Tutorial Sheet 4, Topology 2011

1. Find an open cover of $\mathbb{R}^1$ that does not contain a finite subcover. Do the same for $(0, 1)$.

2. Let X be an infinite set with the finite complement topology.

   (a) Prove that X is not Hausdorff.

   (b) Prove that every subset of X is compact.

   (c) Find an example of a subset of X that is not closed, which is therefore an example of a compact set that is not closed.

3. Are either of the following sets compact?

   (a) The rational numbers, considered as a subset of the real line.

   (b) $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, with finitely many points removed, considered as a subset of $\mathbb{R}^{n+1}$.

4. Find an example of a function $f : X \to Y$, where X is compact and f is a continuos bijection, but such that f is not a homeomorphism. (Note: based on the theorem from class this implies Y cannot be Hausdorff.)

5. Do the real numbers with the half-open interval topology form a compact space? (See tutorial sheet 3 for a definition of this topology.)

6. Prove the Bolzano-Weierstrass Theorem: Any infinite subset of a compact space must have a limit point. (Hint: use a proof by contradiction.)

7. Suppose $f : X \to \mathbb{R}$ where X is compact and f is continuous. Prove that f is bounded and attains its bounds. (This means $f(X) \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ and $\exists\ x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Hint: use the Heine-Borel theorem.)

8. Let (X, τ_X) be a Hausdorff space that is *locally compact*, meaning that each point $x \in X$ has a neighborhood that is compact. Form a new space by adding one extra point, which we denote by ∞: $Y = X \cup \{\infty\}$. Let

$$\tau_Y = \{U \subset Y : U \in \tau_X \text{ or } U = (X \setminus K) \cup \{\infty\} \text{ where } K \text{ is compact as a subset of } X\}.$$

   (a) Prove that τ_Y is a topology on Y.

   (b) Prove that Y is a compact Hausdorff space.

   Note: Y is called the *one-point compactification* of X. (Think about what this space is like if $X = \mathbb{R}$. $\mathbb{R} \cup \{\infty\}$ is actually homeomorphic to the circle S^1.)