Tutorial Sheet 6, Topology 2011

1. Prove that $\mathbb{Q}$, with the subspace topology inherited from $\mathbb{R}$, is totally disconnected, but not discrete.

2. Let $\{A_\alpha\}$ be a collection of connected subspaces of a topological space X. Suppose there is a point $x \in X$ such that $x \in A_\alpha$ for all α. Prove that $\cup_\alpha A_\alpha$ is connected.

3. Let X be the real numbers with the half-open interval topology. What are the components of this space?

4. If X has a finite number of components, show that each component is both open and closed. Find a space for which none of its components are open.

5. Prove that the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is path connected.

6. Prove that a connected open subset X of $\mathbb{R}^n$ is path-connected using the following steps.

   (a) For any $x \in X$ let $U(x)$ be the set of all points in X that can be connected to x with a path. Prove that $U(x)$ is open, by showing for each $y \in U(x)$ there is a $\delta > 0$ such that $B_\delta(y) \subset U(x)$.

   (b) Prove that $U(x)$ is closed by showing its complement is open.

   (c) Conclude X is path connected.

7. (a) If $f : X \to Y$ is continuous and γ is a path in X, prove that $f \circ \gamma$ is a path in Y.

   (b) Conclude that path-connectedness is a topological invariant.

8. Find an example of each of the following:

   (a) A subspace of the real line that is locally connected, but not connected.

   (b) A space that is connected but not locally connected. (Hint: think about the topologist's sine curve.)

   (c) A space that is neither connected nor locally connected.