Tutorial Sheet 6, Topology 2011

1. Prove that $\mathbb{Q}$, with the subspace topology inherited from $\mathbb{R}$, is totally disconnected, but not discrete.

   Solution: It is not discrete because $\{p/q\}$ is not open – if it was $\{p/q\} = U \cap \mathbb{Q}$ for some open set $U \subset \mathbb{R}$. But this isn't possible - the rational numbers are dense, so any open ball contains infinitely many of them.

   To see that it is totally disconnected, let C be a component containing two points, $x_{1,2}$. But there is an irrational number $y \in (x_1, x_2)$ (wlog $x_1 < x_2$), and so $(-\infty, y) \cap \mathbb{Q}$ and $(y, \infty) \cap \mathbb{Q}$ can be used to disconnect C.

2. Let $\{A_\alpha\}$ be a collection of connected subspaces of a topological space X. Suppose there is a point $x \in X$ such that $x \in A_\alpha$ for all α. Prove that $\cup_\alpha A_\alpha$ is connected.

   Solution: Use the theorem from class with the family $\mathcal{F} = \{A_\alpha\}$ and space $Y = \cup_\alpha A_\alpha$. Since $x \in \bar{A}_{\alpha_1} \cap \bar{A}_{\alpha_2}$, no two members of the family are separated, so the union is connected.

3. Let X be the real numbers with the half-open interval topology. What are the components of this space?

   Solution: Let A be any subset containing more than one point. The there exists a $y \in \mathbb{R}$ such that $(-\infty, y) \cap A$ and $[y, \infty) \cap A$ are both nonempty. These sets are also both open, so can be used to disconnect A. Thus, no set containing more than one point can be connected. Hence, the only components are sets containing a single point.

4. If X has a finite number of components, show that each component is both open and closed. Find a space for which none of its components are open.

   Solution: Let $A_1, \ldots, A_n$ be the components of X. By the theorem from lecture, each component is closed. Also, $X = \cup_{i=1}^n A_i$, so $X \setminus A_i = \cup_{j \neq i} A_j$, which is a finite union of closed sets, hence closed. Thus each component is open. And example of space that has no open components is the previous example: the real line with the half-open topology. Alternatively, the rational numbers.

5. Prove that the unit ball $B^n = \{x \in \mathbb{R}^n : |x| \leq 1\}$ is path connected.

   Solution: Use a straight-line path: if $x, y \in B^n$, then $\gamma(t) = tx + (1-t)y$ is a path in B^n, since $|\gamma(t)| \leq |t||x| + |1-t||y| \leq t + 1 - t = 1$.

6. Prove that a connected open subset X of $\mathbb{R}^n$ is path-connected using the following steps.

   (a) For any $x \in X$ let $U(x)$ be the set of all points in X that can be connected to x with a path. Prove that $U(x)$ is open, by showing for each $y \in U(x)$ there is a $\delta > 0$ such that $B_\delta(y) \subset U(x)$.

   Solution: Since X is open, there is a $B_\delta(y) \in X$. This $B_\delta(y)$ is path connected by the argument from the previous problem. Thus, for any $z \in B_\delta(y)$, combine the path from z to y with the path from y to x to get a path from z to x.

   (b) Prove that $U(x)$ is closed by showing its complement is open.

   Solution: We can write $X \setminus U(x) = \cup_{y \notin U(x)} U(y)$. This is a union of open sets, so it is open.

(c) Conclude X is path connected.

Solution: Thus, $U(x)$ is a set that is both open and closed in the connected space X, so $U(x) = X$ and X is path connected. Remark: this can actually be generalized to show that any open connected subset of a path connected space is path connected.

7. (a) If $f : X \to Y$ is continuous and γ is a path in X, prove that $f \circ \gamma$ is a path in Y.

   Solution: $f \circ \gamma$ is continuous because it is the composition of continuous functions, and $f \circ \gamma(t) \in Y$ for all t because the codomain of f is Y. Hence, it is a path in Y.

   (b) Conclude that path-connectedness is a topological invariant.

   Solution: Let $f : X \to Y$ be a homeomorphism where X is path connected, and let $y_1, y_2 \in Y$. There exist points $x_1, x_2 \in X$ such that $f(x_i) = y_i$, $i = 1, 2$, and a path γ in X between them. By part a), $f \circ \gamma$ is then a path in Y between y_1 and y_2, so Y is path connected.

8. Find an example of each of the following:

   (a) A subspace of the real line that is locally connected, but not connected.

   Solution: $[0, 1) \cup (2, 3]$, for example.

   (b) A space that is connected but not locally connected. (Hint: think about the topologist's sine curve.)

   Solution: The topologist's sine cuve is connected, as we proved in class, but it is not locally connected: take a point $(0, y) \in \bar{S}$, $y \neq 0$. Then any small open ball at this point will contain infinitely many line segments from S. This cannot be connected, as each one of these is a component, within the neighborhood.

   (c) A space that is neither connected nor locally connected.

   Solution: The rational numbers.