Tutorial Sheet 9, Topology 2011

1. Prove the following theorem, which was stated in class: If $h : X \to Y$ and $k : Y \to Z$ are continuous functions with $h(x_0) = y_0$ and $k(y_0) = z_0$ for some $x_0 \in X$, then $(k \circ h)_* = k_* \circ h_*$. Also, if $i : X \to X$ is the identity, then i_* is the identity.

   Solution: By definition, since $(k \circ h)(x_0) = z_0$, so $(k \circ h)_* : \pi_1(X, x_0) \to \pi_1(Z, z_0)$ is defined by $(k \circ h)_*(\langle f \rangle) = \langle (k \circ h) \circ f \rangle$. Also, $k_* \circ h_*(\langle f \rangle) = k_*(\langle h \circ f \rangle) = \langle k \circ (h \circ f) \rangle$. Since function composition is associative, these are equal. Also, $i_*(\langle f \rangle) = \langle i \circ f \rangle = \langle f \rangle$.

2. Describe the homomorphism $f_* : \pi_1(S^1, 1) \to \pi_1(S^1, f(1))$ for each of the following continuous functions:

   (a) $f(e^{i\theta}) = e^{i(\theta+\pi)}$, $\theta \in [0, 2\pi]$

   (b) $f(e^{in\theta}) = e^{i(\theta+\pi)}$, $\theta \in [0, 2\pi]$, $n \in \mathbb{Z}$

   (c)
   $$f(e^{in\theta}) = \begin{cases} e^{i\theta} & \text{if} \quad \theta \in [0, \pi] \\ e^{i(2\pi-\theta)} & \text{if} \quad \theta \in [\pi, 2\pi] \end{cases}$$

   Solution:

   (a) This function just rotates the entire circle counterclockwise by π degrees. This doesn't affect the number of times a loop goes around the circle, or the direction in which is goes around. Thus, f_* is the identity.

   (b) For each time a loop α goes around the circle, $f \circ \alpha$ goes around n times (in the same direction). Thus, if α goes around m times, $f_*(\langle \alpha \rangle) = f_*(\langle p \circ \gamma_m \rangle) = \langle p \circ \gamma_{mn} \rangle$.

   (c) This function is the identity on the top half of the circle, and maps the bottom half of the circle onto the top half of the circle. Thus, if a loop approaches and attempts to pass through -1, it is forced by f to turn around a go back up. A similar thing happens if the loop tries to pass through 1. Thus, $f \circ \alpha$ will never go around the circle, and so it will be homotopic to the constant loop. Thus, $f_*(\langle \alpha \rangle) = \langle e \rangle$ for all α. In other words, f_* sends everything to the identity.

3. Compute the fundamental groups of the following spaces:

   (a) A cylinder: $\pi_1(S^1 \times J, x_0)$ where J is any interval in $\mathbb{R}$ and $x_0 = (y_0, z_0) \in S^1 \times J$ is any point.
   Solution: By the theorem from class on product spaces, this fundamental group is isomorphic to $\pi_1(S^1, y_0) \times \pi_1(J, z_0) = \mathbb{Z} \times \{e\}$, which is isomorphic to $\mathbb{Z}$. These are all independent of the base point, because the spaces are path connected.

   (b) The punctured plane: $\pi_1(\mathbb{R}^2 \setminus \{(0, 0)\}, x_0)$, for any point $x_0 \in \mathbb{R}^2 \setminus \{0\}$.
   Solution: Using polar coordinates (r, θ), one can see that the punctured plane $\mathbb{R}^2 \setminus \{(0, 0)\}$ is homeomorphic to $(0, \infty) \times S^1$, and so by part a), the fundamental group is isomorphic to $\mathbb{Z}$.

   (c) A punctured disk: $\pi_1(B^2 \setminus \{0\}, x_0)$, for any point $x_0 \in B^2 \setminus \{0\}$.
   Solution: As in b), the space is homeomorphic to $(0, 1) \times S^1$, so its fundamental group is the integers.