Tutorial Sheet 7, Topology 2011

1. Explain how to construct the torus as an identification space.

   Solution: Let $X = [0,1] \times [0,1]$ (or any rectangle). Define $(x_1, y_1) \sim (x_2, y_2)$ if

   (a) $(x_1, y_1) = (x_2, y_2)$, with $0 < x_1, y_1 < 1$

   (b) $(x_1, y_1) = (0, y)$ and $(x_2, y_2) = (1, y)$, with $0 < y < 1$.

   (c) $(x_1, y_1) = (x, 0)$ and $(x_2, y_2) = (x, 1)$, with $0 < x < 1$.

   (d) $(x_1, y_1), (x_2, y_2) \in \{(0,0), (1,0), (0,1), (1,1)\}$

   The resulting identification space is the torus.

2. Let $f : X \to Y$ be an onto continuous map. Prove that, if f maps open sets to open sets, or if f maps closed sets to closed sets, then f is an identification map.

   Solution: We must show that U is open in Y if and only if $f^{-1}(U)$ is open in X. If U is open in Y, then by continuity $f^{-1}(U)$ is open in X. If $f^{-1}(U)$ is open in X and f maps open sets to open sets, then since f is onto, $f(f^{-1}(U)) = U$, so U is open in Y. The argument for closed sets is similar.

3. Consider the identification space $\mathbb{R}^2/\sim$ under the following equivalence relations. What familiar spaces are they homeomorphic to?

   (a) $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + y_1^2 = x_2 + y_2^2$

   Solution: This is just the real line. To see this, consider $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x + y^2$. Check that g is an identification map such that $(x_1, y_1), (x_2, y_2) \in g^{-1}(c)$ if and only if $x_1 + y_1^2 = x_2 + y_2^2$.

   (b) $(x_1, y_1) \sim (x_2, y_2)$ if $x_1^2 + y_1^2 = x_2^2 + y_2^2$

   Solution: This is just $[0, \infty)$. To see this, consider $g : \mathbb{R}^2 \to \mathbb{R}$, $g(x, y) = x^2 + y^2$. Check that g is an identification map such that $(x_1, y_1), (x_2, y_2) \in g^{-1}(c)$ if and only if $x_1^2 + y_1^2 = x_2^2 + y_2^2$.

4. Let X be a topological space and $I = [0, 1]$. Consider the subset of $X \times I$ given by $A = X \times \{1\}$. Construct $(X \times I)/A$. This is referred to as the cone on X, and is denoted CX.

   Solution: This is just the space where $(x, a) \sim (y, b)$ if $(x, a) = (y, b)$ or if $a = b = 1$. If you visualize this with X as the horizontal axis and I as the vertical one, then X gets pinched to a point at the top of I, so it looks (schematically) like a cone.

5. Consider S^n, and define $x \sim y$ if $x = -y$ (in which case they are referred to as *antipodal*). The space $S^n/\sim$ is referred to as *real projective space* and denoted by P^n. Consider another identification space $(\mathbb{R}^n \setminus \{0\})/\sim$, where in this case $x \sim y$ if x and y lie on the same line through the origin. Prove that P^n and $(\mathbb{R}^n \setminus \{0\})/\sim$ are homeomorphic. In other words, these are two equivalent ways to construct real projective space.

   Solution: Let $p : S^n \to P^n$ be the identification map for real projective space. Define $g : \mathbb{R}^n \setminus \{0\} \to P^n$ to be $g(x) = p(x/|x|)$. One can check that $(\mathbb{R}^n \setminus \{0\})/\sim = \{g^{-1}(y)\}$ and that g is an identification map (because p is). Hence, these two constructions of real projective space are equivalent.

6. Convince yourself by drawing pictures that the Klein bottle is homeomorphic to two Möbius strips glued together at their boundaries.

   Solution: See me if you need help drawing such pictures.

7. Let X be a compact Hausdorff space and let $A \subset X$ be closed. Show that X/A is homeomorphic to the one-point compactification of $X \setminus A$.

   Solution: Recall that $Y = X/A$ is the set of equivalence classes consisting of a single point not in A, or all of A. Recall that the one-point compactification of $Z := X \setminus A$ is $Z \cup \{\infty\}$, where open sets are defined to be the open sets of Z together with sets of the form $(Z \setminus K) \cup \{\infty\}$, where $K \subset Z$ is compact.

   Define
$$h : Y \to Z \cup \{\infty\}, \qquad h(y) = \begin{cases} y & \text{if} \quad y \notin A \\ \infty & \text{if} \quad y \in A \end{cases}$$

   To see that this is a homeomorphism, note that it is one-to-one and onto by construction. Let $O \subset Y$ be open, meaning that $f^{-1}(O)$ is open in X, where f is the identification map. If $f^{-1}(O) \cap A = \emptyset$, then $f^{-1}(O) = O$ and so $h(O) = O$ is open in Z. If $f^{-1}(O) \cap A \neq \emptyset$, then we can write $O = [O \cap (X \setminus A)] \cup [O \cap A]$, and so $h(O) = [O \cap (X \setminus A)] \cup \{\infty\}$. Since $X \setminus O$ is closed, and hence compact, we find $O \cap (X \setminus A)$ is $(X \setminus A) \setminus (X \setminus O) = Z \setminus K$. Hence, h^{-1} is continuous. The proof that h is continuous is similar.

8. Consider $X = \{[0,1] \times \{n\} : n = 1, 2, 3, \ldots\}$ and $Z = \{(x, x/n) : x \in [0,1], n = 1, 2, 3, \ldots\}$. Define $g : X \to Z$ by $g(x, n) = (x, x/n)$. Consider the resulting identification space X^*, whose elements are the sets $\{g^{-1}(z)\}$. Consider the induced map $f : X^* \to Z$. Prove that f is not a homeomorphism. [Hint: First show that, if $g : X \to Z$ is an identification map for any topological spaces X and Z, then C is closed in Z if and only if $g^{-1}(C)$ is closed in X. Then consider the set $A = \{(1/n, n)\}$.]

   Solution: f is a homeomorphism if and only if g is an identification map. We'll show g isn't. If g were an identification map and $g^{-1}(C)$ were closed, then $X \setminus g^{-1}(C)$ would be open. Since $g(X \setminus g^{-1}(C)) = Z \setminus C$ is then open, C must be closed. Let $A = \{(1/n, n)\}$, which is closed in X. But $g(A) = \{(1/n, 1/n^2)\}$, which isn't closed, because it has a limit point at the origin. Hence, g can't be an identification map.