Tutorial Sheet 5, Topology 2011

1. Show that the diagonal map $\Delta : X \to X \times X$, $\Delta(x) = (x, x)$, is continuous, and prove that X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

   Solution: To see that it is continuous. Let U_1 and U_2 be open in X, and note that $\Delta^{-1}(U_1 \times U_2) = U_1 \cap U_2$, which is open.

   Next, assume that X is Hausdorff. Suppose that (p_1, p_2) is a limit point of $\Delta(X)$ but $p_1 \neq p_2$, so $(p_1, p_2) \notin \Delta(X)$. Let U_1 and U_2 be disjoint open sets containing p_1 and p_2. But then $U_1 \times U_2$ is an open set in $X \times X$ containing (p_1, p_2) and such that $(U_1 \times U_2) \cap \Delta(X) = \emptyset$, which is a contradiction.

   Finally, assume that $\Delta(X)$ is closed and let $x_1 \neq x_2$. Then (x_1, x_2) is not a limit point of $\Delta(X)$, so I can find an open set $U \times V \in X \times X$ that contains this point and is disjoint from $\Delta(X)$. But being disjoint from $\Delta(X)$ implies that $U \cap V = \emptyset$, which shows that X is Hausdorff.

2. We know that the projection maps send open sets to open sets. Do they send closed sets to closed sets?

   Solution: No. Consider $\mathbb{R} \times \mathbb{R}$ and define $C = \cup_{n=1}^{\infty} (1/n, n)$. Note that $p_1(C) = \{1/n\}$, which is not closed because 0 is a limit point not in this set.

   I claim that C is a closed set in the product space. Suppose that (p_1, p_2) is a limit point not in the set. If $p_2 \neq n$ for any n, then let N be the closest natural number to p_2 and let $\epsilon = \min(p_2 - N, N+1-p_2)/2$. Then $U \times B_\epsilon(p_2)$ for any open set U containing p_1 is an open set containing this point that is disjoint from C. A similar argument works if $p_1 \neq 1/n$ and $p_1 \neq 0$. If $p_1 = 0$, let N be the closest natural number to p_2. Take $\delta < 1/(2N)$ and $\epsilon = |p_2 - N|/2$ (or $\epsilon = 1/2$, if this difference is zero). Then $B_\delta(0) \times B_\epsilon(p_2)$ is an open set containing $(0, p_2)$ but disjoint from C.

3. Prove that $X \times Y$ is Hausdorff if and only if both X and Y are Hausdorff.

   Solution: Suppose X and Y are Hausdorff and $(x_1, y_1) \neq (x_2, y_2)$. Then wlog $x_1 \neq x_2$, so there are disjoint open sets U_1 and U_2 in X that contain $x_{1,2}$. But then $U_1 \times Y$ and $U_2 \times Y$ are disjoint open sets in $X \times Y$ that contain (x_1, y_1) and (x_2, y_2).

   Next, suppose the product is Hausdorff, and let $x_1 \neq x_2$. Pick any $y \in Y$ and two disjoint open sets O_1 and O_2 in the product space that contain (x_1, y) and (x_2, y). This implies there exist basis elements $U_1 \times V_1 \subset O_1$ and $U_2 \times V_2 \subset O_2$ that contain (x_1, y) and (x_2, y). But since $y \in V_1$ and $y \in V_2$, U_1 and U_2 must be disjoint. Thus, they are the desired open sets.

4. Are the following sets connected?

   (a) The rational numbers, considered as a subset of the real numbers.

   (b) The subset of $\mathbb{R}^2$ defined by

   $$X = \{(x, y) : y = 0\} \cup \{(x, y) : x > 0 \text{ and } y = 1/x\}$$

   (c) Any set with the discrete topology.

Solution: The first is not connected because the sets $(-\infty, \pi) \cap \mathbb{Q}$ and $(\pi, \infty) \cap \mathbb{Q}$ are both open in $\mathbb{Q}$, in the subspace topology, and thus they form a disconnection of the rationals. (The above argument works if π is replaces by any irrational number.) The second set is also not connected; just apply the result in question 5, with $A = \{(x, y) : y = 0\}$ and $B = \{(x, y) : x > 0 \text{ and } y = 1/x\}$. Finally, the third set is also not connected, since $\{x\}$ and $X \setminus \{x\}$ are disjoint open sets that disconnect the space, for any $x \in X$.

5. Prove that a space X is connected if and only if there do not exist nonempty disjoint sets A and B (not necessarily open) such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$ and whose union is X.

   Solution: Suppose there do not exists such sets A and B, and bwoc let X be disconnected. Then there exists disjoint nonempty open sets U and V that separate X. Suppose $\bar{U} \cap V \neq \emptyset$. Then there is a limit point p of U that is also in V. But then V would be an open set containing p, and so the definition of limit point implies that $V \cap U \neq \emptyset$, which isn't true. Therefore, U and V are sets of the form A and B, described above, which is a contradiction. Hence, X is connected.

   Suppose now that X is connected, and bwoc suppose there exist sets as described above. Consider $\text{int}(A)$ and $\text{int}(B)$, which are clearly disjoint open sets. We will show they are nonempty and their union is the entire space, thus contradicting the fact that X is connected. Suppose there exists an $x \in X$, $x \notin \text{int}(A) \cup \text{int}(B)$. Then wlog $x \in A \setminus \text{int}(A)$. But then for any open set U containing x, there is a point $y \in U$ that is not in A – ie $y \in B$. But then x is a limit point of B, and so $A \cap \bar{B} \neq \emptyset$, which isn't true. Thus, $\text{int}(A) \cup \text{int}(B) = X$. To see that both $\text{int}(A)$ and $\text{int}(B)$ are nonempty, suppose instead that $\text{int}(A) = \emptyset$. Since A is nonempty, there exists an $x \in A \setminus \text{int}(A)$. But again the above argument shows this can't happen, because it would imply that $\bar{A} \cap B \neq \emptyset$.

6. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the subspace topology, is a connected space.

   Solution: If we visualize this as a union of horizontal and vertical lines with rational intersection with the axes, we can see how to get from one point to another using a path, so intuitively the space is path connected. To make this rigorous, use the theorem from lecture about a family of connected sets that cover the space, no two of which are separated. (Use $Z(p/q) = (\{p/q\} \times Y) \cup (X \times \{p/q\})$.)

7. Let $f : [a, b] \to \mathbb{R}$ be continuous with $f(a) < 0 < f(b)$. Use the connectedness of $[a, b]$ to prove the intermediate value theorem: there must be a $c \in (a, b)$ such that $f(c) = 0$.

   Solution: bwoc. Since $(-\infty, 0)$ and $(0, \infty)$ are open and $0 \notin f([a, b])$, $f^{-1}(-\infty, 0)$ and $f^{-1}(0, \infty)$ are open, and $[a, b] \subset f^{-1}(-\infty, 0) \cup f^{-1}(0, \infty)$. But since $[a, b]$ is connected, it cannot be written as the union of two open, disjoint sets. Hence this is a contradiction, which proves the result.

8. Prove that the continuous image of a connected set is connected.

   Solution: BWOC, let $f : X \to f(X)$ be continuous with X connected, and suppose $f(X)$ is not connected. There there is a subset $A \subset f(X)$ that is both open and closed. But then $f^{-1}(A)$ is open, and $f^{-1}(f(X) \setminus A)$ is also open. But this implies $f^{-1}(A)$ is closed. Thus, $f^{-1}(A) = \emptyset$ or X. Hence, $A\emptyset$ or $f(X)$.