Tutorial Sheet 4, Topology 2011

1. Find an open cover of $\mathbb{R}^1$ that does not contain a finite subcover. Do the same for $(0,1)$.

   Solution: For the real line, take $B_1(n)$ for all $n \in \mathbb{Z}$. (This is essentially the same as the example in class regarding $\mathbb{R}^2$.) For $(0,1)$ take $(1/n, 1-1/n)$, for $n \in \mathbb{N}$.

2. Let X be an infinite set with the finite complement topology.

   (a) Prove that X is not Hausdorff.

   (b) Prove that every subset of X is compact.

   (c) Find an example of a subset of X that is not closed, which is therefore an example of a compact set that is not closed.

   Solution: a) We'll prove that no two open sets are disjoint, which implies the result. BWOC, let U, V be disjoint open sets. This implies $V \subset (X \setminus U)$, and $X \setminus U$ is a finite set, which implies V is finite. But then $X \setminus V$ is infinite, contradicting the fact that V is open. b) Let $A \subset X$ and let $\{U_\alpha\}$ be an open cover. Pick any single element of the cover, U_{α^*}. It's complement has finitely many elements, so there are only finitely many elements of A, $x_1, \ldots, x_n$, that are not in this set. For each one, find a U_{α_i} containing x_i. Then $U_{\alpha^*}, \{U_{\alpha_i}\}_{i=1}^n$ is a finite subcover. c) Any open set other than X or $\emptyset$ is a set that is compact, but not closed.

3. Are either of the following sets compact?

   (a) The rational numbers, considered as a subset of the real line.

   (b) $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, with finitely many points removed, considered as a subset of $\mathbb{R}^{n+1}$.

   Solution: The first is not compact because it is not closed or bounded. The second is not compact because it is not closed (although it is bounded).

4. Find an example of a function $f : X \to Y$, where X is compact and f is a continuos bijection, but such that f is not a homeomorphism. (Note: based on the theorem from class this implies Y cannot be Hausdorff.)

   Solution: Let $X = [a,b]$ with the usual topology and Y be $[a,b]$ with the finite complement topology. Above we showed that any infinite space with the finite complement topology is not Hausdorff. Let $f(x) = x$, and check it is continuous, but its inverse isn't. (It is clearly one to one and onto.)

5. Do the real numbers with the half-open interval topology form a compact space? (See tutorial sheet 3 for a definition of this topology.)

   Solution: No. The collection $\{[n, n+1)\}$ is an open cover with no finite subcover.

6. Prove the Bolzano-Weierstrass Theorem: Any infinite subset of a compact space must have a limit point. (Hint: use a proof by contradiction.)

   Solution: Suppose S is an infinite set with no limit points in a compact space X. For each $x \in X$, let U_x be an open set containing x that is either i) disjoint from S, if $x \notin S$, or ii) disjoint from $S \setminus x$, if $x \in S$. Then $\{O_x\}$ is an open cover of X, and so there is a finite subcover $\{O_{x_i}\}_{i=1}^n$. But this must also be a cover of S, and each set O_{x_i} contains at most one point of S. Thus, S can contain only finitely many elements, which is a contradiction.

7. Suppose $f : X \to \mathbb{R}$ where X is compact and f is continuous. Prove that f is bounded and attains its bounds. (This means $f(X) \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ and $\exists\ x, y \in X$ such that $f(x) = a$ and $f(y) = b$. Hint: use the Heine-Borel theorem.)

   Solution: Since X is compact, so is $f(X)$, and so by Heine-Borel it is closed and bounded. Consider $a = \inf_{x \in X} f(x)$. Since $f(X)$ is closed, $a \in f(X)$, so there must be some $x \in X$ such that $f(x) = a$. A similar argument works for $b = \sup_{x \in X} f(x)$.

8. Let (X, τ_X) be a Hausdorff space that is *locally compact*, meaning that each point $x \in X$ has a neighborhood that is compact. Form a new space by adding one extra point, which we denote by ∞: $Y = X \cup \{\infty\}$. Let

$$\tau_Y = \{U \subset Y : U \in \tau_X \text{ or } U = (X \setminus K) \cup \{\infty\} \text{ where } K \text{ is compact as a subset of } X\}.$$

   (a) Prove that τ_Y is a topology on Y.

   (b) Prove that Y is a compact Hausdorff space.

   Note: Y is called the *one-point compactification* of X. (Think about what this space is like if $X = \mathbb{R}$. $\mathbb{R} \cup \{\infty\}$ is actually homeomorphic to the circle S^1.)

   Solution: Part a): the empty set is included because it is open in X, and all of Y is included because we can take $K = \emptyset$. Let U and V be open. If they are both open in X, their intersection is clearly still open. If V is open in X and $U = (X \setminus K) \cup \{\infty\}$ for some K, note that K is closed because X is Hausdorff. Thus $X \setminus K$ is open, so $U \cap V = (X \setminus K) \cap V$, which is thus open. If both U and V are of the latter form, then $U \cap V = [X \setminus (K_u \cup K_v)] \cup \{\infty\}$, which is open since $K_u \cup K_v$ is compact. The argument to show that unions of open sets are open is similar.

   Part b): Let $\{U_\alpha\}$ be an open cover of Y. Note that it must contain at least one element of the form $U_\alpha = (X \setminus K) \cup \{\infty\}$, or else ∞ would not be included in the union. Pick one set of this form, $U_{\alpha^*} = (X \setminus K^*) \cup \{\infty\}$, and note that $Y = U_{\alpha^*} \cup K^*$, where K^* is compact. The remaining $U - \alpha$'s cover K^* in X, if we remove ∞ from any set if necessary. (Note this leaves an open cover, since K being closed implies $X \setminus K$ is open.) Take a finite subcover, and add ∞ back to any set where it was removed. This is then a finite subcover of Y. To see that Y is Hausdorff, let $x, y \in Y$ be distinct points. If $x, y \in X$ then since X is Hausdorff we can find disjoint open sets in X, so also open in Y, containing then. Suppose that $y = \infty$. Since X is locally compact, there is a compact neighborhood $K \subset X$ that contains x, and a open set $U \subset K$. Thus, U and $V = (X \setminus K) \cup \{\infty\}$ are two disjoint open sets containing x and y.