Tutorial Sheet 5, Topology 2011

1. Show that the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$, is continuous, and prove that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

2. We know that the projection maps send open sets to open sets. Do they send closed sets to closed sets?

3. Prove that $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

4. Are the following sets connected?

   (a) The rational numbers, considered as a subset of the real numbers.

   (b) The subset of $\mathbb{R}^2$ defined by

   $$X = \{(x, y) : y = 0\} \cup \{(x, y) : x > 0 \text{ and } y = 1/x\}$$

   (c) Any set with the discrete topology.

5. Prove that a space X is connected if and only if there do not exist nonempty disjoint sets A and B (not necessarily open) such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and whose union is X.

6. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the subspace topology, is a connected space.

7. Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < 0 < f(b)$. Use the connectedness of $[a, b]$ to prove the intermediate value theorem: there must be a $c \in (a, b)$ such that $f(c) = 0$.

8. Prove that the continuous image of a connected set is connected.