Tutorial Sheet 7, Topology 2011

1. Explain how to construct the torus as an identification space.

2. Let $f : X \to Y$ be an onto continuous map. Prove that, if f maps open sets to open sets, or if f maps closed sets to closed sets, then f is an identification map.

3. Consider the identification space $\mathbb{R}^2/\sim$ under the following equivalence relations. What familiar spaces are they homeomorphic to?

   (a) $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1^2 = x_2 + y_2^2$

   (b) $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$

4. Let X be a topological space and $I = [0, 1]$. Consider the subset of $X \times I$ given by $A = X \times \{1\}$. Construct $(X \times I)/A$. This is referred to as the cone on X, and is denoted CX.

5. Consider S^n, and define $x \sim y$ if $x = -y$ (in which case they are referred to as *antipodal*). The space $S^n/\sim$ is referred to as *real projective space* and denoted by P^n. Consider another identification space $(\mathbb{R}^n \setminus \{0\})/\sim$, where in this case $x \sim y$ if x and y lie on the same line through the origin. Prove that P^n and $(\mathbb{R}^n \setminus \{0\})/\sim$ are homeomorphic. In other words, these are two equivalent ways to construct real projective space.

6. Convince yourself by drawing pictures that the Klein bottle is homeomorphic to two Möbius strips glued together at their boundaries.

7. Let X be a compact Hausdorff space and let $A \subset X$ be closed. Show that X/A is homeomorphic to the one-point compactification of $X \setminus A$.

8. Consider $X = \{[0, 1] \times \{n\} : n = 1, 2, 3, \dots\}$ and $Z = \{(x, x/n) : x \in [0, 1], n = 1, 2, 3, \dots\}$. Define $g : X \to Z$ by $g(x, n) = (x, x/n)$. Consider the resulting identification space X^*, whose elements are the sets $\{g^{-1}(z)\}$. Consider the induced map $f : X^* \to Z$. Prove that f is not a homeomorphism. [Hint: First show that, if $g : X \to Z$ is an identification map for any topological spaces X and Z, then C is closed in Z if and only if $g^{-1}(C)$ is closed in X. Then consider the set $A = \{(1/n, n)\}$.]