Sets and Subsets

Let $A$ be a set, meaning it is just a collection of objects. The notation $x \in A$ means that $x$ is an element of the set - in other words, $x$ is in $A$. If we write $x \notin A$, then $x$ is not in $A$. The symbol $\emptyset$ denotes the empty set, meaning the set that consists of nothing. Given two sets $A$ and $B$, if we write $B \subset A$, then $B$ is a subset of $A$, meaning that every element of $B$ is also in $A$. For any set $A$ it is true that $\emptyset \subset A$. Sometimes one uses the notation $B \subseteq A$ to emphasize that $B$ may possibly be equal to $A$. Similarly, $B \subset A$ means that $B$ is a subset of $A$ but is not equal to $A$, and $B \nsubset A$ means that $B$ is not a subset of $A$. A set $B$ such that $B \subset A$ is called a proper subset of $A$.

Let $X$ and $A$ be sets, with $A \subset X$. The complement of $A$ in $X$ (often just written “complement of $A$”, if it is understood what $X$ is), is the set of all points that are in $X$ but not in $A$. It is written $A^c = \{ x \in X : x \notin A \}$.

Another way to write this is $X \setminus A$ or $X - A$, which means the set $X$ minus all elements of $A$.

Given a set $X$, the collection of all subsets of $X$ is called the power set of $X$ and is denoted $P(X)$. Note that $P(X)$ is a set of sets. So if $O \in P(X)$, then $O$ itself is a set – in particular, a subset of $X$.

Combining sets

There are at least two key ways to combine sets: intersections and unions. The intersection $A \cap B$ is the collection of elements, or points, that are in both $A$ and $B$:

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}.$$  

Note that $A \cap B \subset A$ and $A \cap B \subset B$. If $A \cap B = \emptyset$, we say that $A$ and $B$ are disjoint, meaning they have no common elements. The union of $A$ and $B$, written $A \cup B$, is the set of points that are in either $A$ or $B$:

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}.$$  

Thus, $A \subset A \cup B$ and $B \subset A \cup B$. The difference between sets is

$$A \setminus B = \{ x : x \in A \text{ and } x \notin B \}.$$  

The following useful facts are referred to as DeMorgan’s laws:

$$(A \cap B)^c = A^c \cup B^c, \quad (A \cup B)^c = A^c \cap B^c.$$  

Another way to combine sets is through their Cartesian product. The set $A \times B$ is the set of ordered pairs $(a, b)$ with $a \in A$ and $b \in B$:

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B \}.$$
Indexed families

Let $I$ be a set and suppose that for each $i \in I$ we are given a set $A_i$. The collection of sets $\{A_i : i \in I\}$ is called a family of sets indexed by $I$. For example, if we have an infinite family of sets $A_1 , A_2 , A_3 , \ldots$, then this family is indexed by the natural numbers, $I = \mathbb{N} = \{1, 2, 3, \ldots\}$. The indexing set can have finitely or infinitely many elements, and it need not be countable. We could have a family $A_\alpha$ for each $\alpha \in \mathbb{R}$, for example. One can consider unions and intersections, based upon this indexing:

$$\bigcup_{i \in I} A_i = \{x : x \in A_j \text{ for some } j \in I\}, \quad \bigcap_{i \in I} A_i = \{x : x \in A_j \text{ for all } j \in I\}.$$ 

Exercises

1. Let $X = \{1, 2, 3, \ldots, 9, 10\}$, $A = \{1, 2, 3, 4, 5, 6\}$, and $B = \{5, 6, 7, 8\}$. Determine $A \cap B$, $A \cup B$, $A^c$, and $A \setminus B$. Is $A \subseteq B$? Is $B \subseteq A$? What is $X \setminus (A \cup B)$?

2. Prove DeMorgan's laws. [Hint: Given two sets $X$ and $Y$, if one can show that $X \subseteq Y$ and $Y \subseteq X$, then it must be true that $X = Y$.]

3. For each $x \in \mathbb{R}$, let $A_x = \{y \in \mathbb{R} : y = nx \text{ for some } n \in \mathbb{Z}\}$. Given two numbers $x_1 , x_2 \in \mathbb{R}$, when is $A_{x_1} \cap A_{x_2} \neq \emptyset$? If $A_{x_1} \cap A_{x_2} \neq \emptyset$, what is $A_{x_1} \setminus A_{x_2}$? What is $\bigcup_{x \in \mathbb{R}} A_x$? What is $\bigcap_{x \in \mathbb{R}} A_x$?