Question 1

(i) Given a metric space (X, d), define what it means for a set to be open in the associated metric topology. **[2 marks]**

(ii) Given a metric space (X, d) and the associated metric topology τ, prove that τ is in fact a topology. **[6 marks]**

(iii) Consider the set $R = [0, 1] \times [0, 1]$ with the subspace topology inherited from $\mathbb{R}^2$, where $\mathbb{R}^2$ has the metric topology. Give an example of a set that is open in R but not in $\mathbb{R}^2$. Provide an argument supporting your answer. **[2 marks]**

(iv) Consider the same set R as in part (iii), with the same topology. Let ∂R denote the boundary of R in $\mathbb{R}^2$ and define an equivalence relation on R as follows: $(x_1, y_1) \sim (x_2, y_2)$ if either (I) $(x_1, y_1) = (x_2, y_2)$ and $(x_1, y_1) \notin \partial R$ or (II) $(x_1, y_1), (x_2, y_2) \in \partial R$. Consider the associated identification space $R/\sim$.

(a) Define what it means for a set in $R/\sim$ to be open. **[2 marks]**

(b) Pick any open set $U \subset R/\sim$ that contains the equivalence class defined by (II). Draw the inverse image of U in R under the associated identification map. **[2 marks]**

(c) The space $R/\sim$ is homeomorphic to a familiar surface. Determine what this surface is, and prove it is homeomorphic to $R/\sim$. **[6 marks]**

Question 2

(i) Consider the rational numbers $\mathbb{Q}$, considered as a subset of $\mathbb{R}$.

(a) If $\mathbb{R}$ is given the discrete topology, is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Is $\mathbb{Q}$ compact? **[3 marks]**

(b) If $\mathbb{R}$ is given the finite complement topology, is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Is $\mathbb{Q}$ compact? **[3 marks]**

Provide brief arguments supporting your answers.

(ii) Suppose X is a compact topological space and $A \subset X$ is closed. Is A necessarily compact? If so, provide a brief reason why. If not, what additional assumption could you place on X to ensure that A would be compact? **[2 marks]**

(iii) If A is a dense subset of a topological space X and $O \subset X$ is open, prove that $O \subseteq \overline{A \cap O}$, where $\overline{A \cap O}$ denotes the closure of $A \cap O$ in X. **[6 marks]**

(iv) Let Y be a subspace of X. Given $A \subseteq Y$, let $\text{int}(A_Y)$ denote the interior of A in Y (ie relative to the subspace topology on Y), and let $\text{int}(A_X)$ denote the interior of A in X (ie relative to the topology on X). Prove $\text{int}(A_X) \subset \text{int}(A_Y)$, and find an example where equality doesn't hold. **[6 marks]**

Question 3

(i) Suppose $f : X \to Y$, where X and Y are topological spaces. Carefully define what it means for f to be continuous. Carefully define what it means for f to be a homeomorphism. **[2 marks]**

(ii) Which of the following are examples of homeomorphic spaces? Provide brief arguments to support your answers.

(a) The unit disk $D = \{(x, y) : x^2 + y^2 \leq 1\}$ and the closed ellipse $E = \{(x, y) : 2x^2 + 3y^2 \leq 1\}$, both with the usual subspace topologies inherited from $\mathbb{R}^2$. **[2 marks]**

(b) The set $(0, 1) \times (0, 1)$, considered as a subspace of $\mathbb{R}^2$ with the usual topology, and the unit sphere S^2, considered as a subspace of $\mathbb{R}^3$ with the usual topology. **[2 marks]**

(c) The set $X = \mathbb{R}$, with the usual topology, and the set $Y = \mathbb{R}$, with the particular point topology. **[2 marks]**

(iii) Let $A \subset X$, $f : X \to Y$ be continuous, and Y be Hausdorff. Suppose there exists a continuous function $g : \bar{A} \to Y$ such that $g(a) = f(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in \bar{A}$. **[6 marks]**

(iv) Let X denote the real numbers with the finite complement topology. Define $f : \mathbb{R} \to X$, $f(x) = x$ where the domain has the usual topology. Prove that f is continuous, but not a homeomorphism. **[6 marks]**

Question 4

(i) Carefully define what it means for a topological space X to be path connected. **[2 marks]**

(ii) Consider the functions $f(t) = (\cos(\pi t), \sin(\pi t))$ and $g(t) = (\cos(\pi t), -\sin(\pi t))$.

(a) Prove that $f, g : [0, 1] \to \mathbb{R}^2$ are homotopic. **[3 marks]**

(b) Are $f, g : [0, 1] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ homotopic? Provide a brief argument to support your answer. **[3 marks]**

(iii) Recall that a function $f : X \to Y$ is said to be null homotopic if f is homotopic to a continuous function that sends all of X to a single point in Y.

(a) Let X be a topological space such that the identity map $Id : X \to X$, $Id(x) = x$, is null-homotopic. Prove that X is path connected. **[6 marks]**

(b) Recall that a loop is a path that begins and ends at the same point. Consider the torus $S^1 \times S^1$. Give an example of a loop on the torus that is is not null-homotopic. Explain why this implies that the torus and the sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, for $n \geq 2$, cannot have isomorphic fundamental groups. **[6 marks]**

Question 5

(i) Carefully define a topological group. [**3 marks**]

(ii) Consider the set $G = S^1$, with the operation of multiplication. Prove that G is a topological group. [**5 marks**]

(iii) Let G be a topological group and H a subgroup (so $H \subset G$ and H is itself a group under the group operation of G). Prove that $\bar{H}$, the closure of H, is a subgroup of G. [**12 marks**]