
   (b) \( S^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \} \), with finitely many points removed, considered as a subset of \( \mathbb{R}^{n+1} \).

4. Find an example of a function \( f : X \to Y \), where \( X \) is compact and \( f \) is a continuous bijection, but such that \( f \) is not a homeomorphism. (Note: based on the theorem from class this implies \( Y \) cannot be Hausdorff.)

5. Do the real numbers with the half-open interval topology form a compact space? (See tutorial sheet 3 for a definition of this topology.)

6. Prove the Bolzano-Weierstrass Theorem: Any infinite subset of a compact space must have a limit point. (Hint: use a proof by contradiction.)

7. Suppose \( f : X \to \mathbb{R} \) where \( X \) is compact and \( f \) is continuous. Prove that \( f \) is bounded and attains its bounds. (This means \( f(X) \subseteq [a,b] \) for some \( a, b \in \mathbb{R} \) and \( \exists x, y \in X \) such that \( f(x) = a \) and \( f(y) = b \). Hint: use the Heine-Borel theorem.)

8. Let \((X, \tau_X)\) be a Hausdorff space that is \( \text{locally compact} \), meaning that each point \( x \in X \) has a neighborhood that is compact. Form a new space by adding one extra point, which we denote by \( \infty \): \( Y = X \cup \{ \infty \} \). Let
   
   \[ \tau_Y = \{ U \subseteq Y : U \in \tau_X \text{ or } U = (X \setminus K) \cup \{ \infty \} \text{ where } K \text{ is compact as a subset of } X \} \]
   
   (a) Prove that \( \tau_Y \) is a topology on \( Y \).
   
   (b) Prove that \( Y \) is a compact Hausdorff space.

Note: \( Y \) is called the \( \text{one-point compactification} \) of \( X \). (Think about what this space is like if \( X = \mathbb{R} \). \( \mathbb{R} \cup \{ \infty \} \) is actually homeomorphic to the circle \( S^1 \).)