Final Exam, F11PE Solutions, Topology, Autumn 2011

Question 1

(i) Given a metric space (X, d), define what it means for a set to be open in the associated metric topology.

Solution: A set $U \subset X$ is open if, for each $x \in U$ there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$, where $B_\epsilon(x) = \{y \in X : d(x, y) < \epsilon\}$. [**2 marks**]

(ii) Given a metric space (X, d) and the associated metric topology τ, prove that τ is in fact a topology.

Solution: The empty set is in τ by definition (since there are no points in $\emptyset$, it is true that around each point in $\emptyset$ we can find an open ball in $\emptyset$) and $X \in \tau$ because $B_\epsilon(x) \in X$ for any $\epsilon > 0$ and any $x \in X$. [**2 marks**] We must check that the intersection of two open sets is open. To see this, let U and V be open, and take any $x \in U \cap V$. Find ϵ_u and ϵ_v such that $B_{\epsilon_u}(x) \subset U$ and $B_{\epsilon_v}(x) \subset V$. If we then take $\delta = \min(\epsilon_u, \epsilon_v)$, we have $B_\delta(x) \subset U \cap V$, and so $U \cap V$ is open. [**2 marks**]. We must check that an arbitrary union of open sets is open. Let $\{U_\alpha\}$ be an arbitrary collection of open sets and take any $x \in \cup_\alpha U_\alpha$. There is some α such that $x \in U_\alpha$, and therefore there is a corresponding $B_\epsilon(x) \subset U_\alpha$. But this ball is then also contained in the union, so the union is open. [**2 marks**]

(iii) Consider the set $R = [0, 1] \times [0, 1]$ with the subspace topology inherited from $\mathbb{R}^2$, where $\mathbb{R}^2$ has the metric topology. Give an example of a set that is open in R but not in $\mathbb{R}^2$. Provide an argument supporting your answer.

Solution: The set $A = B_1(0,0) \cap R$ is open in R by definition of the subspace topology, since $B_1(0,0)$ is open in $\mathbb{R}^2$ in the metric topology. However, it is not open in $\mathbb{R}^2$. This is because $(0, 0) \in A$ but, for any $B_\epsilon(0, 0)$, three quarters of the ball lies outside of A - for example, the point $(0, -\epsilon/2) \in B_\epsilon(0, 0)$ but $(0, -\epsilon/2) \notin A$. Hence, no open ball of $\mathbb{R}^2$ at the origin is contained in A, so A is not open. [**2 marks**]

(iv) Consider the same set R as in part (iii), with the same topology. Let ∂R denote the boundary of R in $\mathbb{R}^2$ and define an equivalence relation on R as follows: $(x_1, y_1) \sim (x_2, y_2)$ if either (I) $(x_1, y_1) = (x_2, y_2)$ and $(x_1, y_1) \notin \partial R$ or (II) $(x_1, y_1), (x_2, y_2) \in \partial R$. Consider the associated identification space $R/\sim$.

(a) Define what it means for a set in $R/\sim$ to be open.

Solution: Let $\pi : R \to R/\sim$ be the map that sends a point to its equivalence class. The set U is open in $R/\sim$ if $\pi^{-1}(U)$ is open in R. [**2 marks**]

(b) Pick any open set $U \subset R/\sim$ that contains the equivalence class defined by (II). Draw the inverse image of U in R under the associated identification map.

Solution: This is look like an open region that goes all the way around the boundary of the rectangle. [**2 marks**]

(c) The space $R/\sim$ is homeomorphic to a familiar surface. Determine what this surface is, and prove it is homeomorphic to $R/\sim$.

Solution: The surface is S^2. Let π be the map defined in the solution of part (b). Since $R \setminus \partial R$ is an open convex set in $\mathbb{R}^2$, it is homeomorphic to all of $\mathbb{R}^2$. Let $h_1 : R \setminus \partial R \to \mathbb{R}^2$ be a homeomorphism. Let p be any point in S^2, and let h_1 be a homeomorphism $h_2 : S^2 \setminus p \to \mathbb{R}^2$. Define $f : R \to S^2$ so that $f(x) = h_2^{-1} \circ h_1(x)$ if $x \notin \partial R$ and $f(x) = p$ in $x \in \partial R$. One can check that f is an identification map such that $f^{-1}(z)$ corresponds exactly to one partition element in $R/\sim$, for each $z \in S^2$. Hence, f induces a homeomorphsim between $R/\sim$ and S^2.

[**6 marks**]

Question 2

(i) Consider the rational numbers $\mathbb{Q}$, considered as a subset of $\mathbb{R}$.

(a) If $\mathbb{R}$ is given the discrete topology, is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Is $\mathbb{Q}$ compact?

Solution: $\mathbb{Q}$ is open, because every set is open in this topology. For the same reason, its complement is open, so $\mathbb{Q}$ is also closed. It is not compact, because it contains infinitely many points. The collection of all sets of the form $\{p/q\}$ is an open cover in this topology, with no finite subcover. [**3 marks**]

(b) If $\mathbb{R}$ is given the finite complement topology, is $\mathbb{Q}$ open? Is $\mathbb{Q}$ closed? Is $\mathbb{Q}$ compact?

Solution: It is not open, because its complement contains infinitely many points. It is not closed, because it contains infinitely many points, so its complement can't be open. It is compact, however. All sets in this topology are compact - any single open set is missing, at most, finitely many points. So given any open cover $\{U_\alpha\}$, one can chose one element U_{α_0}. If it is missing $x_1, \ldots, x_n$ and $x_i \in U_{\alpha_i}$ for each i, then $\{U_{\alpha_i}\}_{i=0}^n$ is a finite subcover. [**3 marks**]

(ii) Suppose X is a compact topological space and $A \subset X$ is closed. Is A necessarily compact? If so, provide a brief reason why. If not, what additional assumption could you place on X to ensure that A would be compact?

Solution: Yes, this is true. A closed subset of a compact set is always compact. (Remark: This is not to be confused with the fact a compact set need not be closed, unless the space is Hausdorff.) [**2 marks**]

(iii) If A is a dense subset of a topological space X and $O \subset X$ is open, prove that $O \subseteq \overline{A \cap O}$, where $\overline{A \cap O}$ denotes the closure of $A \cap O$ in X.

Solution: Take any $x \in O$, and note that x is a limit point of A - since A is dense, every point in X is a limit point of A. Take any open set U containing x. We must show that $U \cap [(A \cap O) \setminus x] \neq \emptyset$. Since $O \cap U$ is open and contains x, $A \cap [(O \cap U) \setminus x] \neq \emptyset$. Since $A \cap (O \cap U) = (A \cap O) \cap U$, this proves the result. [**6 marks**]

(iv) Let Y be a subspace of X. Given $A \subseteq Y$, let $\text{int}(A_Y)$ denote the interior of A in Y (ie relative to the subspace topology on Y), and let $\text{int}(A_X)$ denote the interior of A in X (ie relative to the topology on X). Prove $\text{int}(A_X) \subset \text{int}(A_Y)$, and find an example where equality doesn't hold.

Solution: Let τ be the topology on X. Take $z \in \text{int}(A_X)$. So $z \in O \in \tau$ such that $O \subset A$. But then $z \in O \cap Y$, which is open in Y, and $O \cap Y \subset A$ since $O \subset A$. Thus, $z \in \text{int}(A_Y)$. To see that equality need not hold, consider $X = \mathbb{R}$, $Y = [0,1]$ and $A = (1/2, 1]$. We then have $\text{int}(A_X) = (1/2, 1)$ whereas $\text{int}(A_Y) = (1/2, 1]$. **[6 marks]**

Question 3

(i) Suppose $f : X \to Y$, where X and Y are topological spaces. Carefully define what it means for f to be continuous. Carefully define what it means for f to be a homeomorphism.

Solution: The function is continuous if, for any open set $U \subset Y$, the inverse image $f^{-1}(U)$ is open in X. It is a homeomorphism if it is a continuous bijection and has a continuous inverse [**2 marks**].

(ii) Which of the following are examples of homeomorphic spaces? Provide brief arguments supporting your answers.

(a) The unit disk $D = \{(x,y) : x^2 + y^2 \leq 1\}$ and the closed ellipse $E = \{(x,y) : 2x^2 + 3y^2 \leq 1\}$, both with the usual subspace topologies inherited from $\mathbb{R}^2$.

Solution: These spaces are homeomorphic. One way to see this is to use a result from class that says that any closed, convex set in the plane is homeomorphic to the closed unit disk, and note that E is a closed, convex set. [**2 marks**]

(b) The set $(0,1) \times (0,1)$, considered as a subspace of $\mathbb{R}^2$ with the usual topology, and the unit sphere S^2, considered as a subspace of $\mathbb{R}^3$ with the usual topology.

Solution: These are not homeomorphic. One way to see this is to note that the second set is compact, but the first isn't. Since compactness is a topological invariant, they cannot be homeomorphic. [**2 marks**]

(c) The set $X = \mathbb{R}$, with the usual topology, and the set $Y = \mathbb{R}$, with the particular point topology.

Solution: These are not homeomorphic. One way to see this is that X is Hausdorff but Y is not, and the property of being Hausdorff is a topological invariant. [**2 marks**]

(iii) Let $A \subset X$, $f : X \to Y$ be continuous, and Y be Hausdorff. Suppose there exists a continuous function $g : \bar{A} \to Y$ such that $g(a) = f(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in \bar{A}$.

Solution: BWOC, suppose that there is a $x \in \bar{A} \setminus A$ such that $f(x) \neq g(x)$. Let U_f be open in Y and contain $f(x)$, and let U_g be open in Y and contain $g(x)$, where these open sets are chosen to be disjoint - which is possible since Y is Hausdorff. Since both f and g are continuous, both $f^{-1}(U_f)$ and $g^{-1}(U_g)$ are open and contain x. Thus, $f^{-1}(U_f) \cap g^{-1}(U_g)$ is an open set containing x, so there is a $y \in f^{-1}(U_f) \cap g^{-1}(U_g)$ such that $y \neq x$ and $y \in A$. But then $f(y) = g(y) \in U_f \cap U_g$, which is a contradiction, because the sets were chosen to be disjoint. [**6 marks**]

(iv) Let X denote the real numbers with the finite complement topology. Define $f : \mathbb{R} \to X$, $f(x) = x$ where the domain has the usual topology. Prove that f is continuous, but not a homeomorphism.

Solution: To prove that f is continuous, let U be any open set in X. Thus, $X \setminus U$ contains finitely many elements, and so $f^{-1}(U)$ is the entire real line, except for finitely many points. This is necessarily open in the usual topology, since about any point $x \in f^{-1}(U)$, we can find a $B_\epsilon(x) \subset f^{-1}(U)$, if we just take ϵ to be one half of the shortest distance to the nearest point in the complement, which is finite since there are only finitely many such points. To see it is not a homeomorphism, let $U = (0,1)$, which is open in the real line with the usual topology. But then $X \backslash f(U) = (-\infty, 0) \cup (1, \infty)$ contains infinitely many points, so $f(U)$ is not open. Hence, the inverse is not continuous, so f is not a homeomorphism. **[6 marks]**

Question 4

(i) Carefully define what it means for a topological space X to be path connected.

Solution: X is path-connected if, for any $x, y \in X$, there exists a continuous function $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$. **[2 marks]**

(ii) Consider the functions $f(t) = (\cos(\pi t), \sin(\pi t))$ and $g(t) = (\cos(\pi t), -\sin(\pi t))$.

(a) Prove that $f, g : [0,1] \to \mathbb{R}^2$ are homotopic.

Solution: Note that both f and g are continuous maps from $[0,1]$ to the unit disk, which is a convex subset of $\mathbb{R}^2$. We have proven that any two continuous functions that map into a convex subset of Euclidean space must be homotopic. Hence, f and g are homotopic. Alternatively, you can use the straight-line homotpy $F(s,t) = (1-s)f(t) + sg(t)$. **[3 marks]**

(b) Are $f, g : [0,1] \to \mathbb{R}^2 \setminus \{(0,0)\}$ homotopic? Provide a brief argument to support your answer.

Solution: No, they are not. By drawing a picture, one can see that it is not possible to deform one curve into the other in a continuous way, because the origin is missing. If $F(s,t)$ was such a homotopy, then for each fixed t^*, $F(s,t^*)$ would need to intersect the horizontal axis. If $h(t)$ was that intersection point for each t, it would have to be continuous, since F is. But that wouldn't be possible, because it would have to "jump" over the origin somehow. **[3 marks]**

(iii) Recall that a function $f : X \to Y$ is said to be null homotopic if f is homotopic to a continuous function that sends all of X to a single point in Y.

(a) Let X be a topological space such that the identity map $Id : X \to X$, $Id(x) = x$, is null-homotopic. Prove that X is path connected.

Solution: Let $F(t,x)$ be a homotopy from Id to a constant function $g(x) = x_0$ for all $x \in X$. Given any $y \in X$, let $\gamma_y(t) = F(t,y)$. Then $\gamma_y(0) = y$ and $\gamma_y(0) = x_0$, and γ is continuous because F is. Thus, γ_y is a path from y to x_0. Given any two points x, y we can simply combine the two paths:

$$\gamma(t) = \begin{cases} \gamma_x(1-2t) & t \in [0,1/2] \\ \gamma_y(2t-1) & t \in [1/2,1] \end{cases}$$

[6 marks]

(b) Recall that a loop is a path that begins and ends at the same point. Consider the torus $S^1 \times S^1$. Give an example of a loop on the torus that is is not null-homotopic. Explain why this implies that the torus and the sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, for $n \geq 2$, cannot have isomorphic fundamental groups.

Solution: Consider a loop that goes around the short axis of the torus - so it you cut the torus along the loop you get a cylinder. This loop is not null-homotpic, because you would need to cut the torus as described to shrink it as a point. As a result, the fundamental group of the torus is non-trivial, because not all loops are null-homotopic. However, the sphere (for $n \geq 2$) is simply connected, so its fundamental group is trivial. Thus, the two fundamental groups cannot be isomorphic: one is trival and one isn't. [**6 marks**]

Question 5

(i) Carefully define a topological group.

Solution: A topological group is a set G that is both a Hausdorff topological space and a group. In addition, the two functions $m : G \times G \to G$, $m(x, y) = x \cdot y$, where $\cdot$ is the group operation, and $i : G \to G$, $i(x) = x^{-1}$, must be continuous. [**3 marks**]

(ii) Consider the set $G = S^1$, with the operation of multiplication. Prove that G is a topological group.

Solution: The set G is a Hausdorff topological space because $\mathbb{C}$ is - any metric space with at least two elements is Hausdorff. The space G is a group because inverses exist ($e^{-\mathrm{i}x}$), the identity exists $e^{\mathrm{i}0} = 1$, and the operation is associative $e^{-\mathrm{i}[(x+y)+z]} = e^{-\mathrm{i}[x+(y+z)]}$. The inverse function $i(x) = e^{-ix}$ is continuous because the exponential function is continuous. The multiplication function is continuous for the same reason. [**5 marks**]

(iii) Let G be a topological group and H a subgroup (so $H \subset G$ and H is itself a group under the group operation of G). Prove that $\bar{H}$, the closure of H, is a subgroup of G.

Solution: We must check that $\bar{H}$ is a group under $\cdot$, the operation of G. First, note that $e \in H$ so $e \in \bar{H}$. If $h \in \bar{H}$, is $h^{-1} \in \bar{H}$? If $h \in H$, then yes, clearly. So assume $h \in \bar{H} \setminus H$. BWOC, assume $h^{-1} \notin \bar{H}$. Then $\exists$ an open set U containing h^{-1} such that $U \cap H = \emptyset$. Since i, the inverse function, is continuous, $i^{-1}(U)$ is open in G, and $h \in i^{-1}(U)$. Since h is a limit point of H, there is a $g \in i^{-1}(U) \cap H$, $g \neq h$. But then $g^{-1} \in U \cap H$, which is a contradiction. So $h^{-1} \in \bar{H}$. Finally, if $h_1, h_2 \in \bar{H}$, we need to show that $h_1 \cdot h_2 \in \bar{H}$. If they're both in H, this is clear. Assume $h_2 \in \bar{H} \setminus H$ and $h_1 \in H$, and bwoc assume $h_1 \cdot h_2 \notin \bar{H}$. So there is an open U such that $h_1 \cdot h_2 \in U$ and $U \cap H = \emptyset$. Since m is continuous, $m^{-1}(U)$ is open in $G \times G$, so there is an open set of the form $U_1 \times U_2 \subset m^{-1}(U)$, and $h_1 \in U_1$, $h_2 \in U_2$. Thus, there is a $g \in U_2 \cap H$, which implies $(h_1, g) \in U_1 \times U_2 \subset m^{-1}(U)$. But this is a contradiction, since $h_1 \cdot g \in H$. The proof if $h_1, h_2 \in \bar{H} \setminus H$ is similar. [**12 marks**]