Tutorial Sheet 8, Topology 2011

1. Prove that any two continuous functions $f, g : X \to A$, where A is a convex subset of $\mathbb{R}^n$ and X is an arbitrary topological space, are homotopic.

2. (a) Let $f, g : X \to S^n$ be continuous functions such that $f(x)$ and $g(x)$ are never antipodal (ie $f(x) \neq -g(x)$ for any $x \in X$). Prove that

$$F(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

   is a homotopy between f and g.

   (b) Suppose that $f : S^1 \to S^1$ is continuous and not homotopic to the identity. Prove that $f(x) = -x$ for some $x \in S^1$.

3. Recall that a space is said to be simply connected if its fundamental group is trivial.

   (a) Prove that, in a simply connected space, any two paths that both begin at some point x_0 and both end at some point x_1 are path-homotopic.

   (b) Prove that any convex subset of $\mathbb{R}^n$ is simply connected.

4. Let $f : X \to S^n$ be continuous but not onto, where X is an arbitrary topological space. Show that f is "null homotopic," meaning that it is homotopic to a map that sends all of X to a single point. (In other words, f is homotopic to a constant function.)