Solutions to Tutorial Sheet 8, Topology 2011

1. Prove that any two continuous functions $f, g : X \to A$, where A is a convex subset of $\mathbb{R}^n$ and X is an arbitrary topological space, are homotopic.

   Solution: We can use a straight-line homotopy $F(x,t) = (1-t)f(x) + tg(x)$, which maps $X \times I \to A$ because A is convex.

2. (a) Let $f, g : X \to S^n$ be continuous functions such that $f(x)$ and $g(x)$ are never antipodal (ie $f(x) \neq -g(x)$ for any $x \in X$). Prove that

   $$F(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

   is a homotopy between f and g.

   Solution: This function is continuous because the denominator is never zero. This is guaranteed by the assumption that the functions are never antipodal. If $(1-t)f(x) + tg(x) = 0$, then $f(x) = -(tg(x))/(1-t)$ But since $|f(x)| = |g(x)| = 1$, this implies $t = 1/2$ and $f(x) = -g(x)$, which can't happen.

   (b) Suppose that $f : S^1 \to S^1$ is continuous and not homotopic to the identity. Prove that $f(x) = -x$ for some $x \in S^1$.

   Solution: BWOC, if not, then by part *a)* f would be homotopic to the function $g(x) = x$, which is the identity – hence, a contradiction.

3. Recall that a space is said to be simply connected if its fundamental group is trivial.

   (a) Prove that, in a simply connected space, any two paths that both begin at some point x_0 and both end at some point x_1 are path-homotopic.

   Solution: Note that $f \cdot g^{-1}$ is a loop at x_0, and since the space is simply connected $f \cdot g^{-1} \simeq_p e$. But this implies $g \simeq_p e \cdot g \simeq_p (f \cdot g^{-1}) \cdot g \simeq_p f$. (Note: to see that $f \simeq_p g$ implies $f \cdot h \simeq_p g \cdot g$, if F is the path-homotopy from f to g one can just use the path homotopy

   $$G(s,t) = \begin{cases} F(2s,t) & \text{if} \quad 0 \leq s \leq 1/2 \\ h(2s-1) & \text{if} \quad 1/2 \leq s \leq 1 \end{cases},$$

   which goes from $f \cdot h$ to $g \cdot h$.)

   (b) Prove that any convex subset of $\mathbb{R}^n$ is simply connected.

   Solution: By problem 1, any two paths on the convex subset must be homotopic, and in fact the strait-line homotopy shows they are path-homotopic. Thus, all paths are path-homotopic to the identity, and so the fundamental group is trivial.

4. Let $f : X \to S^n$ be continuous but not onto, where X is an arbitrary topological space. Show that f is "null homotopic," meaning that it is homotopic to a map that sends all of X to a single point. (In other words, f is homotopic to a constant function.)

   Solution: Chose $p \in S^n$ so that $-p \neq f(x)$ for all $x \in X$, which is possible since f is not onto. Then as in the solution to 2a, since $g(x) = p$ is not antipodal to f, they are homotopic.