the description of the projective plane (real projective space of dimension 2) given in Chapter 1. The idea was to attach a disc to a Möbius strip by glueing together their boundary circles. We can now make this precise. Let M denote the Möbius strip and D the disc. Choose a homeomorphism h from the boundary circle of D to that of M and form the identification space $M \cup_h D$. The result is P^2 and is (as we shall see in Chapter 7) independent of the choice of h. We leave the reader to reconcile this description with those listed in 'Projective spaces' above.

One final comment: if Y is an identification space formed from X, then Y is the image of X under a continuous function and therefore inherits properties such as compactness, connectedness, and path-connectedness from X. However, X may be Hausdorff and yet Y not satisfy the Hausdorff axiom. As an example, take X to be the real line with its usual topology, and partition X so that real numbers r and s lie in the same element of the partition if and only if $r - s$ is rational. We invite the reader to check that the corresponding identification space is an *indiscrete* space.

Problems

1. Check that the three descriptions (a), (b), (c) of P^n listed in 'Projective spaces' above do all lead to the same space.

2. Which space do we obtain if we take a Möbius strip and identify its boundary circle to a point?

3. Let $f: X \to Y$ be an identification map, let A be a subspace of X, and give $f(A)$ the induced topology from Y. Show that the restriction $f| A: A \to f(A)$ need not be an identification map.

4. With the terminology of Problem 3, show that if A is open in X and if f takes open sets to open sets, or if A is closed in X and f takes closed sets to closed sets, then $f| A: A \to f(A)$ is an identification map.

5. Let X denote the union of the circles $[x - (1/n)]^2 + y^2 = (1/n)^2, n = 1,2,3,\ldots$, with the subspace topology from the plane, and let Y denote the identification space obtained from the real line by identifying all the integers to a single point. Show that X and Y are not homeomorphic. (X is called the *Hawaiian earring*.)

6. Give an example of an identification map which is neither open nor closed.

7. Describe each of the following spaces: (a) the cylinder with each of its boundary circles identified to a point; (b) the torus with the subset consisting of a meridianal and a longitudinal circle identified to a point; (c) S^2 with the equator identified to a point; (d) $\mathbb{E}^2$ with each of the circles centre the origin and of integer radius identified to a point.

8. Let X be a compact Hausdorff space. Show that the cone on X is homeomorphic to the one-point compactification of $X \times [0,1)$. If A is closed in X,

show that X/A is homeomorphic to the one-point compactification of $X - A$.

9. Let $f: X \to X'$ be a continuous function and suppose we have partitions $\mathcal{P}, \mathcal{P}'$ of X and X' respectively, such that if two points of X lie in the same member of $\mathcal{P}$, their images under f lie in the same member of $\mathcal{P}'$. If Y, Y' are the identification spaces given by these partitions, show that f induces a map $\hat{f}: Y \to Y'$, and that if f is an identification map then so is $\hat{f}$.

10. Let S^2 be the unit sphere in $\mathbb{E}^3$ and define $f: S^2 \to \mathbb{E}^4$ by $f(x,y,z) = (x^2 - y^2, xy, xz, yz)$. Show that f induces an embedding of the projective plane in $\mathbb{E}^4$ (embeddings were defined in Problem 14 of Chapter 3).

11. Show that the function $f: [0,2\pi] \times [0,\pi] \to \mathbb{E}^5$ defined by $f(x,y) = (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^5$.

12. With the notation of Problem 11, show that if $(2 + \cos x)\cos 2y = (2 + \cos x')\cos 2y'$ and $(2 + \cos x)\sin 2y = (2 + \cos x')\sin 2y'$, then $\cos x = \cos x'$, $\cos 2y = \cos 2y'$, and $\sin 2y = \sin 2y'$. Deduce that the function $g: [0,2\pi] \times [0,\pi] \to \mathbb{E}^4$ given by $g(x,y) = ((2 + \cos x)\cos 2y, (2 + \cos x)\sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^4$.

4.3 Topological groups

We leave the notion of an identification space briefly in order to consider spaces which have, in addition to their topology, the structure of a group. A good example is the circle, thought of as the set of complex numbers of unit modulus. Its topology is that induced from the plane and the group structure is simply multiplication of complex numbers. Note that the two functions

$$S^1 \times S^1 \to S^1$$
$$(e^{i\theta}, e^{i\phi}) \mapsto e^{i(\theta + \phi)} \qquad \textit{(group multiplication)}$$

$$S^1 \to S^1$$
$$e^{i\theta} \mapsto e^{-i\theta} \qquad \textit{(inversion in the group)}$$

are continuous, so the topology and the algebraic structure fit together nicely.

(4.9) Definition. *A topological group* G *is both a Hausdorff topological space and a group, the two structures being compatible in the sense that the group multiplication* $\mathrm{m}: \mathrm{G} \times \mathrm{G} \to \mathrm{G}$, *and the function* $\mathrm{i}: \mathrm{G} \to \mathrm{G}$ *which sends each group element to its inverse, are continuous.*

Most of this section will be taken up by examples, including examples of matrix groups. In Section 4.4 we return to identification spaces. We shall define there the action of a topological group on a space, show how an action leads to an identification space, and consider a variety of identification spaces which arise in this way.

Examples of topological groups

1. The real line, the group structure being addition of real numbers.

2. The circle, as described above.

3. Any abstract group with the discrete topology.

4. The torus considered as the product of two circles. We take the product topology and the product group structure. (The product of two topological groups is a topological group; see Problem 13.)

5. The three-sphere considered as the unit sphere in the space of quaternions $\mathbb{H}$. ($\mathbb{H}$ is topologically $\mathbb{E}^4$ and has the algebraic structure of the quaternions.)

6. Euclidean n-space. We choose the notation $\mathbb{R}^n$ to emphasize that we have a topological group (usual addition as group structure) and not simply the topological space $\mathbb{E}^n$.

7. The group of invertible $n \times n$ matrices with real entries. The group structure is matrix multiplication. For the topology we identify each $n \times n$ matrix $A = (a_{ij})$ with the corresponding point

$$(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, a_{31}, \ldots, a_{nn})$$

of $\mathbb{E}^{n^2}$ and take the subspace topology. This topological group is called the *general linear group*, and we denote it by GL(n).† A detailed verification that GL(n) is a topological group will be given in theorem (4.12).

8. The *orthogonal group* O(n) consisting of $n \times n$ orthogonal matrices with real entries. O(n) has both its topology and its group structure induced from GL(n). It is a subgroup (as a topological group) of GL(n). The subgroup of O(n) consisting of those matrices which have determinant $+1$ is called the *special orthogonal group* and written SO(n).

The terms 'isomorphism' and 'subgroup' for topological groups require a few words of explanation. In each case we need to take into consideration both the topological and the algebraic structures. So an isomorphism between two topological groups is a homeomorphism which is also a group isomorphism. In the same spirit, a subset of a topological group is called a subgroup if it is algebraically a subgroup and in addition has the subspace topology. Therefore the integers $\mathbb{Z}$ with the discrete topology form a subgroup of the real line $\mathbb{R}$. If we form the factor group $\mathbb{R}/\mathbb{Z}$ and give it the identification topology (the corresponding partition of $\mathbb{R}$ is that given by the cosets of $\mathbb{Z}$) then we have a topological group isomorphic to the circle. For the map $f:\mathbb{R} \to S^1$ defined by $f(x) = e^{2\pi ix}$ takes open sets to open sets and is an identification map, by theorem (4.3). Two points of $\mathbb{R}$ are identified by f if and only if they differ by an integer, and therefore f induces a homeomorphism of $\mathbb{R}/\mathbb{Z}$† with S^1, by theorem (4.2). It is elementary to check that this homeomorphism is a group isomorphism. For a second example involving the ideas of subgroup and isomorphism, we turn to our matrix groups. Associating each $(n-1) \times (n-1)$ orthogonal matrix A with the $n \times n$ orthogonal matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$$

shows that O($n-1$) is isomorphic to a subgroup of O(n)

Let G be a topological group and x an element of G. The function $L_x: G \to G$ defined by $L_x(g) = xg$ is called *left translation* by the element x. It is clearly one–one and onto, and it is continuous because it is the composition

$$G \to G \times G \xrightarrow{m} G$$
$$g \mapsto (x,g) \mapsto xg.$$

The inverse of L_x is $L_{x^{-1}}$ and therefore L_x is a homeomorphism. Similarly the right translation $R_x: G \to G$ given by $R_x(g) = gx$ is also a homeomorphism.

These translations show that a topological group has a certain 'homogeneity' as a topological space. For if x and y are any two points of a topological group G there is a homeomorphism of G that maps x to y, namely the translation $L_{yx^{-1}}$. Therefore G exhibits the same topological structure locally near each point.

(4.10) Theorem. *Let* G *be a topological group and let* K *denote the connected component of* G *which contains the identity element. Then* K *is a closed normal subgroup of* G.

Remark. If $G = $ O(n) then $K = $ SO(n). We shall prove this later.

Proof. Components are always closed. For any $x \in K$ the set $Kx^{-1} = R_{x^{-1}}(K)$ is connected (since $R_{x^{-1}}$ is a homeomorphism) and contains $e = xx^{-1}$. Since K is the maximal connected subset of G containing e, we must have $Kx^{-1} \subseteq K$. Therefore $KK^{-1} = K$, and K is a subgroup of G. Normality follows in a similar manner. For any $g \in G$ the set $gKg^{-1} = R_{g^{-1}}L_g(K)$ is connected and contains e. Therefore $gKg^{-1} \subseteq K$.

(4.11) Theorem. *In a connected topological group any neighbourhood of the identity element is a set of generators for the whole group.*

† Or GL(n,$\mathbb{R}$) to emphasize that the matrices have real entries. GL(n,$\mathbb{C}$) then denotes the corresponding group of invertible matrices with complex entries.

† We have an unfortunate clash of notation. $\mathbb{R}/\mathbb{Z}$ is used for the identification space whose points are the cosets of $\mathbb{Z}$ in $\mathbb{R}$, and for $\mathbb{R}$ with the single subspace $\mathbb{Z}$ pinched to a point. The first of these is the circle, the second is an infinite bouquet of circles (i.e., an infinite collection of circles all joined together at one point). It should always be clear from the context which of the possibilities we are considering.

Proof. Let G be a connected topological group and let V be a neighbourhood of e in G. Let $H = \langle V \rangle$ be the subgroup of G generated by the elements of V. If $h \in H$ then the whole neighbourhood $hV = L_h(V)$ of h lies in H, so H is open. We claim that the complement of H is also open. For if $g \in G - H$, consider the set gV. If $gV \cap H$ is nonempty, say $x \in gV \cap H$, then $x = gv$ for some $v \in V$. This gives $g = xv^{-1}$, which implies the contradiction $g \in H$ since both x and v^{-1} lie in H. Therefore the neighbourhood $L_g(V) = gV$ of g lies in $G - H$, and we see that $G - H$ is an open set. Now G is connected and so cannot be partitioned into two disjoint nonempty open sets. Since H is nonempty we must have $G - H = \varnothing$, i.e., $G = H$.

(4.12) Theorem. *The matrix group* GL(n) *is a topological group.*

Proof. Let $\mathbb{M}$ denote the set of all $n \times n$ matrices which have real entries, and let $A = (a_{ij})$ represent a typical element of $\mathbb{M}$. We can identify $\mathbb{M}$ with euclidean space of dimension n^2 by associating $A = (a_{ij})$ with the point $(a_{11}, a_{12}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, a_{31}, \ldots, a_{nn})$. The identification gives us a topology on $\mathbb{M}$ and we claim that, with respect to this topology, matrix multiplication $m : \mathbb{M} \times \mathbb{M} \to \mathbb{M}$ is continuous. To see this, we need only examine the well-known formula for the entries of a product matrix: if $A = (a_{ij})$ and $B = (b_{ij})$ then the ijth entry in the product $m(A,B)$ is $\sum_{k=1}^{n} a_{ik}b_{kj}$. Now $\mathbb{M}$ has the topology of the product space $\mathbb{E}^1 \times \mathbb{E}^1 \times \ldots \times \mathbb{E}^1$ (n^2 copies), and for each i, j satisfying $1 \leqslant i,j \leqslant n$ we have a projection $\pi_{ij} : \mathbb{M} \to \mathbb{E}^1$ which sends a given matrix A to its ijth entry. By theorem (3.13), m is continuous if and only if all of the composite functions

$$\mathbb{M} \times \mathbb{M} \xrightarrow{m} \mathbb{M} \xrightarrow{\pi_{ij}} \mathbb{E}^1$$

are continuous. But $\pi_{ij}m(A,B) = \sum_{k=1}^{n} a_{ik}b_{kj}$, a polynomial in the entries of A and B. Therefore $\pi_{ij}m$ is continuous.

The elements of GL(n) are the *invertible* matrices in $\mathbb{M}$. If we give GL(n) the subspace topology from $\mathbb{M}$ then, by the above, matrix multiplication GL(n) $\times$ GL(n) $\to$ GL(n) is continuous. It remains to prove that the inverse function i:GL(n) $\to$ GL(n) is also continuous. We use the same technique: i:GL(n) $\to$ GL(n) $\subseteq \mathbb{E}^1 \times \ldots \times \mathbb{E}^1$ is continuous if and only if all of the composite functions

$$\mathrm{GL}(n) \xrightarrow{i} \mathrm{GL}(n) \xrightarrow{\pi_{jk}} \mathbb{E}^1 \qquad 1 \leqslant j,k \leqslant n$$

are continuous. Now the composition of π_{jk} with i sends a matrix A to the jkth element of A^{-1}, i.e., to (1/det A) (kjth cofactor of A). It should be clear that the determinant of A and the cofactors of A are polynomials in the entries of A. Since det A does not vanish on GL(n), our composition $\pi_{jk}i$ is continuous. This completes the proof that GL(n) is a topological group.

We note in passing that GL(n) is the inverse image of the nonzero real numbers

under the determinant function det : $\mathbb{M} \to \mathbb{R}$. So GL($n$) *is not compact* (it is an open subset of $\mathbb{M}$), and is *not connected* (the matrices with positive and negative determinants partition GL(n) into two disjoint nonempty open sets). How many components has GL(n)?

(4.13) Theorem. O(n) *and* SO(n) *are compact.*

Proof. O(n) consists of those matrices in GL(n) which have their transpose as inverse. It is algebraically a subgroup of GL(n) and we give it the subspace topology. In order to show O(n) compact we show that it corresponds to a closed bounded subset of $\mathbb{E}^{n^2}$ under our identification of $\mathbb{M}$ with $\mathbb{E}^{n^2}$.

Let $A \in$ O(n). Since $AA^t = I$ we have $\sum_{j=1}^{n} a_{ij}a_{kj} = \delta_{ik}$ for $1 \leqslant i,k \leqslant n$. For each choice of i,k we define a map $f_{ik} : \mathbb{M} \to \mathbb{E}^1$ by $f_{ik}(A) = \sum_{j=1}^{n} a_{ij}a_{kj}$. Then O($n$) is the intersection of all sets of the form

$$f_{ik}^{-1}(0) \qquad 1 \leqslant i,k \leqslant n, \qquad i \neq k$$
$$f_{ii}^{-1}(1) \qquad 1 \leqslant i \leqslant n$$

Therefore O(n) is closed in $\mathbb{M}$ since it is the intersection of a finite number of closed sets.

For the boundedness of O(n) we have only to look at the conditions $\sum_{j=1}^{n} a_{ij}a_{ij} = 1$. These imply that the entries of any orthogonal matrix A satisfy $|a_{ij}| \leqslant 1$. This completes the proof that O(n) is compact.

Finally, SO(n) is compact because it is closed in O(n).

We note that SO(2) $\cong S^1$, and SO(3) $\cong P^3$, where $\cong$ means isomorphism of topological groups. Sending the rotation matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

to the point $e^{i\theta}$ of S^1 gives the first of these. For the second, we think of S^3 as the quaternions of norm 1, and note that conjugation in $\mathbb{H}$ by a nonzero quaternion always induces a rotation of the three-dimensional subspace of pure quaternions. This defines a function $\mathbb{H} - \{0\} \to$ SO(3) which is in fact (check these statements!) a homomorphism, onto, and continuous. Its kernel is $\mathbb{R} - \{0\}$. Restricting this function to S^3 gives a continuous epimorphism from S^3 to SO(3) with kernel $\{1, -1\}$. Now the set of cosets $S^3/\{1, -1\}$, with the identification topology, is of course P^3, and therefore we have a continuous group isomorphism $P^3 \to$ SO(3). Since P^3 is compact and SO(3) is Hausdorff, this map is a homeomorphism.

Problems

13. Show that the product of two topological groups is a topological group.

14. Let G be a topological group. If H is a subgroup of G, show that its closure $\bar{H}$ is also a subgroup, and that if H is normal then so is $\bar{H}$.

15. Let G be a compact Hausdorff space which has the structure of a group. Show that G is a topological group if the multiplication function $m: G \times G \to G$ is continuous.

16. Prove that $O(n)$ is homeomorphic to $SO(n) \times \mathbb{Z}_2$. Are these two isomorphic as topological groups?

17. Let A,B be compact subsets of a topological group. Show that the product set $AB = \{ab \mid a \in A, b \in B\}$ is compact.

18. If U is a neighbourhood of e in a topological group, show there is a neighbourhood V of e for which $VV^{-1} \subseteq U$.

19. Let H be a discrete subgroup of a topological group G (i.e., H is a subgroup, and is a discrete space when given the subspace topology). Find a neighbourhood N of e in G such that the translates $hN = L_h(N)$, $h \in H$, are all disjoint.

20. If C is a compact subset of a topological group G, and if H is a discrete subgroup of G, show that $H \cap C$ is finite.

21. Prove that every nontrivial discrete subgroup of $\mathbb{R}$ is infinite cyclic.

22. Prove that every nontrivial discrete subgroup of the circle is finite and cyclic.

23. Let $A,B \in O(2)$ and suppose $\det A = +1$, $\det B = -1$. Show that $B^2 = I$ and $BAB^{-1} = A^{-1}$. Deduce that every discrete subgroup of $O(2)$ is either cyclic or dihedral.

24. If T is an automorphism of the topological group $\mathbb{R}$ (i.e., T is a homeomorphism which is also a group isomorphism) show that $T(r) = rT(1)$ for any rational number r. Deduce that $T(x) = xT(1)$ for any real number x, and hence that the automorphism group of $\mathbb{R}$ is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$.

25. Show that the automorphism group of the circle is isomorphic to $\mathbb{Z}_2$.

4.4 Orbit spaces

The infinite cyclic group $\mathbb{Z}$ can be thought of as a group of homeomorphisms of the real line in a very natural way. Each integer $n \in \mathbb{Z}$ determines a translation $x \mapsto x + n$ of the line.

If we consider the matrix group $O(n)$, then each matrix gives rise to a linear transformation of euclidean n-space. Since the elements of $O(n)$ are invertible, and since orthogonal transformations preserve the euclidean metric (and there-

fore send unit vectors to unit vectors), each orthogonal matrix gives us a homeomorphism from the unit sphere S^{n-1} to itself. This operation of the orthogonal group on the sphere is compatible with the topologies of $O(n)$ and S^{n-1} in the sense that the function

$$O(n) \times S^{n-1} \to S^{n-1}$$
$$(A,\mathbf{x}) \mapsto A\mathbf{x}$$

is continuous. We say that $O(n)$ 'acts' on the space S^{n-1} as a group of homeomorphisms.

If we give $\mathbb{Z}$ its natural topology (the discrete topology induced from $\mathbb{R}$), then both of these examples fit into a general setting.

(4.14) Definition. *A topological group* G *is said to act as a group of homeomorphisms on a space* X *if each group element induces a homeomorphism of the space in such a way that:*

(a) $hg(x) = h(g(x))$† *for all* $g,h \in G$, *for all* $x \in X$;
(b) $e(x) = x$ *for all* $x \in X$, *where* e *is the identity element of* G;
(c) *the function* $G \times X \to X$ *defined by* $(g,x) \mapsto g(x)$ *is continuous.*

If x is a point of the space X, then for each $g \in G$ the corresponding homeomorphism either fixes x or maps it to some new point $g(x)$. The subset of X consisting of all such images $g(x)$, as g varies through G, is called the *orbit* of x and written $O(x)$. If two orbits intersect then they must coincide: the relation defined by $x \sim y$ if and only if $x = g(y)$ for some $g \in G$ is an equivalence relation on X whose equivalence classes are precisely the orbits of the given action. So the orbits define a partition of X. The corresponding identification space is called the *orbit space* and is written X/G. In constructing X/G we 'divide' by G in the sense that we identify two points of X if and only if they differ by one of the homeomorphisms $x \mapsto g(x)$.

In our first example, the orbit of a real number x consists of all points $x + n$ where $n \in \mathbb{Z}$. Therefore in forming $\mathbb{R}/\mathbb{Z}$ we identify two points of $\mathbb{R}$ if and only if they differ by an integer and, as explained in the preceding section, we obtain the circle as orbit space.

The orthogonal action on S^{n-1} is an example of a *transitive action*, that is, an action for which the orbit of any point is the whole space (in this case all of S^{n-1}). The proof is quite easy. Let $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$ be the standard orthonormal basis for $\mathbb{E}^n$ and, given $\mathbf{x} \in S^{n-1}$, construct a second orthonormal basis with $\mathbf{x}$ as first member. If A is the matrix of this new basis with respect to $\mathbf{e}_1,\mathbf{e}_2,\ldots,\mathbf{e}_n$, then A is orthogonal and $A(\mathbf{e}_1) = \mathbf{x}$. Therefore we have shown that the orbit of $\mathbf{e}_1$ is all of S^{n-1}. Whenever we have a transitive action, i.e., only one distinct orbit, then of course the orbit space is a single point.

† We use the same letter for a group element and the homeomorphism induced by it.